Course in Asymptotics

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1. Introduction

1.1 Asymptotic expansions and asymptotic power series

Classical asymptotics typically deals with the qualitative and quantitative description of the behavior of a function (in some direction) near a point, usually a singularity of the function. This description is usually provided in the form of an *asymptotic expansion*, a formal series (that is, there are no convergence requirements) of simpler functions $\tilde{f}_k$,

$$f \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k(t) \qquad (\text{as } t \to t_0) \tag{1.1}$$

in which each successive term is much smaller than its predecessors, written

$$\tilde{f}_{k+1}(t) = o(\tilde{f}_k(t)) \quad \text{or} \quad \tilde{f}_{k+1}(t) \ll \tilde{f}_k(t)$$

denoting

$$\lim_{t \to t_0} \tilde{f}_{k+1}(t)/\tilde{f}_k(t) = 0 \tag{1.2}$$

Functions asymptotic to a series. The relation $f \sim \tilde{f}$ between an actual function and a formal expansion is defined as a sequence of limits, the Poincaré definition of asymptoticity

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = f(t) - \tilde{f}^{[N]}(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{1.3}$$

Condition (1.3) can then be also written as

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.4}$$

where $g(t) = O(h(t))$ means $\limsup_{t \to t_0} |g(t)/h(t)| < \infty$

Note. It is often important to use one-sided limits or more generally to restrict the limiting process to special directions or curves in the complex plane ending at t_0. We allow for this case, and only mention the direction or curve when it matters for the result.

Especially in this case there are some technical advantages in changing over to $t_0 = +\infty$; in this case we shall usually denote by x the variable. We ordinarily use z for variable when the limiting point is zero.

1.1.1 Asymptotic power series

A special role is played by power series, which are series of the form

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k \quad (z \to 0) \tag{1.5}$$

Remark. The prevailing convention allows for some (or even all) of the c_k's to be zero to ensure better algebraic properties. If a c_k is zero then (1.2) fails trivially in which case (1.5) is not, strictly speaking, an asymptotic series.

A function has a given asymptotic *power* series iff (1.3) by

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \tag{1.6}$$

In this sense the power series at zero of e^{-1/x^2} is the zero series. It is certainly incorrect to conclude that the asymptotic behavior of e^{-1/x^2} is zero. We use the boldface notation $\boldsymbol{\sim}$ for the stronger asymptoticity condition in (1.3).

Asymptotic power series form an algebra; addition of asymptotic power series is defined in the usual way:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} c'_k z^k = \sum_{k=0}^{\infty} (Ac_k + Bc'_k) z^k$$

while multiplication is defined as in the convergent case

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} c'_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j c'_{k-j}\right) z^k$$

Remark 1.7 *If the series $\tilde{f}$ is convergent and f is its sum (note the ambiguity of the "sum" notation) $f = \sum_{k=0}^{\infty} c_k z^k$ then $f \sim \tilde{f}$.*

The proof of this remark follows directly from the definition of convergence.

Lemma 1.8 *(Uniqueness of the asymptotic series to a function) If $f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k z^k$ as $z \to 0$ then the $\tilde{f}_k$ are unique.*

Proof. Assume that we also have $f(z) \sim \tilde{F} = \sum_{k=0}^{\infty} \tilde{F}_k z^k$. We then have (cf. (1.3))

$$\tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = o(z^N)$$

which is impossible unless $g_N(z) = \tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = 0$, as it can be easily checked, since g_N is a polynomial of degree N in z.

The proof of the following lemma is immediate:

Lemma 1.9 *(Algebraic properties of asymptoticity to a power series) If* $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ *and* $g \sim \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$ *then*
(i) $Af + Bg \sim A\tilde{f} + B\tilde{g}$
(ii) $fg \sim \tilde{f}\tilde{g}$

Sometimes it is convenient to check an (apparently) weaker condition of asymptoticity:

Lemma 1.10 *There exists a sequence* $p_n \to \infty$ *such that for any* n *there is a* $p(n)$ *such that*

$$f(z) - \tilde{f}^{[p_n]}(z) = o(z^n) \quad as \quad z \to 0$$

then $f \sim \tilde{f}$.

Proof. If $p_n \le n$ for all n there is nothing to show, otherwise, without loss of generality we may assume that $p_n \ge n$ (indeed, otherwise we extract such a subsequence). We then have

$$f(z) - \tilde{f}^{[n]} = (f(z) - \tilde{f}^{[p_n]}) + (\tilde{f}^{[p_n]} - \tilde{f}^{[n]}) = o(z^n) \ (z \to 0)$$

as it can be easily since $z^{-n-1}(\tilde{f}^{[p_n]} - \tilde{f}^{[n]})$ is a polynomial.

We now show that any asymptotic series is asymptotic to some function. A sharper version of the proposition below, the Borel-Ritt lemma, will be proved later.

Proposition 1.11 *Let* $\tilde{f} = \sum_{k=0}^{\infty} a_k z^k$ *be a power series. There exists a function* $f(z)$ *such that* $f(z) \sim \tilde{f}$ *as* $z \to 0$.

Proof. The following elementary proof has some ideas in common with optimal truncation of series, a powerful numerical technique in asymptotics.

By Remark 1.7 we can assume, without loss of generality, that the series has zero radius of convergence.

For every z, we will add "sufficiently many but not too many" terms of the series $\tilde{f}$.

For every z let $N(z)$ be the largest N such that $|a_n| \le 2^{-n}|z|^{-n/2}$ for all $n \le N$. ($N(z) < \infty$, otherwise the series would have nonzero radius of convergence.) It is also easy to see that $N(z)$ is increasing as $|z|$ decreases and that $N(z) \to \infty$ as $z \to 0$. Consider

$$f(z) = \sum_{j=0}^{N(z)} a_n z^n$$

Let N be given and choose z_0 such that $N(z_0) \geq N$. For $|z| < |z_0|$ we have

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| = \left| \sum_{n=N+1}^{N(z)} a_n z^n \right| \leq \sum_{j=N+1}^{N(z)} |z^{j/2}| 2^{-j} \leq |z|^{N/2+1/2}$$

Using now Lemma 1.10, the proof follows.

