MATH 423: PRACTICE PROBLEMS, FINAL

(Note: The actual exam is shorter)

1. (a) Find the solution $u(x, y)$ of the equation $\frac{u_x}{x} + \frac{u_y}{y} - u = 0$ with initial condition $u(0, y) = y^2$.

2. Find the general solution of the equation $xu_x + yu_y + u = 1$.

3. (a) Write a solution of the equation $u_t = u_{xx}$ on $[0, \pi]$ with boundary conditions $u_x(0, t) = 0$, $u_x(\pi, t) = 0$ and initial condition $u(x, 0) = \sin^2(x)$.
(b) Show that the solution found is unique.

4. On the interval $[0, 1]$, consider the eigenvalue problem

$$-X'' = \lambda X$$

$$X(0) = 0 \quad \text{and} \quad X'(1) = 0$$

(a) Is there a zero eigenvalue?
(b) Write down the equation for the positive eigenvalues $\lambda = \beta^2$.
(c) How many positive eigenvalues are there?
(d) How many negative eigenvalues are there?

5. (a) Write the general solution obtained by the method of separation of variables to the equation $tu_t = -u_{xx}$ with boundary conditions $u_x(0, t) = 0$, $u_x(\pi, t) = 0$.
(b) Describe the initial condition(s) $u(x, 0)$ that can be satisfied by the solutions in (a).
(c) For initial condition(s) as in (b), is the solution unique?

6. Calculate the Fourier sine and cosine series on the interval $[0, \pi]$ for the functions
(1) $f(x) = \sin^2(x)$ (2) $f(x) = x$
In all cases, answer the following questions:
(a) At which points in $[0, \pi]$ does the Fourier series converge? (b) Are there points in $[0, \pi]$ where the Fourier series converges, but *not* to $f(x)$?

Useful formulas:
(1) Laplacian in polar coordinates: $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.
(2) Fourier coeffs. on $[0, l]$: $A_n = \frac{2}{l}\int_0^l \phi(x)\cos(m\pi x/l)dx$; $B_n = \frac{2}{l}\int_0^l \phi(x)\sin(m\pi x/l)dx$

7. Find all the eigenvalues and eigenfunctions of the differential operator $i\dfrac{d}{dx}$ on the interval $[0, \pi]$, with boundary conditions $X(0) = X(\pi)$. Are the eigenfunctions orthogonal?

8. Find the constants a, b, c such that $a\cos(x) + b\cos(2x) + c\cos(3x)$ best approximates, in the sense of L^2 of $[0, \pi]$, the function $f(x) = \cos^2(x) - \sin^2(x)$.

9. (a) Find the solutions of Laplace's equation $\Delta u = 0$ in two dimensions inside the unit disk which in polar coordinates (r, θ) are of the form $u(r, \theta) = R(r)\Theta(\theta)$ (use separation of variables).

(b) Find the solutions of Laplace's equation $\Delta u = 0$ inside the unit disk with the boundary condition (in polar coordinates) $u(1, \theta) = \cos\theta$. Write it in Cartesian coordinates as well.

10. (a) Show that if $u(x, y)$ and $v(x, y)$ are twice continuously differentiable if additionally,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then both u and v are harmonic. Such functions are called **harmonic conjugates**.

(b) Show that if f is harmonic and $u(x, y)$ and $v(x, y)$ are harmonic conjugates, then $f\Big(u(x, y), v(x, y)\Big)$ is also harmonic.

(c) Show that if f is harmonic then $f(x^2 - y^2, xy)$ is harmonic.