Math 251 Multivariable Calculus Fall 2009 Workshop 7

You will write up *one* of these problems and turn it in at next week's workshop. You will be graded on your exposition as well as on the mathematical content of your report. During the workshop period, you are encouraged to work with other students in your group. However, the writeup you turn in should represent your own presentation of the solution. It should not be a (rewritten) version of someone else's work. You should:

- Work out the problem and have a clear plan of presentation before you begin to write up your final solution.
- Be neat and write legibly. Don't try to squeeze your report onto this piece of paper.
- Show all steps. Explain what you are doing at each step in complete sentences.

Write as if you intended your explanation to be part of a good textbook, or of a professional lab report in science or engineering.

1. (a) Let D be the set of points in the plane which satisfy the inequalities

$$r_1 \leq r \leq r_2 \quad \text{and} \quad \theta_1 \leq \theta \leq \theta_2$$

in polar coordinates. Find $A(r_1, r_2, \theta_1, \theta_2)$, the area of the region D, as a funtion of r_1 ,r_2, θ_1 and θ_2.

   (b) Treating r_1 and θ_1 as constants, show that

$$\lim_{(r_2,\theta_2)\to(r_1,\theta_1)} \frac{A(r_1, r_2, \theta_1, \theta_2)}{r_1(r_2 - r_1)(\theta_2 - \theta_1)} = 1.$$

   (c) Let $\Delta r = r_2 - r_1$ and $\Delta\theta = \theta_2 - \theta_1$. Use part (b) to conclude that the area of the region D is approximately equal to $r_1 \Delta r \Delta\theta$, when Δr and $\Delta\theta$ are small. (This justifies the expression "$dA = r\,dr\,d\theta$" that we will see later.)

2. Let W be the set of points in the 3–space which satisfy the inequalities

$$\rho_1 \leq \rho \leq \rho_2, \quad \theta_1 \leq \theta \leq \theta_2 \quad \text{and} \quad \phi_1 \leq \phi \leq \phi_2$$

in spherical coordinates.

   (a) Graph W in the case $\rho_1 = 0$, $\rho_2 = 2$, $\theta_1 = 0$, $\theta_2 = 2\pi$, $\phi_1 = 0$ and $\phi_2 = \pi/6$ and describe this set in words using a food analogy.

   (b) Graph W in the case $\rho_1 = 1$, $\rho_2 = 2$, $\theta_1 = -\pi/6$, $\theta_2 = \pi/6$, $\phi_1 = 0$ and $\phi_2 = \pi$ and describe this set in words using a food analogy.

   (c) Again define $\Delta\rho = \rho_2 - \rho_1$, $\Delta\theta = \theta_2 - \theta_1$ and $\Delta\phi = \phi_2 - \phi_1$. Describe the set W when $\Delta\rho$, $\Delta\theta$ and $\Delta\phi$ are small.

(d) Explain why the volume of W is approximately equal to $\rho_1^2 \sin(\phi_1)\Delta\rho\Delta\phi\Delta\theta$ when $\Delta\rho$, $\Delta\theta$ and $\Delta\phi$ are small. (This justifies the expression "$dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$" that we will see later.)

3. Let E be the tetrahedron in the first octant bounded by the plane $x + 2y + 4z = 4$ and the coordinate planes.

   (a) Find the intercepts of the bounding plane and sketch the solid E. Also sketch the region R in the first quadrant of the xy-plane that E projects onto. Describe E by three scanning inequalities on x, y, and z and describe R by two scanning inequalities on x and y.

   (b) Let $f(x, y, z)$ be any continuous function defined on E. Use the information from (a) to express

$$\iiint_E f(x, y, z)\, dV$$

   as an iterated integral (use the order $dz\, dy\, dx$).

   (c) Calculate the integral in (b) for the function $f(x, y, z) = 1$. What is the geometric meaning of the answer? Can you obtain it by a geometric argument (no calculus)?

   (d) Calculate the integral in (b) for the function $f(x, y, z) = x^2 + z^2$.

   (e) What is the average value of $f(x, y, z)$ on E?

4. (a) Consider the triple integral

$$\iiint_E f(x, y, z)\, dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} f(x, y, z)\, dz\, dx\, dy.$$

   Give a set of inequalities on x, y, and z that define E, the region of integration. Describe the *boundary surfaces* of E. Sketch the solid E and the region R in the first quadrant of the xy-plane that E projects onto. Then describe E by a set of inequalities in *cylindrical coordinates* (r, θ, z), and describe R by a set of inequalities in *polar coordinates* (r, θ).

   (b) Consider the triple integral

$$\iiint_E f(x, y, z)\, dV = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} f(x, y, z)\, dz\, dx\, dy.$$

   Give a set of inequalities on x, y, and z that define E, the region of integration. Describe the *boundary surfaces* of E. Sketch the solid E and the region R in the first quadrant of the xy-plane that E projects onto. Then describe E by a set of inequalites in *spherical coordinates* (ρ, ϕ, θ).