Einstein's theory of Special Relativity and the Wave Eqn.

Consider a person on a spaceship moving along the x axis with a speed v. We will denote the speed of light by c. Now lets pretend we have two observers. One in the spaceship a spaceman,Observer 1 and one observer who sits at the origin, Observer 2. The basic assumption in relativity as opposed to the classical Newtonian viewpoint is that time t is not an absolute for both Observer 1 and 2 but time itself is also a coordinate that has to be transformed accordingly. The transformations of the coordinates is given by the famous Lorentz-Fitzgerald transformations which also predict the curious relativistic effects that is much beloved by Science fiction writers. Here are the transformations. The primes are the coordinates in Observer 1's coordinate.

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}. \qquad (1)$$

This tells us how space transforms. Now time also transforms as,

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}. \qquad (2)$$

Notice right away the formulae don't make sense if $v > c$, thus we restrict ourselves always to speeds smaller than the velocity of light. Notice the formulae recover for us the standard facts of classical Mechanics at slow speeds. Suppose $v <<< c$ then the factor

$$1 - v^2/c^2$$

is almost 1. In equation (2) the numerator is t almost for v small compared to c, and thus t and t' are almost the same. Thus the clock time in a speeding car and the clock that is at rest will show no difference. However for speeds close to light speed t and t' don't coincide. Let us now take (1) and differentiate with respect(wrt) to x. We get,

$$\frac{dx'}{dx} = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

We may re-write this as,

$$\Delta x' = \frac{\Delta x}{\sqrt{1 - v^2/c^2}}.$$

Thus for a stick of length Δx in the rest coordinates, the spaceman observes the stick has length $\Delta x'$. When v comes closer to c the denominator becomes small and so the stick becomes longer and longer, so at speeds near light speed you will elongate. Notice that when v is small Δx and $\Delta x'$ are nearly the same. Similarly we can differentiate (2) wrt t. We get,

$$\frac{dt'}{dt} = \frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \sqrt{1 - v^2/c^2}. \qquad (3)$$

Here we used the fact that $dx/dt = v$. So you see that as v comes closer to c, $1-v^2/c^2$ goes to zero so,re-writing (3) as,

$$\Delta t' = \sqrt{1 - v^2/c^2}\Delta t$$

we see easily that for a time interval Δt, in the primed coordinate system $\Delta t'$ is smaller. Thus if you fly around in a spaceship for one hour at speeds approaching light speed you will notice that you will be stretched out for a mile, after one hour as measured on a clock in your spacecraft you come back you will find that 1,000 years have elapsed on Earth(time compression).

So we find from this discussion that t time and space x are not absolutes and depend on the coordinate system. In the classical system x depended on the coordinate system but t time was an absolute. So what is an absolute, never changes with the coordinate system or what is called a **coordinate invariant**.

Enter the Wave Equation.

The wave equation is an invariant and does not change with the Lorentz transformations. By this I mean we have,

$$u_{tt} - c^2 u_{xx} = u_{t't'} - c^2 u_{x'x'}. \qquad (4)$$

Thus if in particular,

$$u_{tt} - c^2 u_{xx} = 0$$

so will we also have,

$$u_{t't'} - c^2 u_{x'x'} = 0.$$

I will now begin the steps to check (4), please complete the details. The computations are a chain rule calculation.

$$u_x = u_{x'}\frac{\partial x'}{\partial x} + u_{t'}\frac{\partial t'}{\partial x}.$$

Using the Lorentz transformations,

$$u_x = u_{x'}\frac{1}{\sqrt{1 - v^2/c^2}} - \frac{v/c^2}{\sqrt{1 - v^2/c^2}}u_{t'}.$$

Now use the chain rule again and show that,

$$u_{xx} = \frac{1}{1 - v^2/c^2}u_{x'x'} + \frac{v^2/c^4}{1 - v^2/c^2}u_{t't'} - 2\frac{v/c^2}{1 - v^2/c^2}u_{x't'}.$$

Similarly calculate u_{tt} in terms of $u_{t't'}$, $u_{x'x'}$ and $u_{t'x'}$. Then substitute these expressions into, $u_{tt} - c^2 u_{xx}$ and you should be able to show (4). So we see even though the different observers may have different times and space coordinates, the fundamental equation that governs light propagation remains the same for both. Notice that both coordinate systems(observers) have the same wave equation with the **same** value of the speed of propagation c. Thus the value of the light speed remains the same in both coordinates though distance may have dilated in one and time contracted etc. These observations form the backbone of the special theory of relativity.

Another basic fact about Lorentz transformations you should verify is:

$$x^2 - c^2 t^2 = (x')^2 - c^2 (t')^2.$$