The Difference of Topology at Infinity in Changing-Sign Yamabe Problems on $\mathbb{S}^3$ (the Case of Two Masses)

ABBAS BAHRI
AND
SAGUN CHANILLO
Rutgers, The State University of New Jersey

1 Introduction, Notation, and Results

In this paper, we study the simplest cases of differences of topology at infinity in Yamabe-type problems with changing-sign solutions. In the past twenty years, there has been a wide range of activity in the study of the positive solutions to the problems of Yamabe-type, using various methods, including the study of differences of topology at infinity. After [1, 6], these differences of topology are starting to be well understood in the framework of positive functions. In sharp contrast to this, very little is known when the hypothesis of positivity is removed.

We believe that completing such a task is important not only per se, but also because it lays the ground for Yang-Mills (Einstein's?) equations. These equations should only represent a complication in the background framework with respect to Yamabe changing-sign problems. In the present work, we are completing this program for the pure Yamabe problem—allowing for sign changes—on $\mathbb{S}^3$ and for only pairs of functions at infinity. In order to formulate our results, we need to introduce some notation and quote slight variations of well-known results.

The partial differential equation that we will be studying is

$$\begin{cases} -\Delta_{\mathbb{S}^3} u + \frac{3}{4} u = u^5 \\ u \in C^\infty(\mathbb{S}^3, \mathbb{R}) \,. \end{cases} \tag{1.1}$$

Equivalently, using one of the stereographic projections from $\mathbb{S}^3$ to $\mathbb{R}^3$, we will be studying the family of solutions to the partial differential equation

$$\begin{cases} -\Delta_{\mathbb{R}^3} \varphi = \varphi^5 \\ \varphi \in H = \left\{ w \in C^\infty(\mathbb{R}^3, \mathbb{R}) \text{ such that } \int_{\mathbb{R}^3} (|\nabla w|^2 + w^6) dx < +\infty \right\} . \end{cases}$$

Both problems have equivalent variational formulations, and we will use the most convenient framework to complete our computations depending on the circumstances; i.e., in order to complete some computations, we will complete a stereographic projection from $\mathbb{S}^3$ into $\mathbb{R}^3$, with conveniently chosen north and south

Communications on Pure and Applied Mathematics, Vol. LIV, 0450–0478 (2001)

poles, and we will use the conformal invariance of several quantities and compute them in $\mathbb{R}^3$.

Recalling the variational formulation of (1.1), we introduce the unit sphere

$$\Sigma = \left\{ u \in H^1(\mathbb{S}^3, \mathbb{R}) : \left(\int\limits_{\mathbb{S}^3} \left(|\nabla u|^2 + \tfrac{3}{4} u^2 \right) dv \right)^{1/2} = |u|_{H^1} = 1 \right\}$$

and on Σ, the variational problem

$$J(u) = \frac{1}{\int_{\mathbb{S}^3} u^6 dv} \,.$$

J is C^3 on Σ but does not satisfy the Palais-Smale condition. It is known, using some subgroups of $O(4)$, to have infinitely many critical points, but no study of the so-called noncompactness has been carried out in this simple framework with the goal of finding a general formula—useful in other cases where infinitely many solutions are not known to exist—for the difference of topology at infinity.

Introducing the family of solutions of the changing-sign Yamabe equation on $\mathbb{S}^3$

$$-\Delta \omega_{i,\sigma} + \tfrac{3}{4} \omega_{i,\sigma} = \omega_{i,\sigma}^5$$

where i runs in $\mathbb{N}$ and σ indicates a position in the conformal group (we will elaborate more on this below), we can characterize the sequences $(u_k) \in \Sigma$ along which J fails to satisfy the Palais-Smale condition.

For this, we introduce, given $\omega_{i,\sigma}$,

DEFINITION 1.1

$$\mu = \left(\int\limits_{\mathbb{S}^3} |\omega_{i,\sigma}|^5 \, dv \right)^{-1/2} .$$

DEFINITION 1.2 We define the concentration point of $\omega_{i,\sigma}$ to be a point b of $\mathbb{S}^3$ such that $\delta(b)$ is minimal, where $\delta(b)$ is defined through the equation

$$\int\limits_{B(b,\delta(b))} |\nabla \omega_{i,\sigma}|^2 \, dv = \frac{S}{2}$$

and S is the Sobolev constant in dimension 3.

We then have the following:

PROPOSITION 1.3 *Let* $(u_k) \in \Sigma$ *be such that* $(J(u_k))$ *is bounded and* $(J'(u_k))$ *tends to zero. Then there exists, up to the extraction of a subsequence,* p *indices* $i_1, \dots, i_p$ *and sequences* $\sigma_k^1, \dots, \sigma_k^p$ *in the conformal group such that*

(i) $|u_k - c(p, \bar{\omega}_1, \dots, \bar{\omega}_p) \sum_{j=1}^p \omega_{i_j, \sigma_k^j} \big|_{H^1} \to 0$ *and*

(ii) $\mu_i^k / \mu_j^k + \mu_j^k / \mu_i^k + \mu_i^k \mu_j^k d(a_i^k, a_j^k)^2 \to \infty \ \forall i \neq j$*, where* a_i^k *is any point of* $\mathbb{S}^3$ *satisfying Definition* 1.2 *for* ω_{i_j, σ_k^j}.

The proof of Proposition 1.3 is omitted here, since it is similar (even simpler, since we do not worry about the positivity of the "bubbles") to the case where u_k is positive [2, 5, 7].

We develop a method similar to the one used in [2, 5]. Given basic functions $\bar{\omega}_1, \dots, \bar{\omega}_p$, denoting by $\omega_1, \dots, \omega_p$ the families of functions derived from $\bar{\omega}_1, \dots, \bar{\omega}_p$ through the action of the conformal group, we introduce the set

$$\begin{aligned} &V(p, \varepsilon, \bar{\omega}_1, \dots, \bar{\omega}_p) = V(p, \varepsilon) = \\ &\qquad \Big\{ u \in \Sigma : \Big| u - c(p, \bar{\omega}_1, \dots, \bar{\omega}_p) \sum_{i=1}^{p} \omega_i \Big|_{H^1} < \varepsilon \\ &\qquad\qquad \text{with } \frac{\mu_i}{\mu_j} + \frac{\mu_j}{\mu_i} + \mu_i \mu_j d(a_i, a_j)^2 > \frac{1}{\varepsilon} \Big\} . \end{aligned}$$

We then introduce, given $u \in V(p, \varepsilon)$, the minimizing problem

$$\min_{\alpha_i > 0,\, \sigma_i} \Big| u - \sum_{i=1}^{p} \alpha_i \omega_i \Big|_{H^1}$$

where σ_i characterizes ω_i with respect to $\bar{\omega}_i$.

We then have the following:

PROPOSITION 1.4 *The preceding minimization problem has a unique solution* $\sum_{i=1}^{p} \alpha_i \omega_i$. *Denoting* $v = u - \sum_{i=1}^{p} \alpha_i, \omega_i$, *we then have*

$$(v, \omega_i)_{H^1} = 0 \,, \quad \left(v, \frac{\partial \omega_i}{\partial \sigma_i^j} \right)_{H^1} = 0 \,, \tag{1.2}$$

where the σ_i^j*'s are coordinates for the action of the conformal group at* ω_i *with respect to* $\bar{\omega}_i$.

We omit the proof of Proposition 1.4 since it is very similar to the case where u_k is positive; see [2, 4, 5].

We study here the case $p = 2$, i.e., the case of two fundamental masses $(\bar{\omega}_1, \bar{\omega}_2)$. We will produce a formula for the difference of topology at infinity in $V(2, \varepsilon, \bar{\omega}_1, \bar{\omega}_2)$ for ε fixed small enough. The result we derive requires four hypotheses or conjectures on the solutions of the changing-sign Yamabe problem on $\mathbb{S}^3$. These are reasonable conjectures that most probably are true. The arguments supporting these conjectures are derived from transversality theory and thus conclusions may be drawn for generic functions. Since we are dealing with specific functions, there is still a possibility that one or several of the assumptions will fail. In such a case, we expect that our result will still hold, with perhaps slight modifications, because the functions ω_i are real analytic and, therefore, the conjecture that we introduce now will still hold, in a modified way. The conjectures are as follows:

(A1) Let $\tilde{\sigma}_i^j$ be parameters along the orthogonal group. For every ω a solution of the (changing-sign) Yamabe problem on $\mathbb{R}^3$, the various $\frac{\partial \omega}{\partial \tilde{\sigma}_i^j}$'s and $x \cdot \nabla \omega(x)$ are linearly independent.

(A2) For every ω a solution of the (changing-sign) Yamabe problem on $\mathbb{S}^3$, $\nabla \omega(x)$ is nonzero at every x such that $\omega(x) = 0$.

(A3) Let σ_i^j be parameters along the conformal group on $\mathbb{S}^3$. For every ω a solution of the (changing-sign) Yamabe problem on $\mathbb{S}^3$, the linearized operator $-\Delta + \frac{3}{4} - 5\omega^4$ has $\operatorname{span}\{\partial\omega/\partial\sigma_i^j\}$ as the null space.

Let N and S be the north and south poles of $\mathbb{S}^3$. The tangent spaces to $\mathbb{S}^3$ at N and S are parallel to the same $\mathbb{R}^3$ and are therefore identified; then

(A4) There exists $c_8 > 0$ such that for every $(\tilde{\omega}_1, \tilde{\omega}_2)$ derived from $(\bar{\omega}_1, \bar{\omega}_2)$ through the action of the orthogonal group of $\mathbb{S}^3$ and satisfying

$$(\Gamma)_\infty \begin{cases} \tilde{\omega}_1(S) = \tilde{\omega}_2(N) = 0 \\ \frac{\partial}{\partial \tilde{\sigma}_1^j}\big(\nabla \tilde{\omega}_1(S) \cdot \nabla \tilde{\omega}_2(N)\big) \text{ is collinear to } \frac{\partial}{\partial \tilde{\sigma}_1^j}(\tilde{\omega}_1(S)) \\ \frac{\partial}{\partial \tilde{\sigma}_2^j}\big(\nabla \tilde{\omega}_1(S) \cdot \nabla \tilde{\omega}_2(N)\big) \text{ is collinear } \frac{\partial}{\partial \tilde{\sigma}_2^j}(\tilde{\omega}(N)) \,, \end{cases}$$

we have

$$|\nabla \tilde{\omega}_1(S) \cdot \nabla \tilde{\omega}_2(N)| \geq c_8 \,.$$

In order to formulate our result about the difference of topology at infinity, we also need to introduce the following: Let $D_u(\bar{\omega}_1)$ be a small disk, the (strict) unstable manifold of $\bar{\omega}_1/|\bar{\omega}_1|_{H^1}$ for J and for some pseudogradient (we need to know $W_u(\bar{\omega}_1)$ only locally, near $\bar{\omega}_1/|\bar{\omega}_1|_{H^1}$), and let

$$c_\infty = c_\infty(\bar{\omega}_1, \bar{\omega}_2) = \left(\int \bar{\omega}_1^6 + \int \bar{\omega}_2^6\right)^2 \,.$$

Let $G_1 \subset O(4)$ be the group of symmetry of $\bar{\omega}_1$ and $G_2 \subset O(4)$ be the group of symmetry of $\bar{\omega}_2$. Let Δ_1 be the standard simplex of dimension 1. We then have the following:

THEOREM 1.5 *Assume that* (A1)–(A4) *hold. Then*

(i) *If $\bar{\omega}_1$ is not derived from $\bar{\omega}_2$ through the action of some element in $O(4)$, then there exists ε_0 and $\varepsilon_1 > 0$ small enough so that, for $\varepsilon < \varepsilon_1$, $(J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0), J_{c_\infty-\varepsilon} \cap V(2, \varepsilon_0))$ is homologically equivalent to*

$$\big(O(4)/_{G_1} \times O(4)/_{G_2} \times D_u(\bar{\omega}_1) \times D_u(\bar{\omega}_2) \times \Delta_1, O(4)/_{G_1} \times O(4)/_{G_2} \\ \times \partial(D_u(\bar{\omega}_1) \times D_u(\bar{\omega}_2) \times \Delta_1) \cup A \times D_u(\bar{\omega}_1) \times D_u(\bar{\omega}_2) \times \Delta_1\big)$$

where

$$A = \big\{(\sigma_1, \sigma_2) \in O(4)/_{G_1} \times O(4)/_{G_2} \textit{ such that } \bar{\omega}_1(\sigma_1(S))\bar{\omega}_2(\sigma(N)) \geq 0\big\} \\ - \big\{(\sigma_1, \sigma_2) \in O(4)/_{G_1} \times O(4)/_{G_2} \textit{ such that } \bar{\omega}_1(\sigma_1(S)) = 0, \\ \bar{\omega}_2(\sigma_2(N)) = 0, \textit{ and } \nabla\bar{\omega}_1(\sigma_1(S)) \cdot \nabla\bar{\omega}_2(\sigma_2(N)) \leq 0\big\} \,.$$

(ii) *If $\bar{\omega}_1$ is derived from $\bar{\omega}_2$ through the action of $O(4)$, all the above formulae have to be changed into the quotient of these spaces through the action of the group of permutations of order 2, π_2, which acts in the natural way on $(O(4)/_G \times D_u(\bar{\omega}))^2$ and permutes α_1 and α_2 in Δ_1.*

The proof of Theorem 1.5 will follow from Lemma 4.3, i.e., from the proof of existence of normal forms for J on $V(2, \varepsilon_0)$ in Section 4. It is worth noting here that while the formula for the difference of topology provided in Theorem 1.5 parallels the one provided in [6], it also differs sharply from it through the addition of various parameters, the definition of A, and also the presence here of the factor $D_u(\omega)$, which stands for the contribution of the unstable manifold of ω on Σ. $D_u(\omega)$ is reduced to zero when ω is positive; this is why we do not see this factor appear in [6].

