Math 251 Multivariable Calculus Fall 2008 Workshop 9

You will write up *one* of these problems and turn it in at next week's workshop. You will be graded on your exposition as well as on the mathematical content of your report. During the workshop period, you are encouraged to work with other students in your group. However, the writeup you turn in should represent your own presentation of the solution. It should not be a (rewritten) version of someone else's work. You should:

- Work out the problem and have a clear plan of presentation before you begin to write up your final solution.
- Be neat and write legibly. Don't try to squeeze your report onto this piece of paper.
- Show all steps. Explain what you are doing at each step in complete sentences.

Write as if you intended your explanation to be part of a good textbook, or of a professional lab report in science or engineering.

1. In this problem we will prove the famous equality which is very important in probability theory
$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$
   (a) Graph the integrand $y = e^{-x^2}$ and explain why there is no simple way to compute this integral. This curve is called the "bell curve" or "normal distribution."
   (b) If $I = \int_{-\infty}^{\infty} e^{-x^2} dx$ show that
$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy.$$
   (c) Describe $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy$ as the limit of $\iint_R e^{-x^2-y^2} dxdy$ over larger and larger squares R. (Recall that we define $\int_0^{\infty} f(t)dt = \lim_{L\to\infty} \int_0^L f(t)dt$ if this limit exists.)
   (d) Prove that this is the same as the limit of $\iint_D e^{-x^2-y^2} dA$ over larger and larger disks D centered at the origin.
   (e) Use polar coordinates to evaluate the limit of $\iint_D e^{-x^2-y^2} dA$ as the radius of the disk goes to infinity.
   (f) Pull all this together to prove the equality $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.
2. Let D be the region in the x-y plane bounded by the spirals $r = \theta$ and $r = 2\theta$ for $0 \le \theta \le 2\pi$.
   (a) Sketch D by plotting the boundary curves in polar coordinates.

(b) Let $f(x, y)$ be any continuous function defined on D. Express

$$\int\int_D f(x, y)\, dA$$

as an iterated integral in polar coordinates (choose the order of integration so that you only need one iterated integral). Then calculate the integral for the function $f(x, y) = x^2 + y^2$.

3. Consider the iterated integral

$$\int_0^{2\sin\beta} \int_{y\cot\beta}^{\sqrt{4-y^2}} \ln(x^2 + y^2)\, dx\, dy,$$

where $0 < \beta < \pi/2$ is a fixed parameter.

(a) Sketch the region of integration D (*Hint:* use the limits of integration to obtain the set of inequalities that define D; try the case $\beta = \pi/4$ first.).

(b) Rewrite the integral with the order of integration reversed (you will need to split D into two parts).

(c) Transform the integral into polar coordinates.

(d) Evaluate the integral using polar coordinates.

4. A rotation of the xy-plane about the origin, with fixed angle θ, is given by

$$x = u\cos\theta - v\sin\theta, \qquad y = u\sin\theta + v\cos\theta$$

(a) Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

(b) Let E denote the area within the ellipse

$$x^2 + xy + y^2 = 3.$$

Use a rotation by $\frac{\pi}{4}$ to obtain an integral that is equivalent to

$$\iint_E y\, dx\, dy.$$

(c) Evaluate the transformed integral.

5. Assume the formula for the volume of a sphere: $V = \dfrac{4}{3}\pi r^3$. Use this and the proper change of variables to obtain a formula for the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$