Math 251 Multivariable Calculus Fall 2008 Workshop 12

You will write up *one* of these problems and turn it in at next week's workshop. You will be graded on your exposition as well as on the mathematical content of your report. During the workshop period, you are encouraged to work with other students in your group. However, the writeup you turn in should represent your own presentation of the solution. It should not be a (rewritten) version of someone else's work. You should:

- Work out the problem and have a clear plan of presentation before you begin to write up your final solution.
- Be neat and write legibly. Don't try to squeeze your report onto this piece of paper.
- Show all steps. Explain what you are doing at each step in complete sentences.

Write as if you intended your explanation to be part of a good textbook, or of a professional lab report in science or engineering.

1. Let $f(x)$ be a positive function for $0 \le x \le 1$. Let $\mathcal{S}$ be the surface obtained by rotating the graph of $y = f(x) \quad (a \le x \le b)$ around the x-axis. Then $\mathcal{S}$ is parameterized by x and θ (the angle of rotation) by the vector equation

$$\Phi(x, \theta) = x\mathbf{i} + f(x)\cos(\theta)\mathbf{j} + f(x)\sin(\theta)\mathbf{k}, \qquad a \le x \le b, \quad 0 \le \theta \le 2\pi.$$

   (a) Suppose $f(x) = 1$. Sketch $\mathcal{S}$, calculate $\mathbf{T}_x$, $\mathbf{T}_\theta$, and the outward normal vector $\mathbf{T}_\theta \times \mathbf{T}_x$. Calculate the area of $\mathcal{S}$ by integrating $dS = ||\mathbf{T}_\theta \times \mathbf{T}_x||\, dx\, d\theta$ over the parameter region. Check your answer by elementary geometry.

   (b) Suppose $f(x) = e^x$ with $0 \le x \le 1$. Sketch $\mathcal{S}$, calculate $\mathbf{T}_\theta$, $\mathbf{T}_x$ and the normal vector $\mathbf{T}_x \times \mathbf{T}_\theta$. Calculate the area of $\mathcal{S}$ by integrating $dS = ||\mathbf{T}_\theta \times \mathbf{T}_x||\, dx\, d\theta$ over the parameter region (use your calculator to evaluate the x integral numerically).

   (c) Let $f(x) = e^x$ as in part (b). Let E be the solid region bounded by the surface $\mathcal{S}$ and the disks D_0: $x = 0$, $y^2 + z^2 \le 1$ and D_1: $x = 1$, $y^2 + z^2 \le e^2$. Let $\mathbf{F} = x\mathbf{i}$. Calculate the *flux integral*

$$\iint_{\partial E} \mathbf{F} \cdot \mathbf{e_n}\, dS,$$

   where $\mathbf{e_n}$ is the unit outward-pointing vector and ∂E is the boundary of E. (*Hint:* Write the integral as a sum of integrals over $\mathcal{S}$, D_0 and D_1. For the integral over $\mathcal{S}$, show that $\mathbf{e_n}\, dS = \mathbf{T}_\theta \times \mathbf{T}_x\, dx\, d\theta$, and then use your calculations from (b). Use geometry to set up and evaluate the integrals over D_0 and D_1.)

2. The definition of element of arc length $ds = ||\mathbf{r}'(t)||\, dt$ of a curve does not depend on the paramerization of the curve (how fast a particle moves along the curve). This is a consequence of the chain rule for functions of one variable. Likewise, the definition of

element of surface area $dS = ||\mathbf{T}_u \times \mathbf{T}_v||\, du\, dv$ does not depend on the parametrization of the surface. This is a consequence of the *chain rule for Jacobians*:

$$\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(r,s)}. \qquad (*)$$

when x, y are functions of u, v and u, v are functions of variables r, s. (Formula $(*)$ follows from the chain rule for partial derivatives and the product formula for determinants.)

(a) Suppose $x = u^2 - v^2$, $y = 2uv$ and $u = r - s$, $v = s/2$. Calculate $\dfrac{\partial(x,y)}{\partial(u,v)}$ and $\dfrac{\partial(u,v)}{\partial(r,s)}$. Then calculate $\dfrac{\partial(x,y)}{\partial(r,s)}$ by multiplying the Jacobians according to the chain rule $(*)$. Check your answer by expressing x, y as explicit functions of r, s and calculating the Jacobian directly.

(b) Use the general chain rule $(*)$ for Jacobians to show that

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)}$$

for any one-to-one change of variable (*Hint:* Take $r = x$ and $s = y$.) Then check this formula by direct calculation for the change of variable $x = u^2/v$, $y = v/u$. Remember to solve for u, v as functions of x, y before calculating $\dfrac{\partial(u,v)}{\partial(x,y)}$.

3. Consider an electric field $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Let $\mathcal{S}$ be the surface of the cube $\mathcal{C}$ with vertices $(\pm1, \pm1, \pm1)$.

(a) Gauss' Law states that the total charge Q inside $\mathcal{S}$ is proportional to the surface integral

$$\iint_{\mathcal{S}} \mathbf{E} \cdot \mathbf{e_n}\, dS,$$

where $\mathbf{e_n}$ is the unit outward normal vector to $\mathcal{S}$. Calculate this integral explicitly (as the sum of 6 integrals over the 6 faces of $\mathcal{S}$).

(b) The *divergence theorem* states that the surface integral in part (a) equals the volume integral

$$\iiint_{\mathcal{C}} \operatorname{div} \mathbf{E}\, dV,$$

Calculate this volume integral directly and verify the divergence theorem for this case.