2 Expansions at Infinity of ω_1, ω_2, and $J(\alpha_1\omega_1 + \alpha_2\omega_2)$

In order to complete our computations, a slight modification of the framework described in Section 1 is convenient. Propositions 1.3 and 1.4 still hold in this modified framework. Given a solution of the Yamabe problem on $\mathbb{S}^3$, ω, we have thought of it as a rescaled version of a basic solution $\bar{\omega}$, which is defined by the condition $|\bar{\omega}|_\infty = 1$ up to a sign change and the action of the orthogonal group. Thus, denoting by T_σ the action for the element σ of the orthogonal group and by $T_{\mu_1,a}$ the action of the conformal vector field defined by the meridians with a as north pole and μ_1 as the coefficient along this action, we can write ω as

$$\omega = T_{\mu_1,a} \circ T_\sigma \bar{\omega}$$

for suitable values of σ, a, and μ_1.

The parameters a and μ_1 can be uniquely defined in a function of ω as soon as the function ω is sufficiently "concentrated"; i.e., there exists b in $\mathbb{S}^3$ such that

$$\int_{B(b,\delta)} |\nabla\omega|^2 \, dv = \frac{S}{2}$$

where S is the Sobolev constant in dimension 3 and δ is small enough. We will refer to a and μ_1 as the point of concentration and the concentration of the "mass" ω. We remark that μ_1 might differ from μ of Definition 1.1, but their ratio belongs to a fixed compact interval in $\mathbb{R}^+ - \{0\}$.

The parameter σ corresponds to a rigid rotation. If $\bar{\omega}$ has some symmetry, typically for positive solutions, for example, σ might not be uniquely defined.

Since our problem is conformally invariant, we can always rescale a single mass ω into its normalized form $\bar{\omega}$. When there are two masses ω_1 and ω_2, or more, this is not possible anymore for both of them. Considering the case of two interacting masses ω_1 and ω_2 with small interaction, i.e., $\int(|\omega_2|^5|\omega_1| + |\omega_1|^5|\omega_2|)dv$ is small (as occurs in the behavior of the sequences that do not satisfy the Palais-Smale condition), we then know that, for one of them, the concentration μ_1 is large and larger than or equal to the concentration of the other mass.

Let us assume that the most concentrated one is ω_2. We can then rescale ω_1 around its concentration point into its normalized form $\bar{\omega}_1$ up to rigid rotation. By conformal invariance, keeping the same notation ω_2 for the second mass, we know that $\int(|\bar{\omega}_1|^5|\omega_2| + |\omega_2|^5|\bar{\omega}_1|)dv$ is still small, so that ω_2 is still concentrated around a point a with a large concentration μ_1. We choose a to be the south pole on $\mathbb{S}^3$. Accordingly, we have a north pole. We would like to have a nice representation for ω_2 after stereographic projection on $\mathbb{R}^3$, and this is easy to complete because, since ω_2 has a large concentration around the south pole, we can equivalently think that its stereographic projection has a large concentration around zero in $\mathbb{R}^3$; it can thus be written as

$$\tilde{\omega}_2(x) = \sqrt{\lambda}\,\underline{\tilde{\omega}}_2(\lambda x)\,, \quad \lambda \text{ large}, \quad \text{where } \underline{\tilde{\omega}}_2(y) = \left(\frac{2}{1+r^2}\right)^{1/2} \bar{\omega}_2'(\pi(y))\,,$$

and $\pi : \mathbb{R}^3 \to \mathbb{S}^3$ is the inverse stereographic projection, which has the analytic form

$$\pi(y) = \frac{2}{1+r^2}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \frac{r^2-1}{2} \end{pmatrix}.$$

$\bar{\omega}_2'$ is similar to $\bar{\omega}_2$, possibly not equal; it is a normalized solution of the Yamabe problem on $\mathbb{S}^3$.

Since no confusion is possible, we replace the notation $\bar{\omega}_2'$ by $\bar{\omega}_2$ for the sake of simplicity. Let $\bar{\omega}_2(\infty)$ be the value of $\bar{\omega}_2$ at the north pole. We then have the following:

LEMMA 2.1 *There are suitable constants c_2, c_4, c_5, c_6, and c_{ij} and a linear form L on $\mathbb{R}^3$ depending only on $\bar{\omega}_2$ such that $\tilde{\omega}_2$ has, for $\lambda|x|$ large, the expansions*

$$\tilde{\omega}_2(x) \sim \bar{\omega}_2(\infty)\frac{\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}} + \frac{c_2\lambda^{3/2}x_1}{(1+\lambda^2r^2)^{3/2}} + O\left(\frac{\lambda^{5/2}r^2}{(1+\lambda^2r^2)^{5/2}}\right), \tag{2.1}$$

$$\begin{aligned}\tilde{\omega}_2(x) \sim{}& \bar{\omega}_2(\infty)\frac{\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}} + \frac{c_2\lambda^{3/2}x_1}{(1+\lambda^2r^2)^{3/2}} + \frac{c_4\sqrt{\lambda}}{(1+\lambda^2r^2)^{3/2}} \\ &+ \frac{\sum c_{ij}\lambda^{5/2}x_ix_j}{(1+\lambda^2r^2)^{5/2}} + \frac{c_5\lambda^{3/2}L(x)}{(1+\lambda^2r^2)^{5/2}} + O\left(\frac{\lambda^{7/2}r^3}{(1+\lambda^2r^2)^{7/2}}\right),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\tilde{\omega}_2(x) ={}& \bar{\omega}_2(\infty)\frac{\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}} + \frac{c_2\lambda^{3/2}x_1}{(1+\lambda^2r^2)^{3/2}} + \frac{c_4\sqrt{\lambda}}{(1+\lambda^2r^2)^{3/2}} \\ &+ \sum\frac{c_{ij}\lambda^{5/2}x_ix_j}{(1+\lambda^2r^2)^{5/2}} + \frac{c_5\lambda^{3/2}L(x)}{(1+\lambda^2r^2)^{5/2}} + \frac{c_6\sqrt{\lambda}}{(1+\lambda^2r^2)^{5/2}} \\ &+ O\left(\frac{\lambda^{7/2}r^3}{(1+\lambda^2r^2)^{7/2}}\right) \quad \textit{for every } x \textit{ in } \mathbb{R}^3\,.\end{aligned} \tag{2.3}$$

Lemma 2.1, which holds in a C^1 sense as will be seen clearly from its proof (C^1 with respect to λ and ∞), allows us to derive a first expansion of $J(\alpha_1\omega_1 + \alpha_2\omega_2)$ in a C^1 sense. The proof of the lemma is provided later.

LEMMA 2.2 *Let α_1 and α_2 be nonnegative such that $|\alpha_1\bar{\omega}_1 + \alpha_2\omega_2|_{H^1} = 1$. Assume that $\int(|\bar{\omega}_1|^5|\omega_2| + |\omega_2|^5|\bar{\omega}_1|)dv$ is small. Let $\bar{\lambda}$ be the concentration of ω_2 in the coordinates defined above. Then $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2)$ expands, in the C^1 sense, as follows: With suitable constants c and c′*

$$J\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big) =$$

$$\frac{\big(\alpha_1^2\int\bar{\omega}_1^6 + \alpha_2^2\int\bar{\omega}_2^6\big)^3}{\int_{\mathbb{S}^3}\big(\alpha_1^6\bar{\omega}_1^6 + \alpha_2^6\bar{\omega}_2^6\big)dv} \times \Bigg[1 + 6\bigg(\frac{\alpha_1\alpha_2}{\alpha_1^2\int\bar{\omega}_1^6 + \alpha_2^2\int\bar{\omega}_2^6} - \frac{\alpha_1^5\alpha_2 + \alpha_2^5\alpha_1}{\alpha_1^6\int\bar{\omega}_1^6 + \alpha_2^6\int\bar{\omega}_2^6}\bigg)$$

$$\times\left(\frac{c\bar{\omega}_2(\infty)\underline{\omega}_1(0)}{\sqrt{\lambda}} + \frac{c'\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)}{\lambda^{3/2}}\right)$$

$$+ O\bigg(\frac{\bar{\omega}_2(\infty)^2}{\lambda} + \frac{|\underline{\omega}_1(0)|^2}{\lambda} + \frac{1}{\lambda^2}\bigg)\Bigg].$$

This expansion can be differentiated with respect to 0, ∞, *and* λ.

PROOF: Expanding, we have

$$J\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big)$$

$$= \frac{\big(\alpha_1^2\int\bar{\omega}_1^6 + \alpha_2^2\int\bar{\omega}_2^6 + 2\alpha_1\alpha_2\int\bar{\omega}_1^5\omega_2\big)^3}{\alpha_1^6\int\bar{\omega}_1^6 + \alpha_2^6\int\bar{\omega}_2^6 + 6\big(\alpha_1^5\alpha_2 + \alpha_2^5\alpha_1\big)\int\bar{\omega}_1^5\omega_2 + O\big(\int\bar{\omega}_1^4 w_2^2 + \omega_2^4\bar{\omega}_1^2 + |\bar{\omega}_1|^3|\omega_2|^3\big)}$$

$$= \frac{\big(\alpha_1^2\int\bar{\omega}_1^6 + \alpha_2^2\int\bar{\omega}_2^6\big)^3}{\alpha_1^6\int\bar{\omega}_1^6 + \alpha_2^6\int\bar{\omega}_2^6}\bigg(1 + 6\bigg(\frac{\alpha_1\alpha_2}{\alpha_1^2\int\bar{\omega}_1^6 + \alpha_2^2\int\bar{\omega}_2^6} - \frac{\alpha_1^5\alpha_2 + \alpha_2^5\alpha_1}{\alpha_1^6\int\bar{\omega}_1^6 + \alpha_2^6\int\bar{\omega}_2^6}\bigg)$$

$$\times\int\bar{\omega}_1^5\omega_2 + O\bigg(\bigg(\int\bar{\omega}_1^5\omega_2\bigg)^2 + \int\bar{\omega}_1^4\omega_2^2 + \omega_2^4\bar{\omega}_1^2 + |\bar{\omega}_1|^3|\omega_2|^3\,dv\bigg)\bigg).$$

We first estimate $\int_{\mathbb{S}^3}\bar{\omega}_1^5\omega_2\,dv = \int_{\mathbb{S}^3}\omega_2^5\bar{\omega}_1\,dv = \int_{\mathbb{R}^3}\tilde{\omega}_2^5\tilde{\bar{\omega}}_1\,dx$. For the sake of simplicity, we remove $\sim$ from above $\tilde{\omega}_2$ and we denote by $\underline{\omega}_1$ the function $\tilde{\bar{\omega}}_1$. We have

$$\int_{\mathbb{R}^3}\omega_2^5\underline{\omega}_1\,dx$$

$$= \underline{\omega}_1(0)\int\omega_2^5 + \int(\underline{\omega}_1 - \underline{\omega}_1(0))\omega_2^5$$

$$= \underline{\omega}_1(0)\int -\Delta\omega_2 + \int\limits_{B_\delta}\left(\nabla\underline{\omega}_1(0)\cdot x + \frac{1}{2}D^2\underline{\omega}_1(0)\cdot x\cdot x + O(|x|^3)\right)\omega_2^5$$
$$+ O\left(\int\limits_{B_\delta^c}|\omega_2|^5\right).$$

B_δ is a small ball of radius δ around the origin. Since $\int\frac{r^2dr}{(1+r^2)^{5/2}}$ is finite, using Lemma 2.1,

$$\int\limits_{B_\delta^c}|\omega_2|^5 = O\left(\frac{\bar{\omega}_2(\infty)^5}{\lambda^{5/2}}\right) + O\left(\frac{1}{\lambda^{7/2}}\right).$$

Clearly,

$$\int\limits_{\mathbb{R}^3} -\Delta\omega_2 = \lim_{R\to+\infty}\int\limits_{\partial B_R} -\frac{\partial\omega_2}{\partial\nu} = \frac{C\bar{\omega}_2(\infty)}{\sqrt{\lambda}}.$$

We also have

$$\int\limits_{B_\delta}|x|^3|\omega_2|^5 = \frac{1}{\lambda^3\sqrt{\lambda}}O\left(\int\limits_{r\le\delta\lambda}\frac{r^5dr}{(1+r^2)^{5/2}}\right) = \delta O\left(\frac{1}{\lambda^{5/2}}\right) = o\left(\frac{1}{\lambda^{5/2}}\right).$$

