

59th Putnam Competition

5 December 1998

- A1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?
- A2. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x -axis and let B be the area of the region lying to the right of the y -axis and to the left of s . Prove that $A + B$ depends only on the arc length, and not on the position, of s .
- A3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

- A4. Let $A_1 = 0$ and $A_2 = 1$. For $n \geq 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n .
- A5. Let F be a finite collection of open discs in \mathbb{R}^2 whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection D_1, \dots, D_n in F such that

$$\bigcup_{j=1, \dots, n} 3D_j \supset E.$$

Here, if D is the disc of radius r and center P , then $3D$ is the disc of radius $3r$ and center P .

- A6. Let A, B, C denote distinct points with integer coordinates in \mathbb{R}^2 . Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC .

B1. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}.$$

B2. Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b) , one on the x -axis, and one on the line $y = x$. You may assume that a triangle of minimal perimeter exists.

B3. Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P a regular pentagon inscribed in C . Determine the surface area of that portion of H lying over the planar region inside P , and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α , and β are real numbers.

B4. Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

B5. Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = \underbrace{1111\dots 11}_{1998 \text{ digits}}.$$

Find the thousandth digit after the decimal point of \sqrt{N} .

B6. Prove that, for any integers a, b, c there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.