Some research ideas accessible to undergraduates

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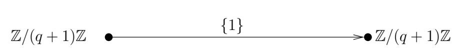
Mentoring undergraduate projects is an appealing prospect for me. The projects I helped the SUSMRI REU students on involved a combination of elementary ring theory and graph theory. Abstract algebra and graph theory strike me as an attractive combination for potential research projects, and my research contains these elements. Here are some ideas from my research directed at an undergraduate (see my research statement for more background and context).

Here's an edge indexed graph:

There's a positive integer – called the edge index – associated to each edge and its reverse. In this example the single edge and its reverse have index $q+1$.

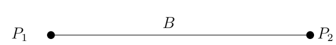
To build a graph of groups from an edge-indexed graph we need to choose groups for every vertex and edge so that the edge groups embed in the vertex groups so that the index of the edge matches the index of the edge group in the initial vertex group.

Here's a graph of groups built from the edge-indexed graph above.



This graph of groups actually encodes a specific group, namely the free product $H = \mathbb{Z}/(q+1)\mathbb{Z} * \mathbb{Z}/(q+1)\mathbb{Z}$.

Here's another example (also built from the edge-indexed graph above).



This example encodes a rank 2 Kac-Moody group over the finite field F_{q+1}, which we'll denote G (we'll leave a definition of this for later).

It turns out that for $q = 2$, H is a subgroup of G, and this is a consequence of the structure of is edge-indexed graph and its graph of groups. Using this fact, we can actually construct more edge-indexed graphs and corresponding graphs of groups that give us subgroups of H and therefore of G. These graphs and groups have to satisfy certain combinatorial or group-theoretic parameters, such as

- The graphs must be bipartite.
- The sum of edge-indices at a single vertex must be 3.
- All the vertex groups must be $\mathbb{Z}/(q+1)\mathbb{Z} = \mathbb{Z}/3\mathbb{Z}$ or trivial, and all edge groups must be trivial.

Actually, these parameters and a few other conditions give us a complete description of all subgroups of H with finite graphs of groups, and there are infinitely many of them.

Using similar methods to those used in the classification and constructions of subgroups of H when $q = 2$, can we answer or make progress on any of the following:

1. For $q = 2$, what subgroups of G, but not of H, can we construct using finite graphs of finite groups? In particular, can we construct examples where the edge groups are not all trivial or some vertex groups are non-cyclic?
2. Can we find a new example for the $q = 2$ case where all the groups are abelian? If so, can we extend the methods used for constructing subgroups of H to constructing an infinite number of subgroups of our new example? (Abelian groups on the vertices and edges give us a nice sufficient condition for guaranteeing that an edge-indexed graph encodes a subgroup.)
3. For what values of q will H be a subgroup of G? When H is a subgroup of G, all the constructions and examples for the $q = 2$ case will extend. Therefore, if H is a subgroup of G, then we have infinitely many subgroups of G
4. Find other examples of subgroups for $q > 2$.

The subgroups obtained in these projects would be cocompact lattice subgroups of a Kac-Moody group.