We are thus left with

$$\int\limits_{B_\delta}\big(\nabla\underline{\omega}_1(0)\cdot x\big)\omega_2^5 \quad\text{and}\quad \frac{1}{2}\int\limits_{B_\delta}\big(D^2\underline{\omega}_1(0)\cdot x\cdot x\big)\omega_2^5.$$

We have

$$\int\limits_{B_\delta}x\omega_2^5 = \int\limits_{\mathbb{R}^3}x\omega_2^5 - \int\limits_{B_\delta^c}x\omega_2^5, \quad \int\limits_{B_\delta^c}x\omega_2^5 = O\left(\frac{1}{\lambda^{5/2}}\int\limits_{|x|\ge\delta}\frac{r^3dr}{r^5}\right) = O\left(\frac{1}{\lambda^{5/2}}\right).$$

However, in ω_2^5, the main term in the expansion when $\lambda|x|$ is large is radial, and its contribution to $\int_{B_\delta^c}x\omega_2^5$ is therefore zero. The remainder terms are then

$$O\left(\frac{1}{\lambda^{7/2}}\int\limits_{|x|\ge\delta}\frac{r^3dr}{r^6}\right) = O\left(\frac{1}{\lambda^{7/2}}\right);$$

thus $\int_{B_\delta^c}x\omega_2^5 = O(1/\lambda^{7/2})$. On the other hand,

$$\int\limits_{\mathbb{R}^3}x\omega_2^5 = \int\limits_{|x|\le M}x\omega_2^5 + O\left(\frac{1}{\lambda^{7/2}}\right) = \int\limits_{|x|=M}\left(\omega_2\frac{\partial x}{\partial\nu} - x\frac{\partial\omega_2}{\partial\nu}\right).$$

Then, by using the expansion of Lemma 2.1, the radial term yields a zero contribution, by oddness. The contribution of $O\big(\frac{\lambda^{5/2}|x|^2}{(1+\lambda^2|x|^2)^{5/2}}\big)$ is

$$\frac{1}{\lambda^{5/2}} O\left(\int\limits_{|x|=M} \frac{r^4}{r^5} d\sigma\right) = \frac{1}{M} O\left(\frac{1}{\lambda^{5/2}}\right) = o\left(\frac{1}{\lambda^{5/2}}\right).$$

We are left with

$$\int\limits_{\mathbb{R}^3} (\nabla \underline{\omega_1}(0)\cdot x) \frac{c_2^5(\lambda^{3/2}x_1)^5}{(1+\lambda^2|x|^2)^{15/2}}\, dx \cdot e_1$$

being, after a suitable rotation, the direction of $\nabla\underline{\omega}_2$ at infinity. We thus have, with an appropriate constant c,

$$\int\limits_{B_\delta} x\omega_2^5 = c\frac{\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)}{\lambda^{3/2}} + o\left(\frac{1}{\lambda^{5/2}}\right).$$

Turning now to $\frac{1}{2}\int_{B_\delta} D^2\underline{\omega}_1(0)x\cdot x\omega_2^5$, we have

$$\begin{aligned}
&\frac{1}{2}\int\limits_{B_\delta} D^2\underline{\omega}_1(0)x\cdot x\omega_2^5\\
&\quad = -\Delta\underline{\omega}_1(0)\int\limits_{B_\delta}\omega_2 - \frac{1}{2}\int\limits_{\partial B_\delta}\left(D^2\underline{\omega}_1(0)x\cdot x\frac{\partial\omega_2}{\partial\nu} - \frac{\partial}{\partial\nu}\big(D^2\underline{\omega}_1(0)x\cdot x\big)\omega_2\right)\\
&\quad = -\frac{\Delta\underline{\omega}_1(0)}{6}\int\limits_{B_\delta}\omega_2\Delta\Big(\sum x_i^2\Big)\\
&\qquad - \frac{1}{2}\int\limits_{\partial B_\delta}\left(D^2\underline{\omega}_1(0)x\cdot x\frac{\partial\omega_2}{\partial\nu} - \frac{\partial}{\partial\nu}\big(D^2\underline{\omega}_1(0)x\cdot x\big)\omega_2\right)\\
&\quad = -\frac{\underline{\omega}_1(0)^5}{6}\int\limits_{B_\delta}\Big(\sum x_i^2\Big)\omega_2^5 + \frac{\underline{\omega}_1(0)^5}{6}\int\limits_{\partial B_\delta}\left(\omega_2\frac{\partial}{\partial\nu}r^2 - r^2\frac{\partial}{\partial\nu}\omega_2\right)\\
&\qquad - \frac{1}{2}\int\limits_{\partial B_\delta}\left(D^2\underline{\omega}_1(0)x\cdot x\frac{\partial\omega_2}{\partial\nu} - \frac{\partial}{\partial\nu}\big(D^2\underline{\omega}_1(0)x\cdot x\big)\omega_2\right).
\end{aligned}$$

Observe that

$$\underline{\omega}_1(0)^5\int\limits_{B_\delta}|x|^2|\omega_2|^5 = \frac{\underline{\omega}_1(0)^5}{\lambda^2\sqrt{\lambda}} O\left(\int\limits_{r\le\delta\lambda}\frac{r^4 dr}{(1+r^2)^{5/2}}\right) = \underline{\omega}_1(0)^5 O\Big(\frac{\log\lambda}{\lambda^{5/2}}\Big).$$

We are thus left with boundary terms, for which we can use the expansions of Lemma 2.1. In these boundary terms, $\frac{c_2\lambda^{3/2}x_1}{(1+\lambda^2r^2)^{3/2}}$ yields a zero contribution, by oddness. $\frac{\bar{\omega}_2(\infty)\sqrt{\lambda}}{(1+\lambda^2 r^2)^{1/2}}$ yields, after integration by parts and an error term that is of order $\underline{\omega}_1(0)^5 O\big(\frac{\log\lambda}{\lambda^{5/2}}\big)$,

$$\bar{\omega}_2(\infty)\int_{B_\delta}\frac{1}{2}D^2\underline{\omega}_1(0)\cdot x\cdot x\frac{\lambda^{5/2}}{(1+\lambda^2|x|^2)^{5/2}}\,dx$$
$$= O\left(\frac{\bar{\omega}_2(\infty)\underline{\omega}_1(0)^5}{\lambda^2\sqrt{\lambda}}\int_{B_\delta}\frac{r^4dr}{(1+r^2)^{5/2}}\right) = O\left(\frac{\bar{\omega}_2(\infty)\underline{\omega}(0)^5}{\lambda^2\sqrt{\lambda}}\log\lambda\right).$$

Using (2.1) from Lemma 2.1, the contribution of $O\big(\frac{\lambda^{5/2}r^2}{(1+\lambda^2r^2)^{5/2}}\big)$ to

$$\frac{\underline{\omega}_1(0)^5}{6}\int_{\partial B_\delta}\left(\omega_2\frac{\partial r^2}{\partial\nu} - r^2\frac{\partial\omega_2}{\partial\nu}\right)$$

is $\underline{\omega}_1(0)^5 O\big(1/\lambda^{5/2}\big)$. We are thus left with its contribution to the other boundary term, and here we use (2.2) from Lemma 2.1, which tells us that

$$\begin{aligned}\omega_2^* &= O\left(\frac{\lambda^{5/2}r^2}{(1+\lambda^2r^2)^{5/2}}\right)\\ &= \sum c_{ij}\frac{\lambda^{5/2}x_ix_j}{(1+\lambda^2r^2)^{5/2}} + \frac{c_4\sqrt{\lambda}}{(1+\lambda^2r^2)^{3/2}} + \frac{c_5\lambda^{3/2}L(x)}{(1+\lambda^2r^2)^{5/2}}\\ &\quad + O\left(\frac{\lambda^{7/2}r^3}{(1+\lambda^2r^2)^{7/2}}\right)\end{aligned}$$

$c_5\frac{\lambda^{3/2}L(x)}{(1+\lambda^2r^2)^{5/2}}$ yields, by oddness, a zero contribution to this boundary term, and $O\big(\frac{\lambda^{7/2}r^3}{(1+\lambda^2r^2)^{7/2}}\big)$ yields a contribution that is $O(1/\lambda^{7/2})$.

We are left with the two other, radial plus quadratic, terms. Using oddness, we can easily see that the contribution of the radial term to

$$\int_{\partial B_\delta}\left(D^2\underline{\omega}_1(0)x\cdot x\frac{\partial\omega_2}{\partial\nu} - \frac{\partial}{\partial\nu}(D^2\underline{\omega}_1(0)x\cdot x)\omega_2\right)d\sigma$$

is $\underline{\omega}_1(0)^5 O(1/\lambda^{5/2})$. With the quadratic term, we have

$$\int_{\partial B_\delta}\Bigg[\big(D^2\underline{\omega}_1(0)x\cdot x\big)\frac{\partial}{\partial\nu}\left(\lambda^{5/2}\sum\frac{c_{ij}x_ix_j}{(1+\lambda^2|x|^2)^{5/2}}\right)$$
$$-\left(\frac{\partial}{\partial\nu}D^2\underline{\omega}_1(0)\cdot x\cdot x\right)\lambda^{5/2}\sum\frac{c_{ij}x_ix_j}{(1+\lambda^2|x|^2)^{5/2}}\Bigg]r^2\,d\sigma$$

$$\begin{aligned}
&= \delta^2 \left\{ \frac{\lambda^{5/2}}{(1+\lambda^2\delta^2)^{5/2}} \sum_{\substack{\ell,m\\ i,j}} \int_{\partial B_\delta} \left(\left(D^2\underline{\omega}_1(0)\right)_{\ell,m} x_\ell x_m \times c_{ij} \frac{\partial r^2}{\partial \nu} \frac{x_i}{|x|} \frac{x_j}{|x|} \right.\right. \\
&\qquad\qquad \left. - \left(D^2\underline{\omega}_1(0)\right)_{\ell,m} \frac{\partial r^2}{\partial \nu} \times \frac{x_\ell}{|x|} \frac{x_m}{|x|} c_{ij} x_i x_j \right) d\sigma \\
&\qquad \left. + \lambda^{5/2} \int_{\partial B_\delta} \left(D^2\underline{\omega}_1(0)\cdot x \cdot x\right) \Bigl(\sum c_{ij} x_i x_j \Bigr) \frac{\partial}{\partial \nu} \frac{1}{(1+\lambda^2 r^2)^{5/2}}\, d\sigma \right\} \\
&= -\frac{5\lambda^{5/2}\lambda^2\delta^7}{(1+\lambda^2\delta^2)^{5/2+1}} \int_{\partial B_\delta} \left(D^2\underline{\omega}_1(0) \cdot \frac{x}{|x|} \cdot \frac{x}{|x|} \right) \left(\sum c_{ij} \frac{x_i}{|x|} \frac{x_j}{|x|} \right) d\sigma \\
&= -\frac{5\lambda^{9/2}\delta^7}{(1+\lambda^2\delta^2)^{7/2}} \sum \int_{\partial B_\delta} c_{ij} \left(D^2\underline{\omega}_1(0)\right)_{ij} \left(\frac{x_i}{|x|}\right)^2 \left(\frac{x_j}{|x|}\right)^2 d\sigma \\
&= -\frac{c''}{\lambda^{5/2}} \sum c_{ij} \left(D^2\underline{\omega}_1(0)\right)_{ij} + O\left(\frac{1}{\lambda^{7/2}}\right) \\
&= -\frac{c''}{\lambda^{5/2}} D^2\underline{\omega}_1(0) \cdot D^2\underline{\omega}_2(\infty) + O\left(\frac{1}{\lambda^{7/2}}\right).
\end{aligned}$$

Summing up, we have derived that

$$\begin{aligned}
\int \underline{\omega}_1 \omega_2^5 = c\frac{\bar{\omega}_2(\infty)\underline{\omega}_1(0)}{\sqrt{\lambda}} + c' \frac{\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)}{\lambda^{3/2}} - \frac{c''}{\lambda^{5/2}} D^2\underline{\omega}_1(0)\cdot D^2\underline{\omega}_2(\infty) \\
+ O\left(\underline{\omega}_1(0)^5 \frac{\log\lambda}{\lambda^{5/2}}\right) + O\left(\frac{\bar{\omega}(\infty)^5}{\lambda^{5/2}}\right) + o\left(\frac{1}{\lambda^{5/2}}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int |\underline{\omega}_1|^3|\omega_2|^3 &\le |\underline{\omega}_1(0)|^3 \int_{|x|\le 1} |\omega_2|^3 + C \int_{|x|\le 1} |x||\omega_2|^3 + \int_{|x|\ge 1} |\underline{\omega}_1|^3|\omega_2|^3\,, \\
\int \underline{\omega}_1^4\omega_2^2 &\le \underline{\omega}_1^4(0) \int_{|x|\le 1} \omega_2^2 + C \int_{|x|\le 1} |x|\omega_2^2 + \int_{|x|\ge 1} \underline{\omega}_1^4\omega_2^2\,, \\
\int \omega_2^4\underline{\omega}_1^2 &\le \underline{\omega}_1^2(0) \int_{|x|\le 1} \omega_2^4 + C \int_{|x|\le 1} |x|\omega_2^4 + \int_{|x|\ge 1} \omega_2^4\underline{\omega}_1^2\,.
\end{aligned}$$

