Math 251 Multivariable Calculus Fall 2008 Workshop 13 – Final Review

This list is meant as a guide for reviewing for the portion of the final exam which will cover the new material. Please note that any kind of problem included in the syllabus, the list of homework problems, or the workshops can be included on the final exam. Also, that this review sheet is longer than the exam itself.

To review for the material covered on the previous exams your should also redo the midterm exam reviews and look over the old exams, quizzes, workshops and homework.

1. A rotation of the xy-plane about the origin, with fixed angle θ, is given by

$$x = u\cos\theta - v\sin\theta, \qquad y = u\sin\theta + v\cos\theta$$

(a) Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$

(b) Let E denote the region within the ellipse

$$x^2 + xy + y^2 = 3$$

Use a rotation by $\frac{\pi}{4}$ to obtain an integral that is equivalent to

$$\iint_E y\,dx\,dy$$

(c) Evaluate the transformed integral.

2. Let D be the region in the xy-plane that is bounded by the coordinate axes and the line $x+y=1$. Use the change of variable $u = x-y$, $v = x+y$ to compute the integral

$$\iint_D (x-y)^5(x+y)^3\,dy\,dx$$

3. Evaluate each line integral:

(a) $\int_C(-y\,dx + x\,dy)$ if C is the parabolic path $y = 4x^2$ from $(1,4)$ to $(0,0)$.

(b) $\int_C(x^2y\,dx - xy\,dy)$ where C is the closed path that begins at $(0,0)$, goes to $(1,1)$ along the parabola $y = x^2$, and then returns to $(0,0)$ along the line $y = x$.

4. Evaluate the line integrals using the fundamental theorem of line integrals.

(a) $\int_C(x\mathbf{i} + y\mathbf{j})\cdot d\mathbf{r}$ where C is any smooth path from $(0,0)$ to $(2,4)$.

(b) $\int_C(e^x\sin y\,dx + e^x\cos y\,dy)$, where C is any smooth curve from $(0,0)$ to $(0,2\pi)$.

5. (a) Over what region on the xy-plane will the line integral

$$\int_C\left[(-yx^{-2} + x^{-1})\,dx + x^{-1}\,dy\right]$$

be independent of path?

(b) Evaluate the integral in (a) if C is defined by $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (2\sqrt{\sec t})\mathbf{j}$ for $0 \leq t \leq \frac{\pi}{6}$.

6. Find the curl and divergence of the following vector fields $\mathbf{F}$. If the field is conservative, find a function f so that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos x\mathbf{j} + z^2\mathbf{k}$

   (b) $\mathbf{F}(x, y, z) = z\mathbf{i} + 2yz\mathbf{j} + (x + y^2 - z^2)\mathbf{k}$

7. A vector field $\mathbf{F}$ is called *irrotational* if $\mathbf{curl}\,\mathbf{F} = 0$, and it is called *incompressible* if $\operatorname{div}\mathbf{F} = 0$.

   (a) Show that any vector field of the form $\mathbf{F}(x, y, z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$ is irrotational.

   (b) Show that any vector field of the form $\mathbf{F}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k}$ is incompressible.

8. In this problem we consider the surface given by

$$\mathbf{r}(u, v) = (u\cos v)\mathbf{i} + (u\sin v)\mathbf{j} + v\mathbf{k} \qquad \text{for} \quad 0 \leq u \leq 1, \quad 0 \leq v \leq \pi.$$

   (a) Sketch the surface.

   (b) Find its surface area.

9. Suppose $\mathcal{S}$ is the portion of the paraboloid $z = x^2 + y^2$ for which $z \leq 4$. Evaluate the surface integral

$$\iint_{\mathcal{S}} \sqrt{1 + 4z}\, dS$$

10. Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface $\mathcal{S}$. In other words, find the flux of $\mathbf{F}$ across $\mathcal{S}$. If the surface is closed, use the positive (outward) orientation.

    (a) $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$. $\mathcal{S}$ is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0, 1] \times [0, 1]$ and has upward orientation.

    (b) $\mathbf{F}(x, y, z) = y\mathbf{j} - z\mathbf{k}$. $\mathcal{S}$ consists of the paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and the disk $x^2 + z^2 \leq 1$, $y = 1$.

11. Use Green's Theorem to evaluate the line integral $\int_C (2y\,dx - x\,dy)$, where C is the half-circle from $(2, 0)$ to $(-2, 0)$ followed by the line segment from $(-2, 0)$ to $(2, 0)$.

12. Use Green's Theorem to find the work done by the force field $\mathbf{F}(x, y) = (3y - 4x)\mathbf{i} + (4x - y)\mathbf{j}$ when an object moves once counterclockwise round the ellipse $4x^2 + y^2 = 4$.

13. Suppose that $\mathbf{F}(x, y, z) = z\mathbf{i} + 2yz\mathbf{j} + (x + y^2)\mathbf{k}$. Let $\mathcal{S}$ be the hemisphere $z = \sqrt{4 - x^2 - y^2}$ with boundary $\mathcal{C}$ the circle $x^2 + y^2 = 4$. Orient $\mathcal{C}$ counterclockwise and orient $\mathcal{S}$ in the positive z direction.

    (a) Calculate the work integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$. (*Hint:* There's an easy way.)

    (b) *Stoke's Theorem* asserts that the work integral in (a) equals $\displaystyle\iint_{\mathcal{S}} (\mathbf{curl}\,\mathbf{F} \cdot \mathbf{e_n})\, dS$.

    Calculate this integral and confirm your answer to (a).

14. Let $\mathbf{F} = \frac{-3}{2}y^2\mathbf{i} - 2xy\mathbf{j} + yz\mathbf{k}$, where $\mathcal{S}$ is that part of the plane $x + y + z = 1$ contained within the triangle $\mathcal{C}$ with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, traversed counterclockwise as viewed from above. Verify Stokes's Theorem by computing both the integrals.

15. Use Stokes's Theorem to compute the line integral $\int_{\mathcal{C}}(x\,dx + z\,dy + x\,dz)$, where $\mathcal{C}$ is the intersection of the plane $x + y = 2$ and the surface $x^2 + y^2 + z^2 = 2(x + y)$, traversed clockwise as viewed from above.

16. Use Stokes's Theorem to evaluate

$$\iint_{\mathcal{S}} (\mathbf{curl}\,\mathbf{F} \cdot \mathbf{e_n})\, dS$$

    for $\mathbf{F}(x, y, z) = xy\mathbf{i} - z\mathbf{j}$ and $\mathcal{S}$ the surface of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, except for the face where $z = 0$. Use the outward unit normal.

17. Use the Divergence Theorem (Gauss's Theorem) to calculate the surface integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

    (that is, calculate the flux of $\mathbf{F}$ across $\mathcal{S}$):

    (a) where $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$ and $\mathcal{S}$ is the boundary of the region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$, oriented outward.

    (b) where $\mathbf{F}(x, y, z) = z^2x\mathbf{i} + (y^3/3 + \tan z)\mathbf{j} + (x^2z + y^2)\mathbf{k}$ and $\mathcal{S}$ is the top half of the sphere $\rho = 1$, oriented upward. (Caution: this is not closed! First compute integrals over $\mathcal{S}_1$ and $\mathcal{S}_2$, where $\mathcal{S}_1$ is the disk $x^2 + y^2 \le 1$ oriented downward, and $\mathcal{S}_2 = \mathcal{S} \cup \mathcal{S}_1$.)