There is certainly no uniqueness in this generality. Given a power series there are many functions asymptotic to it. Indeed there are many functions asymptotic to the (identically) zero power series at zero, in any sectorial punctured neighborhood of zero in the complex plane, and even on the Riemann surface of the log on $\mathbb{C} \setminus \{0\}$, e.g. $e^{-x^{-1/n}}$ has this property in a sector of width $2n\pi$.

1.1.2 Integration and differentiation of asymptotic power series. Abelian and Tauberian theorems

While asymptotic power series can be safely integrated term by term as the next proposition will show, differentiation is more delicate. We will much later see that this asymmetry is largely in suitable spaces of functions and expansions. But for the moment note that the function $e^{-1/z^2} \sin(e^{1/z^4})$ is asymptotic to the zero power series as $z \to 0$ with z real although the derivative is unbounded and thus not asymptotic to the zero series.

Proposition 1.12 *Assume $f(x)$ is integrable near $x = 0$ and that*

$$f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k z^k$$

Then

$$\int_0^z f(s) ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{\tilde{f}_k}{k+1} z^{k+1}$$

Proof. This follows from the fact that $\int_0^z o(s^n) ds = o(z^{n+1})$ as can be seen by immediate estimates.

Sectorial asymptotic power series of analytic function can be differentiated:

Proposition 1.13 *Assume $f(x)$ is analytic in the strip $S_a = \{x : |x| > R, |\Im(x)| < a\}$. Let $\alpha < a$ and and $S_\alpha = \{x : |x| > R, |\Im(x)| < \alpha\}$ and assume that*

$$f(x) \sim \tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k} \quad (|x| \to \infty, x \in S_\alpha)$$

Then, for $\alpha' < \alpha$ we have

$$f'(x) \sim \tilde{f}'(x) := \sum_{k=0}^{\infty} -\frac{kc_k}{x^{k+1}} \quad (|x| \to \infty, x \in S_{\alpha'})$$

Proof. We have $f(x) = \tilde{f}^{[N]}(x) + g_N(x)$ where clearly g is analytic in S_a and $|g(x)| \le Const.|x|^{-N-1}$ in S_α. But then, for $x \in S_{\alpha'}$ and $\delta = \frac{1}{2}(\alpha - \alpha')$ we get for some $C > 0$ which depends on δ but not on x,

$$|g_N'(x)| = \frac{1}{2\pi}\left|\oint_{|x-s|=\delta} \frac{g(s)ds}{(s-x)^2}\right| \le \frac{C}{|x|^{N+1}} \quad (|x| \to \infty,\ x \in S_{\alpha'})$$

By Lemma 1.10, the proof follows.

In many instances the functions (scales) f_k are combinations of exponentials, powers of x, and logarithms. This is not simply a matter of choice or an accident, but reflects some important fact about the relation between asymptotic expansions and functions which will be clarified shortly.

1.1.3 Asymptotics of integrals: first results

Example: Integration by parts and elementary truncation to the least term. A solution of the differential equation

$$f' - 2xf = 1 \tag{1.14}$$

is the complementary error function

$$I(x) = e^{x^2} \int_x^{\infty} e^{-s^2} ds \tag{1.15}$$

Let us find the asymptotic behavior of $I(x)$ for $x \to \infty$. One very simple technique is integration by parts, done in a way that the successive terms decrease rapidly. We have

$$I(x) = \frac{1}{2x} - \frac{e^{x^2}}{2}\int_x^\infty \frac{e^{-s^2}}{s^2}ds = \frac{1}{2x} - \frac{1}{4x^2} + \frac{3e^{x^2}}{4}\int_x^\infty \frac{e^{-s^2}}{s^4}ds = ...$$
$$= \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(k+\frac{1}{2})}{2\sqrt{\pi}x^{2k+1}} + \frac{(-1)^m e^{x^2}\Gamma(m+\frac{1}{2})}{2\sqrt{\pi}}\int_x^\infty \frac{e^{-s^2}}{s^{2m+1}}ds \quad (1.16)$$

where is is easy to see that the series generated in this way is an *alternating* series and by looking at the remainder term we see that the actual value of $I(x)$ is always contained between two successive truncations of the power series obtained, for instance

$$\frac{1}{2x} - \frac{1}{4x^3} \le I(x) \le \frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} \quad (1.17)$$

$$\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} - \frac{5}{16x^7} \le I(x) \le \frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} - \frac{5}{16x^7} + \frac{105}{32x^9} \quad (1.18)$$

Thus the error term when truncating the series is always of the order of magnitude of the first discarded term. The series (1.16) has zero radius of convergence, and thus for large x, although the terms start by decreasing rapidly, they ultimately increase again and tend to infinity. The best approximation based on (1.16) is thus obtained by *optimal truncation*, at the ($x-$dependent) order where the discarded term is minimal. This procedure is called *truncation at the least term* and in an alternating series example like the present one is immediately justified; more analysis is required in general. The least term in our example is of order 10^{-12} when $x = 5$ and of order 10^{-45} when $x = 10$ (!) Although ultimately divergent the series provides very accurate information about the function represented.

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Often solutions of differential or difference equations are presented in the form

$$F(x) = \int_a^b e^{xg(s)}f(s)ds \quad (1.19)$$

and the behavior as $x \to \infty$ of $F(x)$ is needed. Three particular cases are more important: (1) The case where all parameters are real (dealt with by the so-called Laplace method); (2) the case where everything is real except x which is taken to be purely imaginary (stationary phase method) and (3) Everything is complex and f and g are analytic (Steepest descent method). In this latter case, the integral may also come as a contour integral along some path.

(1) The Laplace method. Even when very little regularity can be assumed about the functions, we can still infer something about the large x behavior of (1.19).

Proposition 1.20 *If* $g(s) \in L^\infty([a,b])$ *then*

$$\lim_{x\to\infty}\left(\int_a^b e^{xg(s)}ds\right)^{1/x} = e^{\|g\|_\infty}$$

Proof. This is simply the fact that $\|f\|_n \to \|f\|_\infty$.

Note that this does not give the asymptotic expansion of (1.19) for large x in the sense of (1.3). For that, more regularity needs to be assumed.