$|\underline{\omega}_1|$ is less than $C/(1+r^2)^{1/2}$, and by Lemma 2.1, $|\omega_2|$ is less than $\frac{|\bar{\omega}_2(\infty)|}{r\sqrt{\lambda}} + O\bigl(\frac{1}{\lambda^{3/2}r^2}\bigr)$ for $r \geq 1$. Thus

$$\begin{aligned}
&\int\limits_{|x|\geq 1} \underline{\omega}_1^4\omega_2^2 + \int\limits_{|x|\geq 1} \omega_2^4\underline{\omega}_1^2 + \int\limits_{|x|\geq 1} |\underline{\omega}_1|^3|\omega_2|^3 \\
&\quad\leq C\int\limits_{|x|\geq 1}\biggl[\frac{1}{(1+r^2)^2}\biggl(\frac{\bar{\omega}_2(\infty)^2}{\lambda r^2}+\frac{1}{\lambda^3 r^4}\biggr)+\frac{1}{1+r^2}\biggl(\frac{\bar{\omega}_2(\infty)^4}{\lambda^2 r^2}+\frac{1}{\lambda^6 r^8}\biggr) \\
&\qquad\qquad + \frac{1}{(1+r^2)^{3/2}}\biggl(\frac{\bar{\omega}_2(\infty)^3}{\lambda^{3/2}r^{3/2}}+\frac{1}{\lambda^{9/2}r^6}\biggr)\biggr]r^2\,dr \\
&\quad= O\biggl(\frac{\bar{\omega}_2(\infty)^2}{\lambda}+\frac{1}{\lambda^3}\biggr).
\end{aligned}$$

Using Lemma 2.1, we also know that

$$|\omega_2(x)| \leq \frac{|\bar{\omega}_2(\infty)|\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}} + O\left(\frac{\sqrt{\lambda}(1+\lambda r)}{(1+\lambda^2r^2)^{3/2}}\right).$$

Thus

$$\begin{aligned}
&\int\limits_{r\leq 1} |\omega_2|^3\bigl(|\underline{\omega}_1(0)|^3+Cr\bigr)+\omega_2^4\bigl(\underline{\omega}_1(0)^2+Cr\bigr)+\omega_2^2\bigl(\underline{\omega}_1(0)^4+Cr\bigr) \\
&\quad\leq |\bar{\omega}_2(\infty)|^3\int\limits_{r\leq 1}\bigl(|\underline{\omega}_1(0)|^3+Cr\bigr)\frac{\lambda^{3/2}r^2\,dr}{(1+\lambda^2r^2)^{3/2}} \\
&\qquad+\bar{\omega}_2(\infty)^4\int\limits_{r\leq 1}\bigl(\underline{\omega}_1(0)^2+Cr\bigr)\frac{\lambda^2r^2}{(1+\lambda^2r^2)^2}\,dr \\
&\qquad+\bar{\omega}_2(\infty)^2\int\limits_{r\leq 1}\bigl(\underline{\omega}_1(0)^4+Cr\bigr)\frac{\lambda r^2}{1+\lambda^2r^2}\,dr \\
&\qquad+O\Biggl(\int\limits_{r\leq 1}\biggl[\frac{\lambda^{3/2}(1+\lambda^3r^3)}{(1+\lambda^2r^2)^{9/2}}\bigl(|\underline{\omega}_1(0)|^3+Cr\bigr)+\frac{\lambda^2(1+\lambda^4r^4)}{(1+\lambda^2r^2)^6}\bigl(\underline{\omega}_1(0)^2+Cr\bigr) \\
&\qquad\qquad+\frac{\lambda(1+\lambda^2r^2)}{(1+\lambda^2r^2)^3}\bigl(\underline{\omega}_1(0)^4+Cr\bigr)\biggr]r^2\,dr\Biggr) \\
&\quad= O\biggl(\frac{\underline{\omega}_1(0)^2+\bar{\omega}_2(\infty)^2}{\lambda}+\frac{1}{\lambda^2}\biggr).
\end{aligned}$$

Hence,

$$\int \left(|\underline{\omega}_1|^3|\omega_2|^3 + \underline{\omega}_1^4\omega_2^2 + \underline{\omega}_1^2\omega_2^4\right) \quad \text{is} \quad O\left(\frac{\underline{\omega}_1(0)^2 + \bar{\omega}_2(\infty)^2}{\lambda} + \frac{1}{\lambda^2}\right),$$

and the expansion for J follows from the above estimate and the expansion of $\int \underline{\omega}_1\omega_2^5$ at the C^0-level. Because Lemma 2.1 holds at the C^1-level—i.e., we can differentiate the formulae of Lemma 2.1 with respect to ∞ and $\lambda\frac{\partial}{\partial\lambda}$, keeping all estimates unchanged, while $\frac{\partial\omega_1}{\partial 0}$ behaves as $\underline{\omega}_1$ does—our expansion holds, in fact, in the C^1-sense. □

PROOF OF LEMMA 2.1: $\tilde{\omega}_2(x)$ is equal to $\sqrt{\lambda}\underline{\omega}_2(\lambda x)$ and

$$\underline{\omega}_2(y) = \left(\frac{2}{1+r^2}\right)^{1/2} \bar{\omega}_2'(\pi(y)) = \left(\frac{2}{1+r^2}\right)^{1/2} \varphi(\pi(y))$$

where

$$\pi : \mathbb{R}^3 \to \mathbb{S}^3\,, \qquad y \to \frac{2}{1+r^2}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \frac{r^2-1}{2} \end{pmatrix}; \quad \text{(see [6]).}$$

Extending φ to a neighborhood of $\mathbb{S}^3$ in $\mathbb{R}^3$, we have (N is the north pole)

$$\begin{aligned}\varphi(\pi(y)) = \varphi(N) + D\varphi(N)(\pi(y) - N) + \tfrac{1}{2}D^2\varphi(N)\big(\pi(y) - N\big)\big(\pi(y) - N\big)\\ + O\big(|\pi(y) - N|^3\big)\,.\end{aligned}$$

Thus, since

$$\pi(y) - N = \frac{2}{1+r^2}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ -1 \end{pmatrix},$$

$$\begin{aligned}\left(\frac{2}{1+r^2}\right)^{1/2}\varphi(\pi(y)) = \frac{\sqrt{2}}{(1+r^2)^{1/2}}\varphi(N) + \frac{2\sqrt{2}}{(1+r^2)^{3/2}}D\varphi(N)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ -1 \end{pmatrix}\\ + \frac{1}{2}\left(\frac{2}{1+r^2}\right)^{5/2} D^2\varphi(N)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ -1 \end{pmatrix}\\ + O\left(\frac{1+r^3}{(1+r^2)^{7/2}}\right).\end{aligned}$$

Since $\tilde{\omega}_2(x)$ is $\sqrt{\lambda}\underline{\omega}_2(\lambda x)$, (2.3) and (2.2) in Lemma 2.1 follow as does (2.1). □

3 The v and $\bar{v}$ Parts of u

Next, we want to expand $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+v)$, where v, following the parametrization introduced in Proposition 1.4, satisfies

$$\tag{1.2} \begin{cases} (v, \bar{\omega}_1)_{H^1} = (v, \omega_2)_{H^1} = 0 \\ \left(v, \frac{\partial \bar{\omega}_1}{\partial \tilde{\sigma}_1}\right)_{H^1} = \left(v, \frac{\partial \omega_2}{\partial \tilde{\sigma}_2}\right)_{H^1} = 0;\ \left(v, \frac{\partial \omega_2}{\partial \lambda}\right)_{H^1} = 0;\ \left(v, \frac{\partial \omega_1}{\partial \mu}|_{\omega_1=\bar{\omega}_1}\right) = 0\,, \end{cases}$$

where μ is the rescaling factor for the family of functions corresponding to ω_1. The derivative is taken at $\omega_1 = \underline{\omega}_1$. The $\tilde{\sigma}_i^j$ are variables corresponding to the action of the orthogonal group of $\mathbb{S}^3$ on $\bar{\omega}_1$ and ω_2, respectively.

A straightforward computation yields

$$J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v) = J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2) + (f, v)_{H^1} + Q_0(v, v) + O(|v|_{H^1}^3)\,.$$

This expansion can be differentiated with respect to v; v is small in the H^1-norm, by construction. Q_0 is a quadratic form on the v's satisfying (1.2). We prove in the sequel various results on Q_0 and estimates on $|v|_{H^1}$ and several related quantities. The estimates will be completed in $\mathbb{R}^3$, after stereographic projection. Thus, we will use the notation $\underline{\omega}_1$ and ω_2, which stand for $\tilde{\bar{\omega}}_1$ and $\tilde{\omega}_2$, instead of $\bar{\omega}_1$ and ω_2. However, for the sake of simplicity, we will keep the same notation for v. What v refers to will be clear from the context.

We now introduce the following:

$$\Omega_1 = \left\{x \in \mathbb{R}^3 : \frac{1}{8\sqrt{\lambda}} < |x| < \frac{\sqrt{\lambda}}{8}\right\}, \qquad \Omega_2 = \left\{x \in \mathbb{R}^3 : |x| < \frac{1}{8\sqrt{\lambda}}\right\};$$

P_i, the orthogonal projection from $H = \{w : \nabla w \in L^2(\mathbb{R}^3), w \in L^6(\mathbb{R}^3)\}$ onto $H_0^1(\Omega_i)$;

$$S_1 : H_0^1(\Omega_1) \to H_1 = \left\{w \in H_0^1(\Omega_1) : \int \nabla w \cdot \nabla \underline{\omega}_1 = \int \nabla w \cdot \nabla \frac{\partial \underline{\omega}_1}{\partial \sigma_1^j} = 0\right\},$$

$$S_2 : H_0^1(\Omega_2) \to H_2 = \left\{w \in H_0^1(\Omega_2) : \int \nabla w \cdot \nabla \omega_2 = \int \nabla w \cdot \nabla \frac{\partial \omega_2}{\partial \sigma_2^j} = 0\right\};$$

the orthogonal projectors (the σ_i^j's are coordinates along the conformal group)

$$K = \left\{h : \nabla h \in L^2(\mathbb{R}^3), h \in L^6(\mathbb{R}^3), \Delta h = 0 \text{ in } \Omega_1 \cup \Omega_2\right\}.$$

Let

$$\begin{aligned} v &\overset{T}{\longmapsto} \big(S_1 \circ P_1(v), S_2 \circ P_2(v), v - (P_1 + P_2)(v)\big) = \big(\tilde{v}_1, \tilde{v}_2, h\big) = \tilde{V}\,, \\ H &\longmapsto H_1 \times H_2 \times K\,. \end{aligned}$$

We then have the following:

Lemma 3.1 *Assume* (A3): *For every ω solution of the Yamabe problem on $\mathbb{S}^3$, the linearized operator $-\Delta + \frac{3}{4} - 5\omega^4$ has* span $\left\{\frac{\partial \omega}{\partial \sigma^j}\right\}$ *as the null space. Then, for α_2/α_1 close to* 1,

(i) *$Q_0(v, v) = \int |\nabla v|^2 - S\int(\underline{\omega}_1^4 + \omega_2^2)v^2\,dx + o(|v|^2_{H^1})$, up to a positive constant, and is defined on the set of v's satisfying* (1.2)*, and*

(ii) *T is a linear isomorphism that is bounded and has its inverse bounded independently of λ.*

Q_0 reads as

$$Q_0(\tilde V, \tilde V) = \tilde Q_1(\tilde v_1, \tilde v_1) + \tilde Q_2(\tilde v_2, \tilde v_2) + \int |\nabla h|^2 + o\left(\int |\nabla \tilde v_1|^2 + \int |\nabla \tilde v_2|^2 + \int |\nabla h|^2\right)$$

where

$$\tilde Q_1(\tilde v_1, \tilde v_1) = \int |\nabla \tilde v_1|^2 - 5\int \underline{\omega}_1^4 \tilde v_1^2 \quad \textit{and} \quad \tilde Q_2(\tilde v_2, \tilde v_2) = \int |\nabla \tilde v_2|^2 - 5\int \omega_2^4 \tilde v_2^2$$

are nondegenerate, with spectra bounded away from zero independently of λ. Furthermore, after a change of variables from $\tilde V \to \tilde{\tilde V}$ (both variables belonging to a neighborhood of the origin in $H_1 \times H_2 \times K$), where the change of variables satisfies bounds independent of λ, we find for v small, $v \in H$, satisfying (1.2)*, $J(\alpha_1\bar\omega + \alpha_2\omega_2 + v)$ in the new coordinates reads as $J(\alpha_1\underline{\omega}_1 + \alpha_2\omega_2) + \varphi(\tilde{\tilde V}) + Q(\tilde{\tilde V}, \tilde{\tilde V})$, where $\varphi(\tilde{\tilde V})$ stands, at first order, for $(f, v)_{H^1}$.*

PROOF OF LEMMA 3.1: The proof of (i) is straightforward. Consider now Q_0 on H and let $v_1 = P_1 v$, $v_2 = P_2 v$, and $h = v - (v_1 + v_2)$, where v_1, v_2, and h are connected by (1.2). Then by [2, lemma 3.2, p. 64] and by the proof of [2, proposition 3.1, pp. 65–68], we have

$$Q_0(v, v) = \int |\nabla v_1|^2 - 5\int \underline{\omega}_1^4 v_1^2 + \int |\nabla v_2|^2 - 5\int \omega_2^4 v_2^2 + \int |\nabla h|^2 + o\left(\int |\nabla v_1|^2 + \int |\nabla v_2|^2 + \int |\nabla h|^2\right)$$

because

$$|\underline{\omega}_1| \le \frac{C}{(1 + r^2)^{1/2}} \quad \text{and} \quad |\omega_2| \le \frac{C\sqrt{\lambda}}{(1 + \lambda^2 r^2)^{1/2}}\,.$$

The quadratic form $\tilde Q_1$ is nondegenerate on H_1, and its spectrum is bounded away from zero independently of λ. The same claim holds for $\tilde Q_2$ in H_2. We formally have

$$Q_0(v, v) = \tilde Q_1(v_1, v_1) + \tilde Q_2(v_2, v_2) + \int |\nabla h|^2 + o\left(\int |\nabla v_1|^2 + |\nabla v_2|^2 + |\nabla h|^2\right)$$

up to the fact that the spaces of definition are totally wrong since v_1 and v_2 are not in H_1 and H_2, respectively.

