Math 251 Multivariable Calculus Fall 2008 Workshop 11

You will write up *one* of these problems and turn it in at next week's workshop. You will be graded on your exposition as well as on the mathematical content of your report. During the workshop period, you are encouraged to work with other students in your group. However, the writeup you turn in should represent your own presentation of the solution. It should not be a (rewritten) version of someone else's work. You should:

- Work out the problem and have a clear plan of presentation before you begin to write up your final solution.
- Be neat and write legibly. Don't try to squeeze your report onto this piece of paper.
- Show all steps. Explain what you are doing at each step in complete sentences.

Write as if you intended your explanation to be part of a good textbook, or of a professional lab report in science or engineering.

1. *The paint can with finite volume but infinite surface area.* Consider the curve $z = -1/x$ in the xz–plane for $0 < x \leq 4$, and let $\mathcal{S}$ be the surface obtained by rotating this curve about the z–axis.

   (a) Draw a picture of $\mathcal{S}$ and describe it in words.

   (b) If $\mathcal{D}$ is the region "inside" $\mathcal{S}$, find the volume of $\mathcal{D}$. (You will have to use an improper integral to express this volume.) Notice, in particular, that the volume is finite.

   (c) Express the surface area of $\mathcal{S}$ as an improper integral and show that this surface area is infinite.

   (d) There seems to be a paradox here as your friend might argue the following: "The surface of $\mathcal{D}$ has infinite area so we would never be able to paint it with a finite amount of paint. Yet, if we poured paint into $\mathcal{D}$, we would fill it up with only a finite amount of paint, and filling $\mathcal{D}$ with paint would automatically paint its sides." Explain to your friend what is wrong with her reasoning.

2. In this problem we will consider the following vector field

$$\mathbf{F} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

   (a) Sketch $\mathbf{F}$ and describe its domain. Is the domain connected? Is it simply connected?

   (b) Show that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

   (c) Show that $\mathbf{F}$ is not conservative by finding a closed path $\mathcal{C}$ for which $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} \neq 0$. Why does this not violate Theorems 3 and 4 of Section 16.3?

(d) Let $\mathcal{D}$ be the region of the xy-plane where $x > 0$ and restrict $\mathbf{F}$ to $\mathcal{D}$. Is $\mathcal{D}$ simply connected? Is $\mathbf{F}$ defined everywhere on $\mathcal{D}$. Conclude from Theorem 4 of Section 16.3 that $\mathbf{F}$ must be the gradient of some potential function φ defined on $\mathcal{D}$.

(e) Show that $\varphi(x, y) = \tan^{-1}(y/x)$ is such a potential function on $\mathcal{D}$.

(f) In part (c) we showed that $\mathbf{F}$ cannot be the gradient of any function. Why does part (e) not violate this?

3. *The integral that counts.* For this problem $\mathbf{F}$ will be the same as in the previous problem, namely

$$\mathbf{F} = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

(a) Show that $\text{curl}_z(\mathbf{F})$ is zero everywhere it is defined.

(b) Let $\mathcal{C}_R$ be the circle of radius R centered at the origin and oriented counterclockwise. Show that the circulation of $\mathbf{F}$ around $\mathcal{C}_R$ is 2π for any R.

(c) Why does this not violate Green's Theorem?

(d) Let $\mathcal{C}$ be *any* simple closed curve that loops once around the origin, counterclockwise. Use Green's Theorem to show that $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 2\pi$. (Hint: Choose R small enough so that $\mathcal{C}_R$ is contained within $\mathcal{C}$ and apply Green's Theorem to the region between $\mathcal{C}_R$ and $\mathcal{C}$.) When applying Green's Theorem you must explain why it applies since we saw in parts (a)-(c) that you always have to be careful.

(e) Now suppose that $\mathcal{C}$ is *any* closed curve that does not pass through the origin and let n denote the number of times $\mathcal{C}$ loops around the origin, where n is positive if the 'total looping' is counterclockwise and negative if the 'total looping' is clockwise. Show that

$$\frac{1}{2\pi} \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = n.$$

Hence the integral $\frac{1}{2\pi} \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ *counts* the number of times $\mathcal{C}$ loops around the origin.