Multivariable Calculus
The Gradient

Thursday, June 13, 2002
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1. The gradient
   $\frac{df(\vec{x})}{d\vec{x}} = \nabla f(\vec{x})$
2. Directional derivative
   $\frac{df}{dt} = \nabla f \cdot \vec{u}$
   This is the rate of change of f in the direction $\vec{u}$. In this case $\vec{u} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$ is the slope vector of the tangent line.
   $\frac{df}{dt} = \frac{df(\vec{x})}{d\vec{x}} \cdot \frac{d\vec{x}}{dt}$
3. Extrema
   $\frac{df}{dt} = \nabla f \cdot \vec{d} = 0$
   This must hold for **every** direction $\vec{d}$, and so this is equivalent to $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$.
   $\frac{df}{dt} = \frac{df(\vec{x})}{d\vec{x}} \cdot \frac{d\vec{x}}{dt} = 0$
4. Chain rule
   $\frac{\partial f}{\partial u} = \nabla f \cdot \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle$
   $\frac{\partial f}{\partial u} = \frac{df(\vec{x})}{d\vec{x}} \cdot \frac{\partial \vec{x}}{\partial u}$
5. Total differential
   $df = \nabla f \cdot \langle dx, dy \rangle$
   $df = \frac{df(\vec{x})}{d\vec{x}} \cdot d\vec{x}$

6. Linear approximation

   $\Delta f = \nabla f \cdot \langle \Delta x, \Delta y \rangle$

   $\Delta f = \frac{df(\vec{x})}{d\vec{x}} \cdot \Delta \vec{x}$

7. Lagrange multipliers

   Find extrema of $w = f(x, y)$ subject to the constraint $g(x, y) = 0$.

   $\frac{df}{dt} = \nabla f \cdot \vec{d} = 0$

   $\frac{df}{dt} = \frac{df(\vec{x})}{d\vec{u}} \cdot \frac{d\vec{u}}{dt} = 0$

   This must hold for **every** $\vec{d}$ which is a slope vector for the constraint surface $g(x, y) = 0$. As a necessary and sufficient condition for $\vec{d}$ to be a slope vector is that $\vec{d}$ is perpendicular to the normal vector ∇g, this condition may be also be written as

   $\nabla f = \lambda \nabla g$

   This means that ∇f and ∇g are parallel.

   This generalizes immediately to any number of constraints.