Math 350 Linear Algebra Spring 2008 Final Exam Study Guide

The exam material:

The exam will be approximately distributed with 2/3 covering new material and 1/3 covering old material, with the following sections being covered from each:

New material: 5.2, 6.1-6.6, 6.8, 6.11, 7.1-7.3.

Old material: 1.1-1.6, 2.1-2.6, 3.1-3.4, 4.1-4.4, 5.1.

There will again be less emphasis on the material that was not covered in the homework.

Many of the questions on the exam will be taken directly from the list of practice problems below. In addition you will be asked to prove at least one of the theorems listed below.

Studying for the exam:

Do all the problems below! It might seem like a lot of problems but you have already done many of them on the homework. If you get stuck then work with some of your classmates or come to the review session or office hours. I will happily go over any of these problems or any other questions people have.

Also, redo all the old exams and quizzes. I will post the blank PDF files online so that you can print them and do them. If you get stuck, check the solutions on the webpage.

Then, read the statements of each of the theorems below and try to prove them on your own. If you get stuck then look at the book for a hint then go back to doing it on your own. *Do not simply memorize the proofs!* You should fully understand the proof and reproduce it in your own words.

Theorems to prove:

1.5, 1.9, 2.1, 3.9, 5.1, 6.23.

Practice Problems from the Book:

1.2#9, 1.5#15, 1.6#26, 1.6#28, 2.1#12, 2.1#13, 2.1#20, 2.2#5, 2.3#12, 2.4#9, 2.5#7, 3.2#22, 3.3#9, 4.2#3, 4.3#9, 5.2#3d, 5.2#9, 5.2#11, 6.1#2, 6.1#8, 6.2#9, 6.2#11, 6.4#11, 6.5#24, 6.6#4, 6.8#25, 6.11#6, 7.1#7a-d, 7.2#4a, 7.2#6, 7.3#2d, 7.3#4 (only for 2a,c,d and 3a,c,d and you can use the minimal polynomial from the back of the book).

You should also consider all the problems from the old exams and quizzes as practice problems as well.

Other Practice Problems:

1. Suppose A is an $n \times n$ real matrix with reduced row echelon form R. Label each of the following as true or false. If true, give an explanation of why. If false, give a counterexample.
   (a) $\text{rank}(A) = \text{rank}(R)$.
   (b) $\text{nullity}(A) = \text{nullity}(R)$.
   (c) $\text{R}(A) = \text{R}(R)$.
   (d) $\text{N}(A) = \text{N}(R)$.
   (e) The equations $Ax = 0$ and $Rx = 0$ have the same solutions.

(f) The equations $Ax = b$ and $Rx = b$ have the same solutions.

(g) $\det(A) = \det(R)$.

(h) A and R have the same eigenvalues.

(i) A and R have the same eigenvectors.

(j) A and R are similar.

2. Suppose A and B are similar $n \times n$ real matrices (i. e. there exists some invertible matrix Q such that $A = QBQ^{-1}$). Label each of the following as true or false. If true, give an explanation of why. If false, give a counterexample.

   (a) A and B have the same eigenvalues.

   (b) A and B have the same eigenvectors.

   (c) A and B have the same determinant.

   (d) A and B have the same characteristic polynomial.

   (e) A and B have the same trace.

   (f) A and B have the same rank.

   (g) A and B have the same range.

   (h) A and B have the same nullspace.

   (i) For any n-dimensional real vector space V with basis α, there exists a linear transformation $T : V \to V$ and another basis β of V such that $[T]_\alpha = A$ and $[T]_\beta = B$.

3. Which of the following subsets are subspaces of their respective vector spaces? Give a proof to justify your answers.

   (a) $\{(x, y, z) \in \mathbb{R}^3 \ : \ x + 3y - z = 1\}$

   (b) $\{A \in \mathrm{M}_{n\times n} \ : \ \det(A) = 1\}$

   (c) $\{A \in \mathrm{M}_{n\times n} \ : \ \det(A) = 0\}$

   (d) $\{A \in \mathrm{M}_{n\times n} \ : \ \mathrm{tr}(A) = 1\}$

   (e) $\{A \in \mathrm{M}_{n\times n} \ : \ \mathrm{tr}(A) = 0\}$

   (f) $\{A \in \mathrm{M}_{n\times n} \ : \ 0 \text{ is an eigenvalue of } A\}$

   (g) $\{A \in \mathrm{M}_{n\times n} \ : \ x_0 \text{ is an eigenvector of } A \text{ for eigenvalue } 0\}$ where x_0 is some fixed vector.

4. Suppose V is a real vector space of dimension n and β is a basis for V. Prove that there is an inner product on V such that β is an orthonormal basis. Hence every basis is orthonormal in *some* inner product.

5. Consider $\mathbb{R}^n$ with its standard inner product, $\langle x, y\rangle = x \cdot y$ (the dot product). If A is a real $n \times n$ matrix then directly prove that all of the following are equivalent:

(a) $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$.

(b) $AA^t = I$.

(c) The columns of A form an orthonormal set.

You have to prove this directly, without using any theorems.