You will write up *one* of these problems and turn it in at next week's workshop. You will be graded on your exposition as well as on the mathematical content of your report. During the workshop period, you are encouraged to work with other students in your group. However, the writeup you turn in should represent your own presentation of the solution. It should not be a (rewritten) verion of someone else's work. You should:

- Work out the problem and have a clear plan of presentation before you begin to write up your final solution.
- Be neat and write legibly. Don't try to squeeze your report onto this piece of paper.
- Show all steps. Explain what you are doing at each step in complete sentences.

Write as if you intended your explanation to be part of a good textbook, or of a professional lab report in science or engineering.

1. Consider the function

$$f(x,y) = \begin{cases} (6x^5 - 6xy^4)/(x^2+y^2) & \text{if} \quad (x,y) \neq (0,0) \\ A & \text{if} \quad (x,y) = (0,0) \end{cases}$$

   (a) Give an argument to show that $\lim_{(x,y)\to(-1,2)} f(x,y)$ exists, and find that limit.

   (b) Give an argument to show that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, and find that limit. (You'll need a different argument from the one in part (a). Hint: try polar coordinates.)

   (c) Find the value of A for which $f(x,y)$ is continuous everywhere.

2. A function $u(x,y)$ is called *harmonic* if it satisfies Laplace's equation $u_{xx}+u_{yy} = 0$. (For example, the equation for a soap film stretched over a curved frame in 3 dimensions is given by $z = u(x,y)$ with a harmonic function u).

   (a) For each of the following functions $u(x,y)$, find all values of the constant k (if there are any) so that $u(x,y)$ is harmonic:

$$x^2 + ky^2, \qquad x^3 + kxy^2, \qquad e^{5x}\cos(ky).$$

   (b) Let $u(x,y)$ be a harmonic function. Suppose $u_{xx}(0,0) \neq 0$ and the curve $z = u(x,0)$ in the xz–plane is *concave up* at $x = 0$. Show that the curve $z = u(0,y)$ in the yz–plane must be *concave down* at $y = 0$. (*Hint:* Use the second derivative test for concavity.)

   (c) Determine which of the functions in part (a) satisfy the assumptions of part (b). Sketch the graphs of the curves $z = u(x,0)$ and $z = u(0,y)$ in these cases to confirm the concavity properties that you proved in part (b).

3. A certain function $f(x, y)$ is known to have partial derivatives of the form

$$\frac{\partial f}{\partial x} = 2y\cos(2x) + y^3x^2 + g(y), \qquad \frac{\partial f}{\partial y} = \sin(2x) + x^3y^2 + 4x + 1. \qquad (*)$$

(a) Calculate the second partial derivatives $(f_x)_y$ and $(f_y)_x$ (note that g is a function of y only).

(b) Check that the hypotheses of Clairaut's Theorem are satisfied by f. That theorem tells us that $(f_x)_y = (f_y)_x$. What equation does this give for $g'(y)$? Use this to find the function g (up to an arbitrary additive constant of integration).

(c) Now that you know g, find all functions f with partial derivatives of the form $(*)$. Remember that when you integrate f_x, the *constant* of integration will be a *function* of y, and likewise when you integrate f_y the *constant* of integration will be a *function* of x. Determine these functions by the same method that you used in (b).

4. Let S be the surface with equation $z = 6/(xy)$ and domain

$$\mathcal{D} = \{(x, y) \,:\, x > 0 \text{ and } y > 0\}.$$

(a) Sketch the 3 curves given by the intersection of S with *(i)* the plane $x = 1$; *(ii)* the plane $y = 2$; *(iii)* the plane $z = 3$. Find an equation for the tangent plane to the surface at the point $(1, 2, 3)$. Find the $x-$, $y-$, and $z-$ intercepts of this plane, calculate product of the three intercepts, and sketch the plane.

(b) Let $f(x, y, z) = xyz$. Find the gradient vector $\nabla f(x, y, z)$ and use it to obtain a normal vector $\mathbf{n}$ to the surface S at the point $(1, 2, 3)$. Now use $\mathbf{n}$ to write down equations for the normal line and the tangent plane to S at $(1, 2, 3)$. Check that the equation for the tangent plane is a multiple of the one you obtained in part (a).

(c) Find an equation for the tangent plane to the surface S at the point (a, b, c), where a, b, c are any positive numbers such that $abc = 6$. Show that the product of the $x-$, $y-$, and $z-$ intercepts of the tangent plane is the same number you calculated in (a) (it does not depend on the choice of the point (a, b, c) on the surface).