Proposition 1.21 *(g is maximum at one endpoint) Assume* f *is continuous on* $[a,b]$, $f(a)\neq 0$, g *is in* $C^1[a,b]$ *and* $g' < -\alpha < 0$ *on* $[a,b]$. *Then*

$$J_x := \int_a^b f(s)e^{xg(s)}ds = \frac{f(a)e^{xg(a)}}{x|g'(a)|}(1+o(1)) \quad (x\to+\infty) \tag{1.22}$$

Note: The derivative of g enters in the final result, so regularity is needed.

Proof. Without loss of generality, we may assume $a=0, b=1$. Let ϵ be small enough and choose δ such that if $x<\delta$ we have $|f(x)-f(0)|<\epsilon$ and $|g'(x)-g'(0)|<\epsilon$.
We write

$$\int_0^1 f(s)e^{xg(s)}ds = \int_0^\delta f(s)e^{xg(s)}ds + \int_\delta^1 f(s)e^{xg(s)}ds \tag{1.23}$$

the last integral in (1.23) is bounded by

$$\int_\delta^1 f(s)e^{xg(s)}ds \le \|f\|_\infty e^{xg(0)}e^{x(g(\delta)-g(0))} \tag{1.24}$$

For the middle integral in (1.23) we have

$$\begin{aligned}\int_0^\delta f(s)e^{xg(s)}ds &\le (f(0)+\epsilon)\int_0^\delta e^{x\left[g(0)+(g'(0)+\epsilon)s\right]}ds \\ &\le -\frac{e^{xg(0)}}{x}\frac{f(0)+\epsilon}{g'(0)+\epsilon}\left[1-e^{x\delta(g'(0)+\epsilon)}\right]\end{aligned} \tag{1.25}$$

Combining these estimates, as $x\to\infty$ we thus obtain

$$\limsup_{x\to\infty} xe^{-xg(0)}\int_0^1 f(s)e^{xg(s)}ds \le -\frac{f(0)+\epsilon}{g'(0)+\epsilon} \tag{1.26}$$

A lower bound is obtained in a similar way. Since ϵ is arbitrary, the result follows.

When the maximum is reached inside the interval of integration, a similar analysis requires more regularity.

Proposition 1.27 *(Interior maximum) Assume $f \in C[-1,1]$, $g \in C^2[-1,1]$ has a unique absolute maximum, at $x = 0$, and that $f(0) \neq 0$ and $g''(0) < 0$. Then*

$$\int_{-1}^{1} f(s)e^{xg(s)}ds = \sqrt{\frac{2\pi}{x|g''(0)|}}f(0)e^{xg(0)}(1+o(1)) \quad (x \to +\infty) \tag{1.28}$$

Proof. The proof is similar to the previous one. Let ϵ be small enough and let δ be such that $|s| < \delta$ implies $|g''(s) - g''(0)| < \epsilon$ and also $|f(s) - f(0)| < \epsilon$. We write

$$\int_{-1}^{1} e^{xg(s)}f(s)ds = \int_{-\delta}^{\delta} e^{xg(s)}f(s)ds + \int_{|s|\geq\delta} e^{xg(s)}f(s)ds \tag{1.29}$$

The last term will not contribute in the limit since by assumption for some $\alpha > 0$ and $|s| > \delta$ we have $g(s) - g(0) < -\alpha < 0$ and thus

$$e^{-xg(0)}\sqrt{x}\int_{|s|\geq\delta} e^{xg(s)}f(s)ds \leq 2\sqrt{x}\|f\|_\infty e^{-x\alpha} \to 0 \text{ as } x \to \infty \tag{1.30}$$

On the other hand,

$$\int_{-\delta}^{\delta} e^{xg(s)}f(s)ds \leq (f(0)+\epsilon)\int_{-\delta}^{\delta} e^{xg(0)+\frac{x}{2}(g''(0)+\epsilon)s^2}ds$$
$$\leq (f(0)+\epsilon)e^{xg(0)}\int_{-\infty}^{\infty} e^{xg(0)+\frac{x}{2}(g''(0)+\epsilon)s^2}ds = \sqrt{\frac{2\pi}{|g''(0)-\epsilon|}}(f(0)+\epsilon)e^{xg(0)} \tag{1.31}$$

An opposite inequality is obtained in the same way by noting that

$$\frac{\int_{-a}^{a} e^{-xs^2}ds}{\int_{-\infty}^{\infty} e^{-xs^2}ds} \to 1 \quad \text{as } x \to \infty \tag{1.32}$$

as can be seen by changing variables to $u = sx^{-\frac{1}{2}}$.

With appropriate decay conditions, the interval of integration does not have to be compact. For instance, let $J \subset \mathbb{R}$ be an interval (finite or not) and $[a, b] \subset J$.

Proposition 1.33 *(Interior maximum, noncompact interval) Assume* $f \in C[a,b] \cap L^{\infty}(J)$, $g \in C^2[a,b]$ *has a unique absolute maximum at* $x = c$ *and that* $f(c) \neq 0$ *and* $g''(c) < 0$.

Assume further that g *is measurable in* J *and* $g(c) - g(s) = \alpha + h(s)$ *where* $\alpha > 0$, $h(s) > 0$ *on* $J \setminus [a,b]$ *and* $e^{-h(s)} \in L^1(J)$. *Then,*

$$\int_A^B f(s) e^{xg(s)ds} = \sqrt{\frac{2\pi}{x|g''(c)|}} f(c) e^{xg(c)} (1 + o(1)) \quad (x \to +\infty) \tag{1.34}$$

Proof. This case reduces to the compact interval case by noting that

$$\left| \sqrt{x} e^{-xg(c)} \int_{J \setminus [a,b]} e^{xg(s)} f(s) ds \right| \leq \sqrt{x} \|f\|_{\infty} e^{-x\alpha} \int_J e^{-xh(s)} ds$$
$$\leq Const. \sqrt{x} e^{-x\alpha} \to 0 \text{ as } x \to \infty \tag{1.35}$$

Example. We see that the last proposition applies to the Γ function by writing

$$n! = \int_0^{\infty} e^{-t} t^n dt = n^{n+1} \int_0^{\infty} e^{n(-t + \ln t)} dt \tag{1.36}$$

whence we get Stirling's formula

$$n! = \sqrt{2\pi n} \left(\frac{n}{e} \right)^n (1 + o(1))$$

1.1.4 Watson's Lemma

We note that in many instances integral representations of functions are amenable to Laplace transforms

$$\mathcal{L}F := \int_0^{\infty} e^{-xp} F(p) dp \tag{1.37}$$

The behavior of $\mathcal{L}F$ for large x relates to the behavior for small p of F.

