Math 251 Multivariable Calculus Fall 2009 Workshop 4 – Exam 1 Review

This list is meant as a guide to reviewing. Please note that any kind of problem included in the syllabus, the list of homework problems, or the workshops can be included in the midterm. Also, this review sheet is longer than the exam itself.

1. Consider the points $P = (0, 7, 1)$, $Q = (2, 1, 0)$, and $R = (2, 3, 0)$.
   (a) Give an equation for the plane $\mathcal{P}_1$ which contains the points P, Q, and R.
   (b) Use a suitable vector product to find the area of the triangle PQR.

2. (a) Show that the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar (in other words, that they all lie on the same plane) if and only if
   $$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0 \quad \text{or} \quad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0.$$
   (b) Verify that the vectors $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, and $\mathbf{w} = 7\mathbf{i} - 2\mathbf{k}$ are coplanar.

3. Plane $\mathcal{P}_1$ has equation $-x + 3y + 5z = 0$.
   (a) Find a normal vector for $\mathcal{P}_1$.
   (b) Plane $\mathcal{P}_2$ contains the point $S = (6, -4, 2)$ and is parallel to plane $\mathcal{P}_1$ of part (a). Give the equation for $\mathcal{P}_2$ in standard form.
   (c) Find the distance between the planes $\mathcal{P}_1$ and $\mathcal{P}_2$.

4. Consider the helix $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4t\rangle$ Find the length of the arc of $\mathbf{r}(t)$ from the point $(3, 0, 0)$ to the point $(0, 3, 2\pi)$.

5. (a) For the helix in problem 4, find the vectors $\mathbf{T}$ and $\mathbf{N}$.
   (b) A particle is moving along the helix so that its position at time t is given by $\mathbf{r}(t)$. Find the tangential and normal components of acceleration when $t = 3\pi$.

6. A particle moves with acceleration vector $\mathbf{a}(t) = 8t\mathbf{i}+2\mathbf{j}-\mathbf{k}$. At time $t = 0$ the velocity is $\mathbf{v}(0) = 5\mathbf{i}$ and the position vector is $\mathbf{r}(0) = 7\mathbf{j} + 11\mathbf{k}$. Find the velocity vector $\mathbf{v}(t)$ and the position vector $\mathbf{r}(t)$ of the particle as functions of t.

7. (a) Let $w = f(x, y)$, where f is a function satisfying $f(4, 3) = 0$, $f_x(4, 3) = 5$, and $f_y(4, 3) = 7$. Suppose that x and y depend on variables u, v by the formulas $x = 2uv$ and $y = u^2 - v^2$. Find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ when $u = 2$ and $v = 1$.
   (b) Recall that a function f is said to be *harmonic* if $f_{xx} + f_{yy} = 0$. Find the value or values of m for which the function $g(x, y) = e^{mx}\cos(12y)$ is harmonic.

8. (a) What is the curvature of a circle of radius a? What is the curvature of a straight line? (You should be able to answer without doing any computations.)
   (b) Let $\mathbf{r}(t)$ be the helix of problem 4. Find the curvature of $\mathbf{r}(t)$.

9. Find each of the following limits, if it exists. Otherwise, explain how one can conclude that the limit does not exist.

   (a) $\lim_{(x,y)\to(0,0)} f(x,y)$ where f is defined by

   $$f(x,y) = \begin{cases} (8x^2 - 2y^2)/(5x^2 + y^2) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

   (b) $\lim_{(x,y)\to(0,0)} \dfrac{2x^3 - y^3}{x^2 + y^2}$. (Hint: try polar coordinates.)

10. This problem is about the same function of problem 6 (a), that is,

   $$f(x,y) = \begin{cases} (8x^2 - 2y^2)/(5x^2 + y^2) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

   (a) Give the set of all points (x,y) where f is a continuous function. Explain why.

   (b) In general, the linear approximation $L(a,b)$ is a good approximation to a function $f(x,y)$ if we choose a "good" point (a,b). For the function f that we are dealing with now, what points (a,b) are "good" in this sense? Explain why.

11. If we know that $\dfrac{\partial^2 f}{\partial y \partial x} = \cos(xy) - xy\sin(xy)$, can we conclude anything about $\dfrac{\partial^2 f}{\partial x \partial y}$? If you try to use a theorem be careful to check the hypotheses of the theorem. If you think we cannot conclude anything then give a counterexample to the natural conjecture.

12. Show that the ellipsoid $3x^2+2y^2+z^2 = 9$ and the sphere $x^2+y^2+z^2-8x-6y-8z+24 = 0$ are tangent to each other at the point $P_0 = (1,1,2)$. (This means that they both have the same tangent plane at that point.)

13. Suppose $f(x,y) = x^3y^2$. Use differentials and the values of $f(1,1)$, $f_x(1,1)$, $f_y(1,1)$ to find an approximate value for $f(1.04, 0.98)$. Compare with your calculator value for $f(1.04, 0.98)$

14. Let $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$.

   (a) Calculate the gradient vectors $\nabla u(x,y)$ and $\nabla v(x,y)$.

   (b) Show that $\nabla u(x,y) \cdot \nabla v(x,y) = 0$.

   (c) The level curves $u(x,y) = -2$ and $v(x,y) = 2$ in the xy plane intersect at the point $(1,1)$. Use the result in part (b) to determine the angle between the tangent vectors to these curves at $(1,1)$.

15. For the given function f, find the critical points and classify each as a relative maximum, a relative minimum, or a saddle point.

   $$f(x,y) = x^3 + y^3 + 3x^2 - 18y^2 + 81y + 5$$

16. Find all the critical points of $f(x,y) = x^2y^2$ and show that the Second Derivatives Test fails to classify any of them. Use some other method to determine what each critical point represents.