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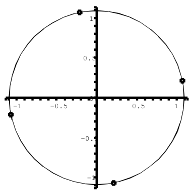
642:528: Midterm 1 and solutions

1. Find all the values of

$$(1+i)^{1/4}$$

Show their location in the complex plane by sketching a figure.

We have $(1+i)=\sqrt{2}e^{i\pi/4}$ and thus $(1+i)^{1/4}=2^{1/8}e^{i\pi/16+2k\pi i}$, k=0,1,2,3.



2 (a). For which values of z does the branch of the logarithm defined in the complex plane with a cut along $\mathbb{R}^+$ differ from the natural branch (i.e., defined with a cut along $\mathbb{R}^-$)?

Log $z=\ln|z|+i$ Arg z where, for the natural branch, Arg z is zero on the positive real line and increases to π as we rotate anticlockwise toward $\mathbb{R}^-$ which is a branch cut, and it decreases to $-\pi$ as we rotate clockwise from $\mathbb{R}^+$ toward $\mathbb{R}^-$.

The branch of the logarithm defined in the complex plane with a cut along $\mathbb{R}^+$ tends to zero as $\mathbb{R}^+$ is approached from above, increases to π as we rotate anticlockwise toward $\mathbb{R}^-$ and tends to 2π as we further rotate anticlockwise toward $\mathbb{R}^+$.

Thus both branches are defined in the open upper half plane and coincide there, and are defined but different in the lower half plane.

(b) Show that, in general, changing the position of the cut affects the value of the associated logarithm by an integer multiple of $2\pi i$.

By definition $e^{\mathrm{Log}\, z}=z$. But $e^{z_1}=e^{z_2}$ if and only if $z_1-z_2=2k\pi i$, $k\in\mathbb{Z}$ and the conclusion follows.

3. For which values of z are the following functions differentiable?

(a) $f(z)=[\Re(z)]^2$;

$f(x+iy)=x^2=u(x,y)+iv(x,y)$ thus $u=x^2$ and $v=0$. The C-R equations give $2x=0, 0=0$. Thus f is differentiable if and only if $x=0$, that is, if and only if .

(b) Any of the functions defined by the equation $f(z)^2-2f(z)=z$.

Solving the equation yields $f(z)=1\pm\sqrt{1+z}$. We make a cut $\mathcal{C}$ at $z=-1$ and the functions are defined and analytic in $\mathbb{C}\backslash\mathcal{C}$. However we make the cut, the function(s) are not differentiable at $z=-1$ as we may see using the C-R equations, or straightforward limits, or noting that complex differentiation implies real differentiation and $\sqrt{1+z}$ has no right-derivative as $z\to-1^+$.

4. Find the most general function which is a harmonic conjugate of $u(x,y)=\cos(x)\sinh(y)$.

We have $i\sin(x+iy)=i\sin(x)\cosh(y)-\cos(x)\sinh(y)$. Thus a harmonic conjugate of $u(x,y)=\cos(x)\sinh(y)$ is $v(x,y)=-\cos(x)\sinh(y)$. We have proved in class that two harmonic conjugates differ by an additive real constant. The answer is $v(x,y)=-\cos(x)\sinh(y)+C$.

5. (a) Find a linear fractional transfomation f that maps the open unit disk D_1 onto the disk $D_2=(x-1)^2+y^2<4$.

D_2 is centered at $z=1$ and has radius 2. Thus it suffices to dilate D_1 by a factor of 2 and then shift it by 1 to the right. This gives $f=2z+1$. This is not the only correct answer. Taking any 3 points on the unit circle and a linear fractional transformation which map these to any 3 points on the second disk produces a correct solution. We will study later "how many" possibilities there are later in the course.

(b) What is the image of the upper half plane under the transformation found in part (a)?

Since $f(z)=2z+1=2x+1+2iy$ and $2y>0$, the image of a point in the upper half plane is still in the upper half plane. The inverse of $f(z)$ is $\frac{1}{2}(w-1)$ which is also defined in upper half plane with values in the upper half plane. The image is thus the whole open upper half plane. However: if you find in (a) a different valid linear fractional transformation, this answer will change accordingly, and you may get a disk instead.

6. Find a conformal map of the domain $\mathcal{D}=\{x+iy: x>0,\ y>0 \text{ and } x^2+y^2>1\}$ onto the open upper half plane.

Use this map to find an equation for the curves along which the potential Φ generated by a charged wire in the shape of the boundary of $\mathcal{D}$ is constant.

(Note: The potential Φ satisfies the equation $\Delta\Phi=0$ in $\mathcal{D}$.

The map $z\to z^2$ opens the first quadrant onto the upper half plane and thus transforms region $\mathcal{D}$ onto the one in the textbook, Fig. 2, page 1177 with $a=1$. The latter regionis mapped onto the upper half plane by the Joukovski transformation $\zeta=z+1/z$. The composed transformation is $f(z)=z^2+1/z^2$. In the upper half plane the constant potential curves are $\mathrm{Im}(\zeta)=$const. This gives for the constant lines of Φ the equation

$$xy-\frac{xy}{(x^2-y^2)^2+4x^2y^2}=\text{const.}$$