It will be shown in the sequel that solutions of generic analytic differential equations, under mild assumptions can be conveniently expressed in terms of Laplace transforms.

For the error function note that

$$\int_x^{\infty} e^{-s^2} ds = x \int_1^{\infty} e^{-x^2 u^2} du = \frac{x}{2} \int_0^{\infty} \frac{e^{-x^2 p}}{\sqrt{p+1}} dp$$

For the Γ function, writing $\int_0^{\infty} = \int_0^1 + \int_1^{\infty}$ in (1.36) we can make the substitution $t - \ln t = p$ and obtain (see §1.1.5)

$$n! = \int_0^\infty e^{-np} W(p) dp$$

Furthermore, note that the integral in Proposition 1.21 can be brought to the form (1.37) by extending f by zero to the whole line and changing variable to $g(t+a) = g(a) + u$. Similarly $u = g(\text{sign}(s)\sqrt{s}) - g(0)$ in Proposition 1.27 brings it to a problem of the form (1.37).

Lemma 1.38 *Let $F \in L^1(\mathbb{R}^+)$ $x = \rho e^{i\phi}, \phi \in (-\pi/2, \pi/2)$ and assume*

$$F(p) \sim p^\beta$$

with $\Re(\beta) > -1$ as $p \to 0^+$. Then

$$\int_0^\infty F(p) e^{-px} dp \sim \Gamma(\beta+1) x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. If $U(p) = p^{-\beta} F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. We choose $C, a > 0$ such that $|F(p)| < C|p^\beta|$ on $[0, a]$. Since

$$\left| \int_a^\infty F(p) e^{-px} \mathrm{d}p \right| \le e^{-xa} \|F\|_1 \tag{1.39}$$

we have, and after the change of variable $s = p/|x|$,

$$x^{\beta+1} \int_0^\infty F(p) e^{-px} \mathrm{d}p = e^{i\phi(\beta+1)} \int_0^\infty s^\beta U(s/|x|) \chi_{[0,a]}(s/|x|) e^{-se^{i\phi}} \mathrm{d}s \\ + O(|x|^{\beta+1} e^{-xa}) \to \Gamma(\beta+1) \quad (|x| \to \infty) \tag{1.40}$$

Watson's Lemma, presented below, states that the asymptotic series at infinity of $(\mathcal{L}F)(x)$ is obtained by formal term-by-term integration of the asymptotic series of $F(p)$ for small p, provided F has such a series.

Lemma 1.41 *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1 + \beta_2 - 1}$ as $p \to 0^+$ for some constants β_i with $\Re(\beta_i) > 0$, $i = 1, 2$. Then*

$$\mathcal{L}F \sim \sum_{k=0}^\infty c_k \Gamma(k\beta_1 + \beta_2) x^{-k\beta_1 - \beta_2}$$

along any ray ρ in the open right half plane H.

Proof. Induction, using Lemma 1.38. □

1.1.5 Example: Gamma function

We start from the representation

$$n! = \int_0^\infty t^n e^{-t} dt = n^{n+1} \int_0^\infty e^{-n(s-\ln s)} ds$$
$$= n^{n+1} \int_0^1 e^{-n(s-\ln s)} ds + n^{n+1} \int_1^\infty e^{-n(s-\ln s)} ds \quad (1.42)$$

On $(0, 1)$ and $(1, \infty)$ separately, the function $s-\ln(s)$ is monotonic and we may write, after inverting $s - \ln(s) = t$ on the two intervals to get $s_{1,2} = s_{1,2}(t)$,

$$n! = n^{n+1} \int_1^\infty e^{-nt}(s_2'(t) - s_1'(t))dt = n^{n+1}e^{-n} \int_0^\infty e^{-np} G'(p)dp \quad (1.43)$$

where $G(p) = s_2(1 + p) - s_1(1 + p))$. In order to determine the asymptotic behavior of $n!$ we need to determine the small p behavior of the function $G'(p)$

Remark 1.44 *The function $G(p)$ is an analytic function in $\sqrt{p}$ and thus $G'(p)$ has a convergent Puiseux series*

$$\sum_{k=-1}^{\infty} c_k p^{k/2} = \sqrt{2}p^{-1/2} + \frac{\sqrt{2}}{6}p^{1/2} + \frac{\sqrt{2}}{216}p^{3/2} - \frac{139\sqrt{2}}{97200}p^{5/2} + ...$$

Thus, by Watson's Lemma, for large n we have

$$n! \sim \sqrt{2\pi n}n^n e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + ...\right) \quad (1.45)$$

Proof. We write $s = 1 + S$ and $t = 1 + p$ and the equation $s - \ln(s) = t$ becomes $S - \ln(1 + S) = p$. Note that $S - \ln(1 + S) = S^2U(S)/2$ where $U(0) = 1$ and $U(S)$ is analytic for small S; with the natural branch of the square root, $\sqrt{U(S)} = H(S)$ is also analytic. We rewrite $S - \ln(1+S) = p$ as $SH(S) = \pm\sqrt{2}\sigma$ where $\sigma^2 = p$. Since $(SH(S))'_{|S=0} = 1$ the implicit function theorem ensures the existence of two functions $S_{1,2}(\sigma)$ (corresponding to the two choices of sign) which are analytic in σ. The concrete expansion may be gotten by implicit differentiation in $SH(S) = \pm\sqrt{2}\sigma$, for instance.

