1. What type of a singularity is $z = 0$ for the following functions? Explain carefully. If the singularity is isolated, determine the radius of convergence of the Laurent series at $z = 0$ and calculate the coefficients c_{-1}, c_0, c_1 and c_2.

(a) $z/\cos z$

A: As a ratio of analytic functions this function is meromorphic. Thus $z = 0$ is a point of analyticity if the denominator does not vanish or a pole (which is an isolated singularity) if it does vanish. In our case, cos(0)=1, thus the function is analytic at zero. Direct calculation gives $c_{-1} = 0, c_0 = 0, c_1 = 1, c_2 = 0$.

(b) $\sin(1/z)$

A: The function sin is entire thus $\sin(1/z)$ is analytic in $\mathbb{C}\setminus\{0\}$ and 0 is an isolated singularity. Since $\sin(t) = \sum_{k=0}^{\infty} \frac{(-t)^{2k+1}}{(2k+1)!}$ we have $\sin(1/z) = \sum_{k=0}^{\infty} \frac{-(-1/z)^{2k+1}}{(2k+1)!}$, a series that converges therefore for $z \neq 0$. We see that $c_{-1} = 1, c_0 = c_1 = c_2 = 0$.

(c) $z^{i\pi}$

By definition $z^{i\pi} = \exp(i\pi \ln z)$ which is easily seen to be multivalued in any annular region around zero, so this singularity is not isolated.

(d) π^z

By definition $\pi^z = \exp(z \ln \pi)$ which is entire for any choice of $\ln \pi$. Ans: $0, 1, \ln \pi, \frac{1}{2}(\ln \pi)^2$.

(e) $ze^{i/z}$

Isolated essential singularity, similar reasoning and calculations as in (b).

(f) $\cosh z/e^{iz}$

A: $\cosh z/e^{iz} = e^{-iz}\cosh z$ is entire, straightforward calculations.

2. Find a formula for the nth coefficient in the Taylor series about $z = 0$ of the function

$$f(z) = \frac{1}{z^2 + 3z + 2}$$

and the radius of convergence of this series.

A: A partial fraction decomposition gives

$f(z) = \frac{1}{1+z} - \frac{1}{2+z} = \frac{1}{1+z} - \frac{1}{2(1+z/2)} = \sum_{n=0}^{\infty} \left[(-z)^n - \frac{1}{2}(-z/2)^n\right]$

and thus the general term is $(-1)^n - \frac{1}{2}(-1/2)^n$, radius of convergence 1.

3. Evaluate the following integrals

(a)
$$\int_0^{2\pi} \frac{1}{(2-\sin x)^2} dx$$

Write $2i \sin x = e^{ix} - e^{-ix}$ and transform the integral into a contour integral around the unit circle. Answer:

$4\pi\sqrt{3}/9$.

(b)
$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+1)^2} dx \qquad (a > 0)$$

A: Write the integral as

$$\Re \int_{-\infty}^{\infty} \frac{\exp(iax)}{(x^2+1)^2} dx \qquad (a > 0)$$

and use the formula expressing such integrals in terms of residues in UHP. Ans: $\frac{1}{2}\pi(1+a)e^{-a}$

(c)
$$\int_0^{\infty} \frac{x^{1/3} \log x}{(x+1)^2} dx$$

A: Make a cut along $-i\mathbb{R}$ and define the log and the cubic root in $\mathbb{C}\backslash(-i\mathbb{R})$. First relate

$$\int_0^{\infty} \frac{x^{1/3} \log x}{(x+1)^2} dx$$

to

$$\int_{-\infty}^{\infty} \frac{x^{1/3} \log x}{(x+1)^2} dx \qquad (\circ)$$

Then take a half-circle in the UHP of radius ε and show $\int_{-\varepsilon}^{\varepsilon}$ as well as the half circle integral have zero contribution in the limit as $\varepsilon \to 0$.

Thus the integral $(\circ)$ equals

$$\int_C \frac{x^{1/3} \log x}{(x+1)^2} dx$$

where the curve C comes from $-\infty$, avoids zero along the half circle and goes to ∞. Now evaluate the latter integral using the residues in the UHP.

Ans: $\int_0^{\infty} \frac{x^{1/3} \log x}{(x+1)^2} dx = 2\pi\sqrt{3}/9$.