In order to overcome this problem, we use the projections S_1 and S_2. Since v satisfies (1.2),

$$\begin{aligned}\int \nabla v_1 \cdot \nabla \underline{\omega}_1 &= -\int \nabla v_2 \cdot \nabla \underline{\omega}_1 - \int \nabla h \cdot \nabla \underline{\omega}_1 \\ &= -\int_{\Omega_1^c} (\nabla v_2 + \nabla h) \cdot \nabla \underline{\omega}_1 - \int_{\partial \Omega_1} \frac{\partial h}{\partial \nu} \underline{\omega}_1 \\ &= o\biggl(\Bigl(\int |\nabla v_2|^2 \Bigr)^{1/2} + \Bigl(\int |\nabla h|^2 \Bigr)^{1/2} \biggr) = o\biggl(\Bigl(\int |\nabla v|^2 \Bigr)^{1/2} \biggr).\end{aligned}$$

Similar estimates hold for the pairs v_1 and $\frac{\partial \underline{\omega}_1}{\partial \sigma_i^j}$, v_2 and ω_2, and v_2 and $\frac{\partial \omega_2}{\partial \sigma_2^j}$; so that, using (A1) and also using the invariance by scaling of various scalar products,

$$S_1(v_1) + S_2(v_2) + h = v + o\biggl(\Bigl(\int |\nabla v|^2 \Bigr)^{1/2} \biggr).$$

Thus we can now write, in a rigorous way, that

$$\begin{aligned}Q_0(v, v) = \tilde{Q}_1\bigl(S_1(v_1), S_1(v_1)\bigr) + \tilde{Q}_2\bigl(S_2(v_2), S_2(v_2)\bigr) + \int |\nabla h|^2 \\ + o\Bigl(\int |\nabla S_1(v_1)|^2 + \int |\nabla S_2(v_2)|^2 + \int |\nabla h|^2 \Bigr).\end{aligned}$$

Conversely, given s_1 in H_1, s_2 in H_2, and h harmonic in Ω_1 and Ω_2, there exists (using (A1)) a unique function φ in

$$F = \operatorname{span} \left\{ \mu \frac{\partial \underline{\omega}_1}{\partial \mu}, \lambda \frac{\partial \omega_2}{\partial \lambda}, \frac{\partial \underline{\omega}_1}{\partial \tilde{\sigma}_1^j}, \frac{\partial \omega_2}{\partial \tilde{\sigma}_2^j} \right\}$$

so that

$$s_1 + s_2 + h + \varphi \quad \text{satisfies (1.2).}$$

We also have, denoting Q_F the projector onto F and recalling that (A1) holds,

$$\begin{aligned}\int |\nabla \varphi|^2 &\le C \Bigl(\int |\nabla Q_F(s_1)|^2 + \int |\nabla Q_F(s_2)|^2 + \int |\nabla Q_F(h)|^2 \Bigr) \\ &\le o\Bigl(\int \bigl(|\nabla s_1|^2 + |\nabla s_2|^2 + |\nabla h|^2 \bigr) \Bigr),\end{aligned}$$

since (s_1, s_2, h) belong to $H_1 \times H_2 \times K$. Thus, the map T is invertible and we may consider $\tilde{Q}$ to be defined on $H_1 \times H_2 \times K$, with a quadratic form very close to $\tilde{Q}_1(s_1, s_2) + \tilde{Q}_2(s_2, s_2) + \int |\nabla h|^2$. Since each $\tilde{Q}_i$ is nondegenerate and is the linearized operator of the Yamabe problem at $\underline{\omega}_1$ and ω_2, respectively, the result follows. □

Using Lemma 3.1, we find a unique $\bar{v}$ satisfying (1.2) such that

$$\frac{\partial}{\partial v} J\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v\big)|_{\bar{v}} = 0\,.$$

In order to estimate $\bar{v}$ in the H^1-norm, we have the following:

LEMMA 3.2 (i) *For any v satisfying* (1.2),

$$|(f,v)_{H^1}| \le C|v|_{H^1}\left(\frac{|\underline{\omega}_1(0)| + |\bar{\omega}_2(\infty)|}{\sqrt{\lambda}} + \frac{1}{\lambda^{3/2}}\right).$$

(ii)

$$|\bar{v}|_{H^1} \le C\left(\frac{|\underline{\omega}_1(0)| + |\bar{\omega}_2(\infty)|}{\sqrt{\lambda}} + \frac{1}{\lambda^{3/2}}\right).$$

PROOF: Since $\int \bar{\omega}_1^5 v = \int \omega_2^5 v = 0$,

$$\begin{aligned}|(f,v)_{H^1}| &\le C\int \big(|\bar{\omega}_1|^4|\omega_2| + |\omega_2|^4|\underline{\omega}_1|\big)|v|\\ &\le C|v|_{L^6}\left(\int \big(|\bar{\omega}_1|^4|\omega_2| + \omega_2^4|\bar{\omega}_1|\big)^{6/5}\right)^{5/6}.\end{aligned}$$

Using Lemma 3.1, we then have

$$|\bar{v}|_{H^1} \le C'\left(\int \big(\bar{\omega}_1^4|\omega_2| + \omega_2^4|\bar{\omega}_1|\big)^{6/5}\right)^{5/6}.$$

Thus, we need to estimate $\big(\int |\underline{\omega}_1|^{24/5}|\omega_2|^{6/5}\big)^{5/6} + \big(\int |\omega_2|^{24/5}|\underline{\omega}_1|^{6/5}\big)^{5/6}$. We will use (2.3) of Lemma 2.1.

Clearly, $\underline{\omega}_1(x) = \underline{\omega}_1(0) + O(|x|)$. Thus,

$$\left(\int |\omega_2|^{24/5}|\underline{\omega}_1|^{6/5}\right)^{5/6} \le C\left(|\underline{\omega}_1(0)|\left(\int |\omega_2|^{24/5}\right)^{5/6} + \left(\int |\omega_2|^{24/5}|x|^{6/5}\right)^{5/6}\right).$$

We know that $|\omega_2(x)|$ is bounded above by $\frac{C\sqrt{\lambda}}{(1+\lambda^2 r^2)^{1/2}}$. Thus,

$$\left(\int |\omega_2|^{24/5}\right)^{5/6} \le \frac{C}{\lambda^{(3-\frac{12}{5})\frac{5}{6}}}\left(\int \frac{r^2\,dr}{(1+r^2)^{12/5}}\right)^{5/6} \le \frac{C'}{\sqrt{\lambda}}\,.$$

We also have

$$\left(\int |\omega_2|^{24/5}|x|^{6/5}\right)^{5/6} \le \frac{C}{\lambda^{3/2}}\left(\int \frac{r^{2+\frac{6}{5}}}{(1+r^2)^{12/5}}\right) \le \frac{C''}{\lambda\sqrt{\lambda}}\,.$$

Thus,

$$\left(\int |\omega_2|^{24/5}|\underline{\omega}_1|^{6/5}\right)^{5/6} \le C\left(\frac{|\underline{\omega}_1(0)|}{\sqrt{\lambda}} + \frac{1}{\lambda^{3/2}}\right).$$

For $\left(\int |\underline{\omega}_1|^{24/5}|\omega_2|^{6/5}\right)^{5/6}$, we use the formula of (2.3) from Lemma 2.1. Outside a ball of radius M, $\omega_2(x) - \frac{\bar{\omega}(\infty)\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}}$ is bounded above by $\frac{C}{\lambda^{3/2}}$. Thus, the contribution of $\omega_2 - \frac{\bar{\omega}_2(\infty)\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}}$ outside a ball of radius M is bounded above by

$$\frac{C}{\lambda^{3/2}}\left(\int\limits_{B^c(0,M)} \underline{\omega}_1^{24/5}\right)^{5/6} = O\left(\frac{1}{\lambda^{3/2}}\right).$$

On the other hand,

$$\begin{aligned}\left(\int |\underline{\omega}_1|^{24/5}\left(\frac{\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}}\right)^{6/5}\right)^{5/6} &\le \frac{C}{\sqrt{\lambda}} + \frac{C}{\lambda^2}\left(\int\limits_{B(0,1)} \frac{\lambda^3 r^2\,dr}{(1+\lambda^2r^2)^{3/5}}\right)^{5/6}\\ &\le \frac{C}{\sqrt{\lambda}} + \frac{C}{\lambda^2}\left(\int\limits_{r\le\lambda} \frac{r^2\,dr}{(1+r^2)^{3/5}}\right)^{5/6}\\ &\le \frac{C}{\sqrt{\lambda}} + \frac{C}{\lambda^2}(\lambda^{3-6/5})^{5/6} = \frac{C_1}{\sqrt{\lambda}}.\end{aligned}$$

Thus,

$$|\bar{\omega}_2(\infty)|\left(\int |\underline{\omega}_1|^{24/5}\left(\frac{\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}}\right)^{6/5}\right)^{5/6} \le C\frac{|\bar{\omega}_2(\infty)|}{\sqrt{\lambda}}.$$

Next, we estimate

$$\left(\int\limits_{B(0,M)} |\underline{\omega}_1|^{24/5}|\omega_2 - \frac{\bar{\omega}_2(\infty)\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}}|^{6/5}\right)^{5/6}.$$

We observe that

$$\left|\omega_2 - \frac{\bar{\omega}_2(\infty)\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}}\right| \le \frac{C\sqrt{\lambda}}{1+\lambda^2r^2}.$$

Thus,

$$\begin{aligned}\left(\int\limits_{B(0,M)} |\underline{\omega}_1|^{24/5}\left|\omega_2 - \frac{\bar{\omega}_2(\infty)\sqrt{\lambda}}{(1+\lambda^2r^2)^{1/2}}\right|^{6/5}\right)^{5/6} &\le C\left(\int\limits_{r\le M} \frac{\lambda^{3/5}}{(1+\lambda^2r^2)^{6/5}} r^2\,dr\right)^{5/6}\\ &\le \frac{C\lambda^{1/2}}{\lambda^{5/2}}\left((M\lambda)^{3-\frac{12}{5}}\right)^{5/6} = \frac{C}{\lambda^{3/2}}.\end{aligned}$$

Summing up, we have

$$\left(\int |\underline{\omega}_1|^{24/5}|\omega_2|^{6/5}\right)^{5/6} \le C\left(\frac{|\bar{\omega}_2(\infty)|}{\sqrt{\lambda}} + \frac{1}{\lambda^{3/2}}\right),$$

and thus

$$|(f, v)_{H^1}| \le C|v|_{H^1}\left(\frac{|\underline{\omega}_1(0)| + |\bar{\omega}_2(\infty)|}{\sqrt{\lambda}} + \frac{1}{\lambda^{3/2}}\right).$$

Using Lemma 3.1(i) and (ii), we find a unique optimal $\bar{v}$ in the space of v's, and we have

$$|\bar{v}|_{H^1} \le C\left(\frac{|\underline{\omega}_1(0)| + |\bar{\omega}_2(\infty)|}{\sqrt{\lambda}} + \frac{1}{\lambda^{3/2}}\right).$$

□

4 The Morse Lemmas at Infinity

We thus have been able to prove, for α_2/α_1 close to 1, the existence of $\bar{v}$ satisfying (1.2) such that

$$J\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v\big) = J\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}\big) + Q(V, V)$$

where V is essentially $v - \bar{v}$. In fact, in V there is an additional lower-order term since, in the expansion of J, there is an $O(|v|^3_{H^1})$. V also satisfies (1.2), and therefore, by Lemma 3.1, $Q(V, V)$ is a nondegenerate quadratic form.