1.1.6 Borel-Ritt Lemma: end of proof in a half plane

Proposition 1.46 *Given a formal power series $\tilde{f} = \sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}}$ there exists an entire function $f(x)$, of exponential order one, which is asymptotic to $\tilde{f}$ in the right half plane, i.e., if $\phi \in (-\pi/2, \pi/2)$ then*

$$f(x) \sim \tilde{f} \ \ as\ x = \rho e^{i\phi}, \quad \rho \to +\infty$$

Proof. Let $\tilde{F} = \sum_{k=0}^{\infty} \frac{c_k}{k!} p^k$, let $F(p)$ be a function asymptotic to $\tilde{F}$ as in Proposition 1.11. Then clearly the function

$$f(x) = \int_0^1 e^{-xp} F(p) dp$$

is entire, bounded by $Const. e^{|x|}$, and, by Watson's Lemma has the desired properties.

Exercises.
(1) How can this method be modified to give a function analytic in a sector of opening $2\pi n$ for an arbitrary fixed n which is asymptotic to $\tilde{f}$?

(2) Assume F is bounded on $[0, 1]$ and has an asymptotic expansion $F(t) \sim \sum_{k=0}^{\infty} c_k t^k$ as $t \to 0^+$. Let $f(x) = \int_0^1 e^{-xp} F(p) dp$

(a) Find necessary and sufficient conditions on F such that $\tilde{f}$, the asymptotic power series of f for large x, is a convergent series for $|x| > R > 0$.

(c) Show that in case (a) there is a convergent representation of f in the form $\tilde{f} + e^{-x}\tilde{f}_1$ where $\tilde{f}_1$ is also a convergent series for $|x| > R > 0$.

(b) Assume that $\tilde{f}$ *converges to* $f(x)$. Show that f is zero.

(3) The width of the sector in Proposition 1.46 cannot be extended to a more than a half plane: Show that if f is entire and bounded in a sector of opening exceeding π, and of exponential order one then it is constant. (This follows immediately from the Phragmen-Lindelöf principle; an alternative proof can be derived from elementary properties of Fourier transforms and contour deformation.) The exponential order has to play a role in the proof: check that the function $\int_0^\infty e^{-px-p^2} dp$ is bounded for $\arg(x) \in (-\frac{3\pi}{4}, \frac{3\pi}{4})$. How wide can such a sector be made?

1.1.7 Oscillatory integrals and the stationary phase method

In this setting, an integral of a function against a rapidly oscillating exponential becomes small as the frequency of oscillation increases. Again we first look at the case where there is minimal regularity; the following is a version of the Riemann–Lebesgue lemma.

Proposition 1.47 *Assume* $f \in L^1[0, 1]$. *Then* $\int_0^{2\pi} e^{ixt} f(t) dt \to 0$ *as* $x \to \infty$.

It is enough to show the result on a set which is dense in L^1. Since trigonometric polynomials are dense in $C[0, 2\pi]$ in the sup norm, and thus in $L^1[0, 2\pi]$, it suffices to look at trigonometric polynomials, thus at e^{ikx} for fixed k, where the integral can be expressed explicitly and gives

$$\int_0^{2\pi} e^{ixs}e^{iks}ds = O(x^{-1}) \text{ for large } x. \qquad \square$$

No rate of decay follows without further knowledge about the regularity of f. We have the following characterization:

Proposition 1.48 *For $\eta \in (0,1]$ let the $H^\eta[0,1]$ be the Hölder continuous functions of order η on $[0,1]$, i.e., the functions with the property that there is some C such that for all $x, x' \in [0,1]$ we have $|f(x) - f(x')| \le C|x - x'|^\eta$.*

(i) We have $f \in H^\eta[0,1] \Rightarrow \int_0^1 f(s)e^{ixs}ds = O(x^{-\eta})$ as $x \to \infty$.

(ii) If $f \in L^1(\mathbb{R})$ and $|x|^\eta f(x) \in L^1(\mathbb{R})$ with $\eta \in (0,1]$, then its Fourier transform $\hat{f} = \int_{-\infty}^{\infty} f(x)e^{-ixs}ds$ is in $H^\eta(\mathbb{R})$.

(iii) Let $f \in L^1(\mathbb{R})$. If $x^n f \in L^1(\mathbb{R})$ with $n \in \mathbb{N}$ then $\hat{f} \in C^{[n]}(\mathbb{R})$; If $e^{|Ax|} f \in L^1(\mathbb{R})$ then $\hat{f}$ extends analytically in a strip of width $|A|$ centered on $\mathbb{R}$.

Note. The rate of decay may improve if the lack of regularity is due to behavior at isolated points for otherwise smoother functions. Such is for instance the function $f(x) = \sqrt{x}$, which is in $H^{1/2}[0,1]$ but not in $H^\eta[0,1]$ if $\eta > 1/2$, and yet $\int_0^1 e^{ixs}\sqrt{s}ds = O(x^{-1})$ as shown at the end of the proof of Proposition 1.56.

Proof. (i) We have as $x \to \infty$

$$\begin{aligned}
&\left|\int_0^1 f(s)e^{ixs}ds\right| = \\
&\left|\sum_{j\in[0,\frac{x}{2\pi}-1)} \left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds + \int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right| + O(x^{-1}) \\
&\quad = \left|\sum_{j\in[0,\frac{x}{2\pi}-1)} \int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} \big(f(s) - f(s+\pi/x)\big)e^{ixs}ds\right| + O(x^{-1}) \\
&\qquad\qquad \le \sum_{j\in[0,\frac{x}{2\pi}-1)} C\left(\frac{\pi}{x}\right)^\eta \frac{\pi}{x} \le Cx^{-\eta} + O(x^{-1}) \qquad (1.49)
\end{aligned}$$

(ii) We see that

$$\frac{\hat{f}(s) - \hat{f}(s')}{(s-s')^\eta} = \int_{-\infty}^{\infty} \frac{e^{ixs} - e^{ixs'}}{(xs - xs')^\eta} x^\eta f(x)dx$$

is bounded.
(iii) Follows in the same way as (ii), using dominated convergence.

