An Optimal Fence

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Problem: Consider an ordered sequence of positive reals; e.g.

9 9 9 8 8 7 6 6 6 6 5 5 4 3 3 3 2 2 1 1.

Split them into 2 parts by placing a ”fence”; e.g.

9 9 9 8 8 7 6 | 6 6 6 5 5 4 3 3 3 2 2 1 1.

There are 7 elements on the left, 13 on the right. The mean of the left is 8, the mean of the right is 47/13. I would like to maximize $7 \cdot 8^2 + 13 \cdot (47/13)^2$. Actually my previous statement is not accurate; more correctly I would like to maximize $n_1 \cdot {M_1}^2 + n_2 \cdot {M_2}^2$. In this case this value is approximately 618.34.

Clearly we can compute this value by moving the fence and picking the position of the fence that yields the highest value. The above choice for the ”fence” is definitely not the best (618.34 is not maximal).

Prove or disprove that if the optimal fence is determined, then the adjacent valuesto the left and right of the fence are not identical. The sequence is assumed to have at least 2 distinct values.

Solution: The conjecture is true; in fact, a stronger statement is true.

Lemma 1 *Let $x_1, \ldots, x_n$ be a sequence of increasing real numbers, not all alike. Let M be a value of m, $1 \le m \le n-1$, for which*

$$f(m,n) = (x_1 + ... + x_m)^2/m + (x_{m+1} + ... + x_n)^2/(n-m)$$

takes the greatest value. Then if $x_M = x_{M+1}$, $M = 1$ or $M = n-1$.

Proof: Let $A = x_1 + ... + x_M$, $B = x_{M+1} + ... + x_n$, $a(t) = (A + t \cdot x_M)^2/(M+t)$, and $b(t) = (B + t \cdot x_{M+1})^2/((n-M)+t)$. I claim that if all the elements in our sequence are not identical, then $a(t)$ and $b(t)$ are convex for $1 \ge t \ge -1$, and at least one of the two is strictly convex. To see this, note that

$$a(t) = ((A - M \cdot x_M) + x_M \cdot (M+t))^2/(M+t) = C/(M+t) + P(t),$$

where $P(t)$ is a 1st degree polynomial in t and C is a non-negative constant. Note that $C = 0$ if and only if $A - M \cdot x_M = 0$, i.e. $x_1 = \ldots = x_M$. Differentiating twice, we see that

$a''(t) = 2C/(M+t)^3 \implies a(t)$ is convex if $t > -M$, strictly so if the first M elements are not identical. The same argument applies to $b(t)$, and there we deduce that $b(t)$ is convex if $t > n-M$, strictly so if the last $n-M$ elements are not identical. Since we are assuming that all the elements are not identical, we see that the sum of the two will be strictly convex if $t > -M$ and $t > n-M$.

Assuming $1 < M < n-1$ and applying Jensen's Inequality, we get

$$(a(-1)+a(1)) + (b(-1)+b(1)) \geq 2 \cdot a(0) + 2 \cdot b(0)$$

$$\implies ((a(-1)+b(1)) - (a(0)+b(0))) + ((a(1)+b(-1)) - (a(0)+b(0))) > 0.$$

This implies that either $(a(-1)+b(1))-(a(0)+b(0)) > 0$ or $(a(1)+b(-1))-(a(0)+b(0)) > 0$. But if $x_M = x_{M+1}$ this is equivalent to either $f(M-1,n) > f(M,n)$ or $f(M+1,n) > f(M,n)$, thus the only possiblities are that either $M = 1$ or $M = n-1$. □

Let M' be the value of m, $1 \leq m \leq n$, for which $f(m,n)$ takes on the maximum value.

The above lemma does most of the work. To finish, we need only demonstrate that if all the elements in our sequence aren't identical and if $x = x_1 = x_2$, then $M' \neq 1$; likewise for $x = x_{n-1} = x_n$. Let $B = x_3 + \ldots + x_n$.

Now

$$x^2 + (B+x)^/(n-1) < (2x)^2/2 + B^2/(n-2) \iff (B - x(n-2))^2 > 0,$$

but since $x_1 = x_2$ and since our sequence is increasing, $(B - x(n-2))^2 = 0$ would imply that all the elements in our sequence are identical. Contradiction by assumption, hence $f(2,n) > f(1,n)$. For the other case, the proof is almost identical. □