We study, in what follows, $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})$, and following the method of [3], we find a change of variables in the $(\tilde{\sigma}_1, \tilde{\sigma}_2, \lambda)$-space that brings $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})$ into a nice normal form.

We first prove the following:

LEMMA 4.1 *Assume* (A1): *For every solution ω of the Yamabe problem in $\mathbb{R}^3$, the various $\frac{\partial\omega}{\partial\tilde{\sigma}^j}$ and $x \cdot \nabla\omega(x)$ are independent, where the $\tilde{\sigma}^j$ are parameters along the orthogonal group.*

Then, for every

$$w_0 = w_0^1\lambda\frac{\partial\omega_2}{\partial\lambda} + \sum w_{0,j}^2\frac{\partial\omega_2}{\partial\tilde{\sigma}_2^j} + \sum w_{0,j}^1\frac{\partial\bar{\omega}_1}{\partial\tilde{\sigma}_1^j}$$

that corresponds to the variation $\lambda w_0^1 = \delta\lambda$ in the λ-concentration and the various variations $w_{0,j}^i$ in the parameters corresponding to the orthogonal group and that induces the variation $\partial\bar{v}(w_0)$ in the $\bar{v}$-component, we have

$$\begin{aligned} -J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}\big)\cdot\big(w_0 + \partial\bar{v}(w_0)\big) &\ge \\ -J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2)\cdot w_0 + O\left(\frac{\bar{\omega}_1(0)^2 + \bar{\omega}_2(\infty)^2}{\lambda} + \frac{1}{\lambda^3}\right)&\left(|w_0^1| + \sum_{i,j}|w_{0,j}^i|\right). \end{aligned}$$

PROOF OF LEMMA 4.1: Let

$$E = \operatorname{span}\left\{\underline{\omega}_1, \omega_2, \mu\frac{\partial\underline{\omega}_1}{\partial\mu}, \lambda\frac{\partial\omega_2}{\partial\lambda}, \frac{\partial\omega_2}{\partial\tilde{\sigma}_2^j}, \frac{\partial\omega_1}{\partial\tilde{\sigma}_1^j}\right\}$$

and

$$F = \operatorname{span}\left\{\mu\frac{\partial\underline{\omega}_1}{\partial\mu}, \lambda\frac{\partial\omega_2}{\partial\lambda}, \frac{\partial\omega_2}{\partial\tilde{\sigma}_2^j}, \frac{\partial\omega_1}{\partial\tilde{\sigma}_1^j}\right\}.$$

Let Q_E and Q_F be the orthogonal projections from H^1 onto these spaces. Since $J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v})$ is, by definition of $\bar{v}$, orthogonal to the space of v's, i.e., to the orthogonal of E, and since, by differentiation of (1.2), $\partial\bar{v}(w_0)$ is orthogonal to $\bar{\omega}_1$ and ω_2, we have

$$Q_E(\partial\bar{v}(w_0)) = Q_F(\partial\bar{v}(w_0))$$

and

$$\begin{aligned} &J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}\big)\cdot\partial\bar{v}(w_0) \\ &\quad = J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}\big)\cdot Q_F\big(\partial\bar{v}(w_0)\big) \\ &\quad = J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big)\cdot Q_F\big(\partial\bar{v}(w_0)\big) + O\big(|\bar{v}|_{H^1}|Q_F\big(\partial\bar{v}(w_0)\big)|_{H^1}\big)\,. \end{aligned}$$

We need to estimate $Q_F(\partial\bar{v}(w_0))$. Using our relations, we have

$$Q_F(\partial\bar{v}(w_0))\cdot\partial\omega_i = -\bar{v}\cdot\frac{\partial}{\partial w_0}(\partial\omega_i)$$

where $\partial\omega_i$ stands for one of the functions spanning F.

Using (A2) and the fact that the functions $\mu\frac{\partial\bar{\omega}_1}{\partial\mu}$ and $\frac{\partial\bar{\omega}_1}{\partial\sigma_1^j}$ on one hand and the $\frac{\partial\omega_2}{\partial\sigma_2^j}$ and $\lambda\frac{\partial\omega_2}{\partial\lambda}$ on the other are nearly orthogonal (by assumption, we are in a neighborhood of infinity), we derive that

$$|Q_F(\partial\bar{v}(w_0))|_{H^1} \le C|\bar{v}|_{H^1}|w_0|_{H^1}\,.$$

Indeed, by conformal invariance,

$$\left|\frac{\partial}{\partial w_0}(\partial\omega_i)\right|_{H^1} \le C|w_0|_{H^1}\,.$$

Thus,

$$\begin{aligned} &J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v})\cdot\partial\bar{v}(w_0) = \\ &\qquad J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big)\cdot Q_F(\partial\bar{v}(w_0)) + O\big(|\bar{v}|_{H^1}^2|w_0|_{H^1}\big)Q_F(\partial\bar{v}(w_0)) \end{aligned}$$

is orthogonal to $\bar{\omega}_1$ and ω_2. We also have

$$\int\bar{\omega}_1^5 Q_F(\partial\bar{v}(w_0)) = \int\omega_2^5 Q_F(\partial\bar{v}(w_0)) = 0\,.$$

Thus,

$$\begin{aligned} &J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big)\cdot Q_F(\partial\bar{v}(w_0)) \\ &\quad = O\left(\int\big(\underline{\omega}_1^4|\omega_2| + \omega_2^4|\underline{\omega}_1|\big)|Q_F(\partial\bar{v}(w_0))|\right) \end{aligned}$$

$$= O\left(|\bar{v}|_{H^1}|w_0|_{H^1}\left(\int(\underline{\omega}_1^4|\omega_2| + \omega_2^4|\underline{\omega}_1|)^{6/5}\right)^{5/6}\right)$$

$$= O\left(|\bar{v}|_{H^1}|w_0|_{H^1}\left(\frac{|\underline{\omega}_1(0)| + |\bar{\omega}_2(\infty)|}{\sqrt{\lambda}} + \frac{1}{\lambda^{3/2}}\right)\right)$$

and, using Lemma 3.2,

$$J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}) \cdot \partial\bar{v}(w_0) = O\left(\left(\frac{\underline{\omega}_1(0)^2 + \bar{\omega}_2(\infty)^2}{\lambda} + \frac{1}{\lambda^3}\right)|w_0|_{H^1}\right).$$

Turning then to $J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}) \cdot w_0$, we expand to find

$$J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}\big) \cdot w_0 = \\ J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big) \cdot w_0 + J''\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big) \cdot \bar{v} \cdot w_0 + O\big(|\bar{v}|_{H^1}^2|w_0|_{H^1}\big).$$

Using the fact that $J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2) \cdot \bar{v} = O(|\bar{v}|_{H^1}^2)$, since $J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}) \cdot \bar{v} = 0$, and using the fact that $\bar{v}$ is H^1-orthogonal to w_0, a direct computation shows that

$$J''(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2) \cdot \bar{v} \cdot w_0 = O\left(\int(\alpha_1\underline{\omega}_1 + \alpha_2\omega_2)^4 w_0\bar{v}\right) + O(|\bar{v}|_{H^1}^2|w_0|_{H^1}).$$

Observe that

$$\int \underline{\omega}_1^4\frac{\partial\underline{\omega}_1}{\partial\tilde{\sigma}_1^j}\bar{v} = 0, \quad \int \omega_2^4\frac{\partial\omega_2}{\partial\sigma_2^j}\bar{v} = \int \omega_2^4\frac{\partial\omega_2}{\partial\lambda}\bar{v} = 0.$$

Thus,

$$J''\big(\alpha_1\bar{\omega}_1 + \alpha_2\big) \cdot \bar{v} \cdot w_0 \\ = \left(|w_0^1|\left|\int \underline{\omega}_1^4\lambda\frac{\partial\omega_2}{\partial\lambda}\bar{v}\right| + \sum|w_{0,j}^2|\left|\int \underline{\omega}_1^4\frac{\partial\omega_2}{\partial\sigma_2^j}\bar{v}\right| + \sum|w_{0,j}^1|\left|\int \omega_2^4\frac{\partial\underline{\omega}_1}{\partial\tilde{\sigma}_1^j}\bar{v}\right|\right. \\ \left. + \int\big(|\underline{\omega}_1|^3|\omega_2| + |\omega_2|^3|\underline{\omega}_1|\big)|w_0||\bar{v}| + |\bar{v}|_{H^1}^2|w_0|_{H^1}\right).$$

Using the conformal invariance and the scaling, this reduces to

$$J''\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big) \cdot \bar{v} \cdot w_0 = \\ O\left(\left(|w_0^1| + \sum|w_{0,j}^2|\right)\int \underline{\omega}_1^4|\omega_2||\bar{v}|\right. \\ \left. + \left(\sum|w_{0,j}^1|\right)\int \omega_2^4|\underline{\omega}_1||\bar{v}| + |\bar{v}|_{H^1}^2|w_0|_{H^1}\right).$$

Using the estimates derived in the proof of Lemma 3.2, we have

$$J''\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big) \cdot \bar{v} \cdot w_0 = \\ O\left(\left(|w_0^1| + \sum|w_{0,j}^2| + \sum|w_{0,j}^1| + |w_0|_{H^1}\right)\left(\frac{|\underline{\omega}_1(0)| + |\bar{\omega}_2(\infty)|}{\sqrt{\lambda}} + \frac{1}{\lambda^{3/2}}\right)^2\right).$$

Under (A2),

$$|w_0|_{H^1} \quad \text{is equivalent to} \quad |w_0^1| + \sum |w_{0,j}^2| + \sum |w_{0,j}^1| \,.$$

Thus,

$$\begin{aligned} &J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}\big) \cdot w_0 = \\ &\quad J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big) \cdot w_0 \\ &\qquad + O\left(\left(|w_0^1| + \sum |w_{0,j}^1| + \sum |w_{0,j}^2|\right)\left(\frac{\underline{\omega}_1(0)^2 + \bar{\omega}_2(\infty)^2}{\lambda} + \frac{1}{\lambda^3}\right)\right). \end{aligned}$$

and combining the estimates on $J'(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})\cdot\partial\bar{v}(w_0)$ with the above remark on $|w_0|_{H^1}$ yields

$$\begin{aligned} &J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}\big) \cdot \big(w_0 + \partial\bar{v}(w_0)\big) = \\ &\quad J'\big(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2\big) \cdot w_0 \\ &\qquad + O\left(\left(|w_0^1| + \sum |w_{0,j}^1| + \sum |w_{0,j}^2|\right)\left(\frac{\underline{\omega}_1(0)^2 + \bar{\omega}_2(\infty)^2}{\lambda} + \frac{1}{\lambda^3}\right)\right). \end{aligned}$$

□

Next, we prove the following:

LEMMA 4.2 *Assume* (A1)*, the additional condition* (A2) (*for every solution ω of the Yamabe problem on $\mathbb{S}^3$, the differential of ω is nonzero on the zero set of ω*)*, and* (A4)*: There exists a constant $c_8 > 0$ such that $|\nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty)| \geq c_8$ on the set of couples $(\bar{\omega}_1, \bar{\omega}_2)$ satisfying*

$$(\Gamma)_\infty \begin{cases} \underline{\omega}_1(0) = \bar{\omega}_2(\infty) = 0 \\ \frac{\partial}{\partial\tilde{\sigma}_1^i}(\nabla\omega_1(0) \cdot \nabla\underline{\omega}_2(\infty)) \textit{ is collinear to } \frac{\partial}{\partial\tilde{\sigma}_1^i}(\underline{\omega}_1(0)) \\ \frac{\partial}{\partial\tilde{\sigma}_2^i}(\nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty)) \textit{ is collinear to } \frac{\partial}{\partial\tilde{\sigma}_2^i}(\bar{\omega}_2(\infty))\,. \end{cases}$$

Then we can construct w_0 depending smoothly on $(\alpha_i, \sigma_i^j, \lambda)$ so that, with an appropriate positive constant c,

$$\begin{aligned} &-J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}) \cdot (w_0 + \partial\bar{v}(w_0)) \geq \\ &\qquad\qquad c|w_0|_{H^1}\left(\frac{|\underline{\omega}_1(0)| + |\bar{\omega}_2(\infty)|}{\sqrt{\lambda}} + \frac{1}{\lambda\sqrt{\lambda}}\right). \end{aligned}$$

PROOF: We first observe—and this is obvious from the proof—that the expansion of Lemma 2.2 can be differentiated with respect to λ and $\tilde{\sigma}_j^i$, because Lemma 2.1 holds for the derivatives of ω_2 with respect to λ and $\tilde{\sigma}_2^i$. (This claim is a straightforward consequence of the proof of Lemma 2.1, which is based on a scaling argument and an expansion around the north pole of the function φ on

$\mathbb{S}^3$.) Consider now the expansion of $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega)$. If $\underline{\omega}_1(0)$ or $\bar{\omega}_2(\infty)$ are both not small, the proof of Lemma 4.2 is straightforward. If one of $\underline{\omega}_1(0)$ or $\bar{\omega}_2(\infty)$ is small while the other is not small, we can choose $w_0^1 = 0$ and an appropriate $w_{0,j}^i$ nonzero with all the other ones zero. Lemma 2.2 will then imply that

$$-J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2) \cdot w_0 \geq \frac{c|w_0|_{H^1}}{\sqrt{\lambda}} .$$

Using Lemma 4.1, Lemma 4.2 follows.