Notes In part (i), compactness of the interval is crucial. Indeed, the function $f(x) = 1$ on the interval $[n, n + e^{-n^2}]$ for $n \in \mathbb{N}$ and zero otherwise is in $L^1(\mathbb{R})$ and further has the property that f and $e^{|Ax|}f \in L^1(\mathbb{R})$ for any A, and thus $\hat{f}$ is entire. Thus f is the Fourier transform of an entire function, $\hat{f}$, and nevertheless does not decay pointwise as $x \to \infty$.
(2) It is worth mentioning that in Laplace type integrals it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral. This is for instance seen in Watson's Lemma when $\beta = 0$, but in Fourier-like integrals, continuity does not ensure $O(x^{-1})$ decay. When the conditions for the steepest descent method studied in the next section apply, a better control of decay of a Fourier type integral may be achieved by transforming it into a Laplace-like one.

Proposition 1.50 *Assume $f \in C^n[a, b]$. Then, if $m < n$ we have*

$$\int_a^b e^{ixt} f(t)dt = e^{ixa} \sum_{k=1}^{m} c_k x^{-k} + e^{ixb} \sum_{k=1}^{m} d_k x^{-k} + o(x^{-m})$$
$$= e^{ixt} \left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{m-1} \frac{f^{(m)}(t)}{(ix)^m} \right)\Bigg|_a^b + o(x^{-m}) \quad (1.51)$$

Proof. This follows by integration by parts since

$$\int_a^b e^{ixt} f(t)dt = e^{ixt} \left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{m-1} \frac{f^{(m-1)}(t)}{(ix)^m} \right)\Bigg|_a^b + \frac{(-1)^m}{(ix)^m} \int_a^b f^{(m)}(t) e^{ixt} dt \quad (1.52)$$

Corollary 1.53 *(1) Assume $f \in C^\infty[a, b]$ is periodic with period $b - a$. Then $\int_a^b f(t)e^{int} = o(n^{-m})$ for any $m > 0$ as $n \to +\infty, n \in \mathbb{Z}$.*

(2) Assume $f \in C_0^\infty[a, b]$, a smooth function which vanishes with all derivatives at the endpoints; then $\hat{f}(x) = \int_a^b f(t)e^{ixt} = o(x^{-m})$ for any $m > 0$ as $x \to +\infty$.

Exercises. (a) Show that if is analytic in a neighborhood of $[a, b]$ but is not an entire function, then both series in (1.51) have zero radius of convergence.

(b) In Corollary 1.53 (2) show that $\limsup_{x\to\infty} e^{\epsilon|x|}|\hat{f}(x)| = \infty$ for any $\epsilon > 0$ unless $f = 0$.

Oscillatory integrals with monotonic phase.

Proposition 1.54 *Let the real valued functions $f \in C^m[a, b]$ and $g \in C^{m+1}[a, b]$ and assume $g' \neq 0$ on $[a, b]$. Then*

$$\int_a^b f(t)e^{ixg(t)}dt = e^{ixg(a)}\sum_{k=1}^{m} c_k x^{-k} + e^{ixg(b)}\sum_{k=1}^{m} d_k x^{-k} + o(x^{-m}) \qquad (1.55)$$

as $x \to \infty$, where the coefficients c_k and d_k can be computed by Taylor expanding f and g at the endpoints of the interval of integration.

Proof. Since $g' \neq 0$ we may invert $g(t) = G$ in $C^{[m+1]}[a,b]$, change variables in the integral and write

$$\int_a^b f(t)e^{ixg(t)}dt = \int_{g(a)}^{g(b)} f(G(g))e^{ixg}G'(g)dg$$

and apply Proposition 1.50 to the latter integral. The computation of the coefficients c_k and d_k is straightforward.

Stationary phase method. We consider now the case when $g(s)$ has a stationary point inside the interval $[a,b]$. In this case the main contribution to the integral on the lhs of (1.55) comes from a neighborhood of the stationary point of g since around that point the oscillations that determine the integral to be small are less rapid. We have the following result:

Proposition 1.56 *Assume the real valued functions $f, g \in C^\infty[a,b]$ and that $g'(c) = 0$ $g''(x) \neq 0$ on $[a,b]$. Then for any $m \in \mathbb{N}$ we have*

$$J = \int_a^b f(s)e^{ixg(s)}ds = e^{ixg(c)}\sum_{k=1}^{2m} c_k x^{-k/2} + e^{ixg(a)}\sum_{k=1}^{m} d_k x^{-k} + e^{ixg(b)}\sum_{k=1}^{m} e_k x^{-k} + o(x^{-m}) \qquad (1.57)$$

for large x, where the coefficients of the expansion can be calculated by Taylor expansion around a, b and c of the integrand. In particular, we have

$$c_1 = \sqrt{\frac{2\pi i}{g''(c)}}f(c)$$

Proof. It is convenient split the integral into $\int_a^c + \int_c^b$ and reduce to the case when the extremum is at one endpoint. By a change of variables we make $c = 0$ and $b = 1$, and by subtracting out $g(0)$ we arrange $g(0) = 0$. We analyze the case $g'' > 0$, the other case being similar. Let $g''(0) = 2\alpha$. On the interval $[0,1]$ g is monotonic and we change variables to $g(s) = G$.

We can write

$$g(s) = \int_0^s (s-t)g''(t)dt = s^2\int_0 (1-u)g''(su)du := s^2\alpha H(s)$$

with $H \in C^\infty$, $H > 0$ on $[0,1]$ and $H(0) = 1$. Thus $h(t) = \sqrt{H} \in C^\infty$, $h' \neq 0$ on $(0,1]$, and $(sh)'(0) = 1$ and $(sh)^{-1} \in C^\infty$ implying that the equation $g(s) = t$ has a solution of the form $G(\sqrt{t})$ with $G \in C^\infty$. With $b = g(1)$ we get

$$J = \int_0^b t^{-1/2} F(\sqrt{t}) e^{ixt} dt$$

with $F \in C^\infty$. As above, we can write $F(\sqrt{t}) = F(0) + \sqrt{t} F_1(\sqrt{t})$, $F_1 \in C^\infty$. Now

$$\int_0^b t^{-1/2} e^{ixt} dt = \int_0^\infty t^{-1/2} e^{ixt} dt - \int_b^\infty t^{-1/2} e^{ixt} dt$$
$$= \sqrt{\frac{\pi}{ix}} + \frac{i}{x\sqrt{b}} e^{ixb} - \frac{i}{2x} \int_b^\infty t^{-3/2} e^{ixt} \qquad (1.58)$$

By integration by parts,

$$\int_0^b F_1(\sqrt{t}) e^{ixt} dt = \frac{1}{ix} F_1(\sqrt{t}) e^{ixt} \Big|_0^b - \frac{1}{ix} \int_0^b t^{-1/2} F_2(\sqrt{t}) e^{ixt} dt$$

with F_2 in C^∞. The proof is completed by induction.
Note It is easy to see that in the settings of Watson's Lemma and of Propositions 1.50, 1.54 and 1.56 the asymptotic expansions are differentiable, in the sense that the integral transforms are differentiable and their derivative is asymptotic to the formal derivative of the associated expansion.

