02–03–2004

Spring 2004 Midterm 1 and solutions

1. Find all the roots of the equation

$$z^3 = \frac{1+i}{1-i}$$

in the complex plane. (Your answer should be exact.) Draw a sketch representing these roots.

Answer: We have $\frac{1+i}{1-i} = \frac{(1+i)^2}{(1+i)(1-i)} = i = e^{i\pi/2}$. Thus the three roots are $e^{i\pi/6+2k\pi i/3}$, $k = 0, 1, 2$, that is: $e^{i\pi/6}$, $e^{i\pi/6+2\pi i/3}$ and $e^{i\pi/6+4\pi i/3}$. These evaluate to $\frac{1}{2}(\sqrt{3}+i)$, $\frac{1}{2}(-\sqrt{3}+i)$ and $-i$ respectively.

2. Assume $f(x+iy) = u(x,y) + iv(x,y)$ is an analytic function which does not vanish anywhere in $\mathbb{C}$ and let

$$H(x,y) = \frac{u(x,y)}{u(x,y)^2 + v(x,y)^2}$$

Find a harmonic conjugate of $H(x,y)$.

Answer: since f is nowhere zero, the function $1/f = 1/(u+iv)$ is analytic in $\mathbb{C}$. We have

$$\frac{1}{u+iv} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

Thus a harmonic conjugate of H is $-\dfrac{v(x,y)}{u(x,y)^2+v(x,y)^2}$. The *general* harmonic conjugate differs from it by an arbitrary real constant.

3. Consider the region $\mathcal{D} = \left\{z = x+iy : y > 0 \text{ and } x^2+y^2 > 1\right\}$ (see Figure 1) and let $f(z) = z + 1/z$.

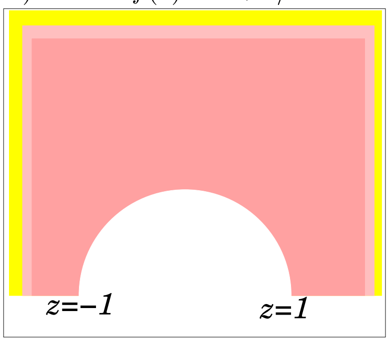

Figure 1

(a) What is the image under f of the half-circle $z = e^{it}, t \in [0, \pi)$ (lying on the boundary of $\mathcal{D}$)?

(You will probably find it useful to work with the polar representation of z.)

If $z = e^{it}$ then $z + 1/z = e^{it} + e^{-it} = 2\cos t \in \mathbb{R}$. When $t \in [0, \pi)$ we have 2 cos t $\in (-2, 2]$. Thus, the image is the segment $(-2, 2]$.

(b) What is the image under f of the half-circle $z = 2e^{it}, t \in [0, \pi)$?

As above, we have $f(2e^{it}) = u + iv = (2 + \frac{1}{2})\cos t + (2 - \frac{1}{2})i \sin t$. Thus $\dfrac{u^2}{(2+1/2)^2} + \dfrac{v^2}{(2-1/2)^2} = 1$, the upper half of an ellipse, the point $z = -5/2$ excluded.

(c) What does $\mathcal{D}$ become under the transformation f? (That is, what is the set $\{f(z) : z \in \mathcal{D}\}$?)

Taking ae^{it} with $a > 1$ instead of $2e^{it}$ in the calculation in (b), we obtain the upper half of an ellipse with semiaxis (continuous and) increasing in a. The major semiaxis is $a + 1/a$ and increases from 2 to infinity while the minor semiaxis $a - 1/a$ increases from 0 to infinity. The domain thus covered is the upper half plane.

(b) Is f one-to-one between $\mathcal{D}$ and $f(\mathcal{D})$? Explain.

If $z_1 + 1/z_1 = z_2 + 1/z_2$ then $z_1 - z_2 = (z_1 - z_2)/(z_1 z_2)$ that is, $z_1 = z_2$ or $z_1 z_2 = 1$. In $\mathcal{D}$ we have $|z| > 1$ thus $z_1 z_2 = 1$ is not possible. It follows that f is one-to-one.

4. (a) Find *all* the points in the complex plane for which the series

$$S(z) = \sum_{n=0}^{\infty} z^{2n}$$

converges.

The ratio test immediately implies that $S(z)$ is absolutely convergent if $|z| < 1$ and divergent if $|z| > 1$. If $|z| = 1$ then the general term of the series, z^{2n}, has absolute value and the series fails a necessary condition of convergence. Therefore the series converges if and only if $|z| < 1$.

(b) For those z for which $S(z)$ is convergent, express $S(z)$ in terms of simple functions (polynomials, fractions etc.). Find where is this expression analytic. Explain carefully.

S is a geometric series with progression z^2. Its sum for $|z| < 1$ is $1/(1 - z^2)$. Since $(1 - z^2)$ is analytic in $\mathbb{C}$, $1/(1 - z^2)$ analytic wherever the denominator is nonzero, that is in $\mathbb{C} \setminus \{-1, 1\}$.