If both $\underline{\omega}_1(0)$ and $\bar{\omega}_2(\infty)$ are small, we use the $\tilde{\sigma}_j^i$-parameters. Then, taking the derivative of the expansion of Lemma 2.2 with respect to these parameters, either

$$(\Gamma) \begin{cases} \underline{\omega}_1(0) = O(\frac{1}{\lambda});\ \bar{\omega}_2(\infty) = O(\frac{1}{\lambda}) \\ c\frac{\bar{\omega}_2(\infty)}{\sqrt{\lambda}}\frac{\partial}{\partial\tilde{\sigma}_1^i}(\bar{\omega}_1(0)) - c'\frac{\partial}{\partial\tilde{\sigma}_1^i}\frac{\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)}{\lambda^{3/2}} = o\left(\frac{1}{\lambda^{3/2}}\right) \\ c\frac{\underline{\omega}_1(0)}{\sqrt{\lambda}}\frac{\partial}{\partial\tilde{\sigma}_2^i}(\bar{\omega}_2(\infty)) - c'\frac{\partial}{\partial\tilde{\sigma}_2^i}\frac{\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)}{\lambda^{3/2}} = o\left(\frac{1}{\lambda^{3/2}}\right), \end{cases}$$

or (Γ) is not satisfied. Then, we can build $w_{0,j}^i$, i.e., w_0, so that the inequality of Lemma 4.2 holds. Thus, assuming (Γ), under (A3), $\underline{\omega}(0)$ and $\bar{\omega}_2(\infty)$ are $O(1/\lambda)$. Multiplying (Γ) by $\lambda^{3/2}$ and going to the limit as λ tends to $+\infty$, we derive that we have a solution of

$$(\Gamma)_\infty \begin{cases} \underline{\omega}_1(0) = \bar{\omega}_2(\infty) = 0 \\ C_1\frac{\partial}{\partial\tilde{\sigma}_1^i}\underline{\omega}_1(0) - c'\frac{\partial}{\partial\tilde{\sigma}_1^i}(\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)) = 0 \\ C_2\frac{\partial}{\partial\tilde{\sigma}_2^i}\bar{\omega}_2(\infty) - c'\frac{\partial}{\partial\tilde{\sigma}_2^i}(\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)) = 0 \end{cases}$$

where C_1 and C_2 are limits of $\lambda\underline{\omega}_1(0)$ and $\lambda\bar{\omega}_2(\infty)$. C_1 and C_2 behave as Lagrange multipliers; i.e., their values are not prescribed. Thus, $(\Gamma)_\infty$ amounts to twelve constraints on the twelve parameters $\tilde{\sigma}_j^i$ related to $O(4)$ and affecting $(\bar{\omega}_1, \omega_2)$ (once basic forms of solutions to the Yamabe problem have been assigned for $\bar{\omega}_1$ and $\bar{\omega}_2$).

Thus, we expect generically to have only isolated solutions to $(\Gamma)_\infty$, up to rigid motions of the couple $(\bar{\omega}_1, \bar{\omega}_2)$, and since $\nabla\underline{\omega}_1(0)\cdot\nabla\omega_2(\infty)$ is invariant through these rigid motions, we may assume (A4), i.e., that there exists a constant $c_8 > 0$ such that, for λ large enough,

$$|\nabla\underline{\omega}_1(0)\cdot\nabla\omega_2(\infty)| \geq c_8 > 0$$

as soon as (Γ) is satisfied.

Differentiating then the expansion of Lemma 2.2 with respect to λ, we derive

$$J'(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2)\cdot\lambda\frac{\partial\omega_2}{\partial\lambda} = O\left(\frac{\underline{\omega}_1(0)\bar{\omega}_2(\infty)}{\sqrt{\lambda}}\right) - c'\frac{\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)}{\lambda^{3/2}} + O\left(\frac{1}{\lambda^{5/2}}\right).$$

Since $\underline{\omega}_1(0)\bar{\omega}_2(\infty)$ is $O\left(\frac{1}{\lambda^2}\right)$ and $|\nabla\underline{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)|$ is larger than c_8, w_0 can again be built, using Lemma 4.1, so that the inequality of Lemma 4.2 holds. It is easy to patch, using a partition of unity, the various local values of w_0 and derive a smooth w_0. $\square$

We now provide the various Morse lemmas at infinity that cover all the possible situations we may encounter. Coming back to Lemma 3.1, $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v)$ reads as $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2) + \varphi(\tilde{\tilde{V}}) + Q(\tilde{\tilde{V}}, \tilde{\tilde{V}})$ with $\tilde{\tilde{V}}$ belonging to a neighborhood of the origin in $H_1 \times H_2 \times K$. Minimizing $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v)$ with respect to v is equivalent to minimizing $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2) + \varphi(\tilde{\tilde{V}}) + Q(\tilde{\tilde{V}}, \tilde{\tilde{V}})$ with respect to $\tilde{\tilde{V}}$. After a change of variables $\tilde{\tilde{V}} \to V$, which holds in a neighborhood of the origin in $H_1 \times H_2 \times K$, $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v)$ reads as $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + \bar{v}) + Q(V, V)$. The notation $Q(V, V)$, which is now clear, will be used in the statement of the following:

LEMMA 4.3 *Assume that* (A1)–(A4) *hold.*

- (i) *Near a couple* $(0, \infty)_0$ *such that neither* $(\bar{\omega}_1(0))_0$ *nor* $(\bar{\omega}_2(\infty))_0$ *is zero,* $J(\alpha_1\bar{\omega}_1 + \alpha_1 + \alpha_2\omega_2 + v)$ *can be read, up to an additive universal constant and another multiplicative constant depending only on* $(\alpha_1, \alpha_2, \bar{\omega}_1, \bar{\omega}_2)$, *as* $c/\sqrt{\lambda} + Q(V, V)$.
- (ii) *Near a couple* $(0, \infty)_0$ *such that* $(\bar{\omega}_1(0))_0$ *is zero while* $(\bar{\omega}_2(\infty))_0$ *is nonzero,* $X = \bar{\omega}_1(0)$ *can be taken to be a coordinate and* $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v)$ *can again be read, up to an additive universal constant and another multiplicative constant depending only on* $(\alpha_1, \alpha_2, \bar{\omega}_1, \bar{\omega}_2)$, *as* $cX/\sqrt{\lambda} + Q(V, V)$ *and* $cY/\sqrt{\lambda} + Q(V, V)$ *in the reverse case, when* $(\bar{\omega}_2(\infty))_0$ *is zero while* $(\bar{\omega}_1(0))_0$ *is nonzero.*
- (iii) *Near a couple* $(0, \infty)_0$ *such that* $(\bar{\omega}_1(0))_0$ *is zero as well as* $(\bar{\omega}_2(\infty))_0$, $\bar{\omega}_1(0) = X$ *and* $\bar{\omega}_2(\infty) = Y$ *can be taken to be two coordinates.*
  - (a) *Assuming* $(0, \infty)_0$ *and* $(\bar{\omega}_1, \bar{\omega}_2)$ *do not satisfy* $(\Gamma)_\infty$, *there are constants* c_0, $\bar{c}_1$, *and* $\bar{c}_2$ *and a nonsingular linear form* L *on* $\mathbb{R}^{10}$, *where* $Z \in \mathbb{R}^{10}$ *stands for the remaining ten dimensions of the orthogonal group's action on* (ω_1, ω_2) *after* X *and* Y *have been chosen, so that* (*up to constants as above*), $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v)$ *reads as*

$$\frac{-\lambda XY - c_0 - \bar{c}_1 X - \bar{c}_2 Y - L(Z)}{\lambda^{3/2}} + Q(V, V)\,.$$

  - (b) *Assuming now that* $(0, \infty)$ *and* $(\bar{\omega}_1, \bar{\omega}_2)$ *satisfy* $(\Gamma)_\infty$, *there exists a constant* c_0 (*nonzero*) *such that* (*up to constants as above*) $J(\alpha_1\bar{\omega}_1 + \alpha_2\omega_2 + v)$ *reads as*

$$\frac{-\lambda XY - c_0}{\lambda^{3/2}} + Q(V, V)\,.$$

PROOF: (i) and (ii) follow easily from the proof of (iii); we thus look into (iii). The line of proof of Lemma 4.3(iii) is identical, given Lemma 4.2, to the line of proof in [3, appendices 1 and 2].

Let us consider the following three functionals: $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})$, $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)$, and $\psi(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)$ defined as follows: Consider the function $\nabla\bar{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)$ defined on the space of variables (X, Y, Z). Expanding, we have

$$\nabla\bar{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)(X,Y,Z) = \nabla\bar{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)(0,0,0)+\frac{\bar{c}_1}{c'}X+\frac{\bar{c}_2}{c'}Y+L(Z)+\bar{Q}(X,Y,Z)\,,$$

where $\bar{Q}$ is of degree larger than or equal to 2 and L is a linear form on Z. Set

$$\nabla\bar{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)(0,0,0)=\frac{c_0}{c'}\,,$$

$$\begin{aligned}\bar{Q}(X,Y,Z) &= \bar{Q}(0,0,Z)+o(|X|+|Y|)\\ &= Q_1(Z,Z)+o\big(|X|+|Y|+|Z|^2\big)\,.\end{aligned}$$

$\psi(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)$ is then defined to be

$$\frac{-\lambda XY-c_0-\bar{c}_1X-\bar{c}_2Y-L(Z)}{\lambda^{3/2}} \quad \text{in case (iii)(a),}$$

$$\frac{-\lambda XY-c_0-\bar{c}_1X-\bar{c}_2Y-Q_1(Z,Z)}{\lambda^{3/2}} \quad \text{in case (iii)(b).}$$

By Lemma 4.2 and its proof

$$-J'\big(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v}\big)\cdot\big(w_0+\partial\bar{v}(w_0)\big)\geq c|w_0|_{H^1}\left(\frac{|\bar{\omega}_1(0)|+|\bar{\omega}_2(\infty)|}{\sqrt{\lambda}}+\frac{1}{\lambda\sqrt{\lambda}}\right)$$

and

$$-J'\big(\alpha_1\bar{\omega}_1+\alpha_2\omega_2\big)\cdot w_0\geq c|w_0|_{H^1}\left(\frac{|\bar{\omega}_1(0)|+|\bar{\omega}_2(0)|}{\sqrt{\lambda}}+\frac{1}{\lambda\sqrt{\lambda}}\right).$$

Using the expansion of Lemma 3.1, the above definition of ψ, and the arguments of Lemma 4.2 again, it is easy to see that we also have

$$-\psi'm\big(\alpha_1\bar{\omega}_1+\alpha_2\omega_2\big)\cdot w_0\geq c|w_0|_{H^1}\left(\frac{|\bar{\omega}_1(0)|+|\bar{\omega}_2(\infty)|}{\sqrt{\lambda}}+\frac{1}{\lambda\sqrt{\lambda}}\right).$$

In case (iii)(b), because we know that c_0 is nonzero (we are at $(\bar{\omega}_1, \bar{\omega}_2, 0, \infty)$ satisfying $(\Gamma)_\infty$), we can remove $\bar{c}_1X$, $\bar{c}_2Y$, and $Q_1(Z,Z)$ from the expansion of ψ. Thus, w_0 is a common pseudogradient for the three functionals $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})$, $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)$, and $\psi(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)$, and we can use its flow lines to transmute one functional into the other and conversely. The only problem is with the Palais-Smale condition on the flow lines of w_0: We have to check that, along a flow line of w_0, before reaching the value $\lambda=+\infty$ or exiting the domain of definition of each local w_0, we have changed, along the flow line, $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})$ into $\psi(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)$, and vice versa. By construction, $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})$ and

$\psi(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)$ differ by an amount that is $o(1/\lambda^{3/2})$. Indeed, by a preliminary change of variables in the (X, Y, Z)-space, we can get rid of $O(\lambda^{-1}(\bar{\omega}_1(0)^2+\bar{\omega}_2(\infty)^2))+o(\lambda^{-3/2}|Z|^2)=O(\lambda^{-1}(X^2+Y^2))+o(\lambda^{-3/2}|Z|^2)$ in the expansion of J by using the term $-\lambda^{-1/2}XY-\lambda^{-3/2}L(Z)-\lambda^{-3/2}Q_1(Z,Z)$. In addition, we can get rid of $Q_1(Z,Z)$ if the linear form $L(Z)$ is nonsingular.

Thus, if along a flow line of w_0, either of these functionals undergoes a change of the order of $c/\lambda_0^{3/2}$, starting with an initial large λ_0, while λ is still not $+\infty$ and we are still in the domain of definition of the local λ_0, we will have established Lemma 4.3.