1.1.8 Elementary introduction to transseries

The asymptotic expansions seen in the previous examples have as a common feature that they are written in terms of powers of the variable, exponentials and logs, e.g.

$$\int_x^\infty e^{-s^2} ds \sim e^{-x^2} \left(1 + \frac{1}{2x} - \frac{1}{4x^2} + \frac{5}{8x^3} - \ldots\right) \qquad (1.59)$$

$$n! \sim \sqrt{2\pi} e^{n \ln n - n - \frac{1}{2} \ln n} \left(1 + \frac{1}{12n} + \ldots\right) \qquad (1.60)$$

$$\int_1^x \frac{e^t}{t} \sim e^x \left(\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \ldots\right) \qquad (1.61)$$

Hardy noted that "No function has yet presented itself whose asymptotic expansion cannot be expressed in terms of exponentials, power series and logs". The modern conjecture of Écalle states that functions of natural origin can be isomorphically represented by "transseries" in the same way as an analytic function is locally given by a convergent Taylor series. Transseries are *formal* combinations of exponentials, power series and logs which are asymptotic. It is convenient to take the limit setting $x \to +\infty$. Here is an example of a transseries:

$$e^{\sum_{k=0}^{\infty} \frac{k! e^x}{x^k}} + \ln\ln x \left(1 + \ln(x)^{-1} + ...\right) + e^{-x}\left(\frac{1}{x} + \frac{1}{x^2} + ...\right), \quad x \to +\infty$$

Transseries will be studied in more carefully later; for the moment we give a short heuristic description.

1. Transseries have an exponential level (height) which is the highest order of composition of the exponential, and similarly a logarithmic depth; both of these are finite; $\exp(\exp(x^2)) + \ln x$ has height 2 and depth 1.
   It is convenient to first construct transseries without logs and then define the general ones by composition to the right with an iterated log.
2. Transseries of level zero are simply finitely generated *asymptotic* power series. That is, given $\alpha_1, ..., \alpha_n$ with $\Re(\alpha_i) > 0$ a level zero transseries is a sum of the form
   $$S = \sum_{k_i \geq M_i} c_{k_1,...,k_n} x^{-\alpha_1 k_1 - ... - \alpha_n k_n} \tag{1.62}$$
   where $M_1, ..., M_n$ are *integers*, positive or negative; the terms of S are therefore nonincreasing in k_i and bounded *above* by $O(x^{-\alpha_1 M_1 - ... - \alpha_n M_n})$.
3. A term of the form $m = x^{-\alpha_1 k_1 - ... - \alpha_n k_n}$ is a level zero (trans)monomial.
4. The lower bound for k_i easily implies that there are only finitely many terms with the same monomial. Indeed, the equation $\alpha_1 k_1 + ... + \alpha_n k_n = p$ does not have solutions if $\Re(\alpha_i) k_i > |p| + \sum_{j \neq i} |\alpha_j||M_j|$.
5. A transmonomial is small $m = o(1)$ and large if $1/m$ is small. m is neither large nor small iff $m = 1$ i.e., $-\alpha_1 k_1 - ... - \alpha_n k_n = 0$; this is a degenerate case and for some purposes it is not considered a monomial.
6. A level zero transseries can be decomposed as $L + const + s$ where L, which could be zero, is the purely large part in the sense that it contains only large monomials and s is small.
   Assuming the coefficient of $x^{-\alpha_1 M_1 - ... - \alpha_n M_n}$ is nonzero, we can write
   $$S = const\, x^{-\alpha_1 M_1 - ... - \alpha_n M_n}(1 + s)$$
   where s is small.
7. Operations are defined on level zero transseries in the natural way. The product of level zero transseries is a level zero transseries where as in 4 above the lower bound for k_i entails that there are only finitely many terms with the same monomial in the product.

8. It is easy to see that the expression $(1-s)^{-1} := 1 - s + s^2 - ...$ is well defined and this allows definition of division via

$$1/S = const^{-1} x^{\alpha_1 M_1 + \ldots + \alpha_n M_n} (1-s)^{-1}$$

9. $x^{\alpha_1 M_1 + \ldots + \alpha_n M_n}$ is the leading order and *const* is the leading constant.
10. It can be checked that level zero transseries form a differential field. Composition $S(s)$ is also well defined whenever s is a *large* transseries.
11. Level one. The exponential e^x has no asymptotic *power* series at infinity (in particular, its power series about zero is not of the form (1.62) and e^x is taken to be its own expansion. It is a new element.
12. A level one transmonomial is of the form $\mu = me^L$ where m is a level zero transmonomial and L is a purely large level zero transseries. μ is *large* if the leading constant of L is positive and small otherwise. If L is large and positive then e^L is, by definition, much larger than any monomial of level zero. We define naturally $e^{L_1} e^{L_2} = e^{L_1 + L_2}$.
13. A level one transseries is of the form

$$S = \sum_{k_i \geq M_i} c_{k_1, \ldots, k_n} \mu_1^{-k_1} \cdots \mu_n^{-k_n} := \sum_{\mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \tag{1.63}$$

where μ_i are *large* level one transmonomials.
With the operations defined naturally as above, level one transseries form a differential field.