Along the flow lines of w_0, normalized so that $|w_0|_{H^1}=1$, we have, since $w_0=w_0^1\lambda(\partial\omega_2/\partial\lambda)+\sum w_{0,j}^2(\partial\omega_2/\partial\sigma_2^j)+\sum w_{0,j}^1(\partial\bar{\omega}_1/\partial\sigma_1^j)$,

$$\dot{\lambda}=O(\lambda)\quad\text{and}\quad-\frac{d}{dt}\big(J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})\big)\geq\frac{c_1}{\lambda^{3/2}}\,.$$

Thus, with λ_0 as the initial concentration

$$\lambda=\lambda_0O(e^t)\,,\quad t\in(-\infty,\infty)\,,$$

and

$$\begin{aligned}&J\big(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v}\big)(0)\geq\\&\qquad J\big(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v}\big)(t)+\frac{c_{10}}{\lambda_0^{3/2}}\int_0^t e^{-\frac{3\tau}{2}}\,d\tau\quad\text{for } t\geq 0\,,\\&J\big(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v}\big)(t)\geq\\&\qquad J\big(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v}\big)(0)+\frac{c_{10}}{\lambda_0^{3/2}}\int_t^0 e^{-\frac{3\tau}{2}}\,d\tau\quad\text{for } t\leq 0\,.\end{aligned}$$

The same inequalities hold with $\psi(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)$ instead of $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})$. The only restriction on these computations lies with the validity of the expansions in the (X, Y, Z)-space, in particular the one concerning $\nabla\bar{\omega}_1(0)\cdot\nabla\underline{\omega}_2(\infty)$ and the use of the (X, Y, Z) coordinates. Along the flow lines of w_0, we have

$$\dot{\overline{(X,Y,Z)}}=O(1)\quad\text{by construction (in fact, }\dot{Y}=O(\lambda^{-1}))\,,$$

so that the time needed to exit the neighborhood where such expansions hold is bounded from below by a fixed and positive $\gamma>0$.

Using the inequalities above and the existence of γ, we see that the variation of $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})(t)$ around $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})(0)$ is at least $c_{11}/\lambda_0^{3/2}$ in both the positive and the negative directions. The same holds with $\psi(\alpha_1\bar{\omega}_1+\alpha_1\omega_2)$. Since $J(\alpha_1\bar{\omega}_1+\alpha_2\omega_2+\bar{v})(0)$ and $\psi(\alpha_1\bar{\omega}_1+\alpha_2\omega_2)(0)$ differ by $o(1/\lambda_0^{3/2})$ when λ_0 tends to $+\infty$, and since w_0 is a pseudogradient for both functionals, we may transmute one functional into the other using the flow lines of w_0. $\square$

PROOF OF THEOREM 1.5: According to Lemma 4.3, in $V(2, \varepsilon_0, \bar\omega_1, \bar\omega_2)$, for ε_0 small enough, $J(\alpha_1\bar\omega_1 + \alpha_2\omega_2 + v)$ reads in suitable local coordinates as

$$\frac{\left(\alpha_1^2 \int \bar\omega_1^6 + \alpha_2^2 \int \bar\omega_2^6\right)^3}{\alpha_1^6 \int \bar\omega_1^6 + \alpha_2^6 \int \bar\omega_2^6} \left(1 + h(X, Y, Z, \lambda) + |V^+|^2 - |V^-|^2\right)$$

where $h(X, Y, Z, \lambda)$ assumes the various forms of Lemma 4.3 and $Q(V, V)$ has been diagonalized into $|V^+|^2 - |V^-|^2$. We recall that

$$c_\infty = \left(\int \bar\omega_1^6 + \int \bar\omega_2^6\right)^2 .$$

We want to study, up to excision, the pair $(J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0), J_{c_\infty-\varepsilon} \cap V(2, \varepsilon_0))$ for ε tending to zero, ε_0 small. Taking ε_0 small enough, we can use Proposition 1.4 and Lemma 4.3. When λ tends to $+\infty$ and $|V|^2$ tends to zero while α_2/α_1 tends to 1, $J(\alpha_1\bar\omega_1 + \alpha_2\omega_2 + v)$ tends to c_∞. Thus, $J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0)$ can be replaced with the full-parameter space $P_A(\alpha_1, \alpha_2, X, Y, Z, V^+, V^-, \lambda)$ with λ large enough $\lambda \geq A, |V^+|, |V^-|$ small: Using the pseudogradient w_0 of Lemmas 4.1–4.3, this space deforms onto $J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0)$ in $J_{c_\infty+\varepsilon'} \cap V(2, \varepsilon_0)$, with ε' small, ε' tending to zero when A tends to $+\infty$. Since $J_{c_\infty+\varepsilon'} \cap V(2, \varepsilon_0)$ deforms through w_0 on $J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0)$ and since, using various values for ε, ε', and A, we can build a nested family

$$J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0) \hookrightarrow P_A \hookrightarrow J_{c_\infty+\varepsilon'} \cap V(2, \varepsilon_0) \hookrightarrow P_{A'}$$

with $P_{A'}$ deforming onto P_A, and $J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0)$ and P_A being homologically equivalent. We can use P_A in lieu of $J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0)$ in the computation of the difference of topology.

On the other hand, $J_{c_\infty+\varepsilon} \cap V(2, \varepsilon_0)$ can be replaced by $J_{c_\infty} \cap V(2, \varepsilon_0)$ because, from Proposition 1.3, the Palais-Smale condition is satisfied below the level c_∞ (until the next critical level at infinity), and using the decreasing flow lines of w_0, we may assume, even if we start at the level c_∞, that we are a tiny bit below this level.

Thus, we are considering the pair $(P_A, J_{c_\infty} \cap V(2, \varepsilon_0))$. The differential equation $\partial V^+/\partial s = -V^+$ retracts by deformation this pair onto the pair $(\widetilde{P}_A, \widetilde{J}_{c_\infty} \cap V(2, \varepsilon_0))$, where

$$\widetilde{P}_A = \left\{\left(\alpha_1, \alpha_2, X, Y, Z, V^-, \lambda\right), \lambda \geq A, |V^-| \leq \frac{1}{A}, \left|\frac{\alpha_2}{\alpha_1} - 1\right| < \frac{1}{A}\right\}$$

and

$$\widetilde{J}_{c_\infty} \cap V(2, \varepsilon_0) = \left\{\left(\alpha_1, \alpha_2, X, Y, X, V^-, \lambda\right) = z \in \widetilde{P}_A : \widetilde{J}_{c_\infty}(z) = \frac{\left(\alpha_1^2 \int \bar\omega_1^6 + \alpha_2^2 \int \bar\omega_2^6\right)^3}{\alpha_1^6 \int \bar\omega_1^6 + \alpha_2^6 \int \bar\omega_2^6} \times \left(1 + h(X, Y, Z, \lambda) - |V^-|^2\right) \leq c_\infty\right\}.$$

Let us first assume, in order to study $\widetilde{J}_{c_\infty} \cap V(2, \varepsilon_0)$, that $V^- = 0$ and $\alpha_1 = \alpha_2$. This part, D_∞, of $\widetilde{J}_{c_\infty} \cap V(2, \varepsilon_0)$, reads as

$$h(X, Y, Z, \lambda) \leq 0\,;$$

X and Y are $\underline{\omega}_1(0)$ and $\bar{\omega}_2(\infty)$; Z represents the transversal parameters in the orthogonal group.

(1) Considering Lemma 4.3(i) and (ii), we are led directly to include in D_∞ the subset of $\widetilde{P}_A$ defined by

$$C = \left\{z \in \widetilde{P}_A : XY \geq 0,\ X \text{ or } Y \text{ nonzero}\right\}.$$

(2) Studying Lemma 4.3(iii)(b), we see that the condition $h(X, Y, Z, \lambda) \leq 0$ reads

$$XY \geq \frac{c_0}{\lambda}\,.$$

Since c_0 is nonzero, we distinguish two cases:

(α) When c_0, which represents $\nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty)$, is positive, then this set has two connected contractible components, since $(0, 0)$ in the (X, Y)-space is not included in it. It has the same homotopy type (up to multiplication by a ball in the Z-space and by $[A, +\infty)$); then $\{(X, Y) : XY \geq 0\} - \{(0, 0)\}$.

(β) When $\nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty)$ is negative, then this set is reduced to a contractible set, since $(0, 0)$ ties the two former components. It has the same homotopy type as $\{(X, Y) : XY \geq 0\}$.

(3) (α) and (β) can be extended to include all cases where $\nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty)$ is nonzero, since Lemma 4.3(iii)(b) holds under this hypothesis.

(4) The case remains where $\nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty)$ is zero and Lemma 4.3(iii)(a) holds. Then $h(X, Y, Z, \lambda) \leq 0$ reads

$$\left(X + \frac{\bar{c}_2}{\lambda}\right)\left(Y + \frac{\bar{c}_1}{\lambda}\right) \geq \frac{\bar{c}_1\bar{c}_2}{\lambda^2} - \frac{L(Z)}{\lambda}\,.$$

$|Z|$ is small, independently of how large λ is, i.e., independently of A; i.e., $|Z| \leq \theta$, while $\lambda \geq A$. Taking A very large and using the fact that L is nonsingular, we easily see that the above set completes the transition between (α) and (β); i.e., as $L(Z)$ changes from $-\gamma_0$ to γ_0, when γ_0 is a small positive constant, the corresponding section to this set evolves from (α) to (β) or vice versa.

Observe that L is the differential of the function $\nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty)$ restricted to $\{X = 0\} \cap \{Y = 0\}$. By assumption, the level zero for this function on $\{X = 0\} \cap \{Y = 0\}$ is regular, since L is nonzero. It is therefore easy to identify the transition set from (α) to (β) with the level zero of $\nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty)$ on $\{X = 0\} \cap \{Y = 0\}$.

Summing up three cases, we see that D_∞ is the union of C and of a subset of $\{(X, Y, Z) \in \widetilde{P}_A : X = Y = 0\} \times [A, +\infty)$ that is very close and has the same homotopy type as $\{(X, Y, Z) \in \widetilde{P}_A : X = Y = 0, \nabla\underline{\omega}_1(0) \cdot \nabla\underline{\omega}_2(\infty) > 0\} \times [A, +\infty)$.

In $\widetilde{J}_{c_\infty} \cap V(2, \varepsilon_0) - D_\infty$, since V^- is nonzero or α_1/α_2 is not 1, there is a decreasing pseudogradient, which is nonzero as long as α_1 and α_2 are nonzero and V^- is small. Along the flow lines of this pseudogradient, we deform $\widetilde{J}_{c_\infty} \cap V(2, \varepsilon_0) - D_\infty$ onto $\widetilde{P}_A \times \partial(\Delta_1 \times D_u(Q))$, where Δ_1 is the standard one simplex (for (α_1, α_2)) and $D_u(Q)$ is the unstable disk of Q for this pseudogradient. By Lemma 3.1, $D_u(Q)$ is a product equal to $D_u(\widetilde{Q}_1) \times D_u(\widetilde{Q}_2)$. Since this deformation acts only on the $(\alpha_1, \alpha_2, V^-)$-space, it does not exit D_∞, and the previous arguments characterizing D_∞ and the present argument that displays a retract by deformation of $\widetilde{J}_{c_\infty} \cap V(2, \varepsilon_0) - D_\infty$ combine to provide the statement of Theorem 1.5 after canceling the $[A, +\infty)$ factor (which is useless) and identifying $\widetilde{P}_A$ (up to $[A, +\infty)$) with the orthogonal group $O(4)^2 \times \Delta_1$, modded out by π_2 in the event $\bar{\omega}_1$ and $\bar{\omega}_2$ can be brought one onto the other by an element of $O(4)$.

□

Bibliography

[1] Ahmedou, M. O.; El Mehdi, K. O. Computation of the difference of topology at infinity for Yamabe-type problems on annuli-domains. I, II. *Duke Math. J.* **94** (1998), no. 2, 215–229, 231–255.

[2] Bahri, A. *Critical points at infinity in some variational problems.* Pitman Research Notes in Mathematics Series, 182. Longman Scientific & Technical, Harlow; Wiley, New York, 1989.

[3] Bahri, A. An invariant for Yamabe-type flows with applications to scalar-curvature problems in high dimension. A celebration of John F. Nash, Jr. *Duke Math. J.* **81** (1996), no. 2, 323–466.

[4] Bahri, A.; Brezis, H. Non-linear elliptic equations on Riemannian manifolds with the Sobolev critical exponent. *Topics in geometry*, 1–100. Progr. Nonlinear Differential Equations Appl., 20. Birkhäuser, Boston, 1996.

[5] Bahri, A.; Coron, J.-M. On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. *Comm. Pure Appl. Math.* **41** (1988), no. 3, 253–294.

[6] Bahri, A.; Li, Y. Y.; Rey, O. On a variational problem with lack of compactness: the topological effect of the critical points at infinity. *Calc. Var. Partial Differential Equations* **3** (1995), no. 1, 67–93.

[7] Struwe, M. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.* **187** (1984), no. 4, 511–517.

ABBAS BAHRI
Rutgers University
Department of Mathematics
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
E-mail: `abahri@math.rutgers.edu`

SAGUN CHANILLO
Rutgers University
Department of Mathematics
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
E-mail: `chanillo@math.rutgers.edu`

Received January 2000.