14. We define, for a small transseries, $e^s = \sum_{k=0}^{\infty} s^k / k!$. If s is of level zero, then e^s is of level zero too.
15. The construction proceeds similarly, by induction and a general exponential-free transseries is one obtained at *some level* of the induction. They form a differential field.
16. It can be shown, by induction, that $S' = 0$ *iff* $S = const$.
17. *Dominance*: If $S \neq 0$ then there is a largest transmonomial $\mu_1^{-k_1} \cdots \mu_n^{-k_n}$ in S, with nonzero coefficient, C. Then $\mathrm{Dom}(S) = C\mu_1^{-k_1} \cdots \mu_n^{-k_n}$. If S is a nonzero transseries, then $S = \mathrm{Dom}(S)(1+s)$ where s is purely small, i.e., all the transmonomials in s are small. It can be shown (the construction will be given later) that a base of monomials can then be chosen such that all M_i in s are positive.
18. Topology.
    a) If $\tilde{S}$ is the space of transseries generated by the monomials $\mu_1, ..., \mu_n$ then, by definition, the sequence $S^{[j]}$ converges to S given in (1.63) if for any $\mathbf{k}$ there is a $j_0 = j_0(\mathbf{k})$ such that $c_{\mathbf{k}}^{[j]} = c_{\mathbf{k}}$ for all $j \geq j_0$.
    b) This topology is metrizable, by taking for instance $\mathrm{dist}(S_1, S_2) = \mathrm{dist}(S_1 - S_2, 0) := d(S_1 - S_2)$ with $d(S) = \sum_{c_{\mathbf{k}} \neq 0} e^{-k_1 - \ldots - k_n}$.
    c) In this topology, addition and multiplication are continuous, but multiplication by scalars is not.

d) It is easy to check that any Cauchy sequence is convergent and transseries form a complete linear metric space.
e) Contractive mappings: A function (operator) $\mathcal{A} : \tilde{S} \to \tilde{S}$ is contractive if for some $\alpha < 1$ and any $S_1, S_2 \in \tilde{S}$ we have $d(A(S_1)-A(S_2)) \le \alpha\, d(S_1 - S_2)$.
f) *Fixed point theorem.* It can be proved in the usual way that if $\mathcal{A}$ is contractive, then the equation $S = S_0 + \mathcal{A}(S)$ has a unique fixed point.
Examples –This is a convenient way to show the existence of multiplicative inverses. It is enough to invert $1 + s$ with s small. We choose a basis such that all M_i in s are positive. Then the equation $y = -s - sy$ is contractive.
–The equation $y = 1/x - y'$ is contractive within level zero transseries; It has a unique solution.

19. If $L_n = \log(\log(\ldots\log(x)))$ n times, and T is an exponential-free transseries then $T(L_n)$ is a general transseries. They form a differential field, closed under integration, composition to the right with large transseries, and many other operations; this closure is proved as part of the general induction.
20. The theory of differential equations in transseries has many similarities with the usual theory. For instance it is easy to show, using an integrating factor and 16 above that the equation $y' = y$ has the general solution Ce^x and that the equation $y" = xy$ has at most two linearly independent solutions. We will find two such solutions in the examples below.

*

Example 1. Solve, in the field of transseries, the Airy equation

$$f'' = xf$$

The equation is linear with entire coefficients; the solutions are entire too. At any point in $\mathbb{C}$ the solutions convergent Taylor series can be found and give the general solution; the only interesting point is the point at infinity, which is an *irregular* singular point. Indeed, with $x = 1/z$ we get

$$g'' + 2z^{-1}g' - z^{-5}g = 0 \tag{1.64}$$

and we see that the order of the pole exceeds the order of the equation.
There are no nontrivial level zero transseries solutions, since if x^p is the leading order we would get $x^{p+1} = O(x^{p-2})$ for large x.

We now try level one transseries, namely $y = e^W$ where W is a large (not necessarily purely large) level zero transseries, of leading order $x^m, \Re(m) > 0$. (The WKB method relies on this substitution.) This yields

$$W'^2 + W'' = x$$

With this choice of W we have $W'^2 \gg W''$. Indeed, if instead $W'' = O(W'^2)$ then integration yields $W = O(\log x)$ contrary to the assumption. We obtain formal solutions by putting the small term to the right hand side, solving for W' and iterating by starting with $W'' = 0$ then with W'' obtained from the first order approximation etc. That is, the iteration scheme is

$$W'_n = \pm\sqrt{x - W''_{n-1}}$$

Within the contractive mapping formalism, we see that, for instance

$$W' = \sqrt{x} - \frac{W''}{\sqrt{x} + \sqrt{x - W''}}$$

is contractive within level zero transseries with $W' \ll x$. Iteration yields

$$W = \pm\frac{2}{3}x^{3/2} - \frac{1}{4}\ln x \pm \frac{5}{48x^{3/2}} + ...$$

and thus

$$f = x^{-1/4}e^{\pm\frac{2}{3}x^{3/2}}\left(1 - \frac{5}{48x^{3/2}} + ...\right)$$

The type of exponential growth is related to the factorial power of divergence of the power series. For illustration we return to (1.64), adding an inhomogeneous term,

$$g'' + 2z^{-1}g' - z^{-5}g = 1 \tag{1.65}$$

The presence of a pole of higher order than the equation makes the power series expansion $\sum_k c_k z^k$ of a solution diverge ($c_k \propto k^p,\ p > 0$), since at the level of the recurrence for the c_k it implies that coefficients with larger k are given in terms of earlier ones multiplied by powers of n. In our specific case we get

$$c_{n+3} = n(n+1)c_n$$

with the solution

$$c_{3k} = const.\Gamma(k + 1/3)\Gamma(k)$$

roughly,

$$c_k \propto (k!)^{2/3} \tag{1.66}$$

1.1.9 Representability in terms of Laplace transforms

We divide by the exponential and change variable $\frac{2}{3}x^{3/2} = s$ to linearize the exponent and ensure that the transformed function has an asymptotic series with factorial divergence. Such a series can be obtained by Watson's lemma from a convergent series. Inverse Laplace transform in then likely to regularize the equation.

Taking $f(x) = e^{\frac{2}{3}x^{3/2}}h(\frac{2}{3}x^{3/2})$ we get

$$h'' + \left(2 + \frac{1}{3s}\right)h' + \frac{1}{3s}h = 0 \tag{1.67}$$

and with $H = \mathcal{L}^{-1}(h)$ we get

$$p(p-2)H' = \frac{5}{3}(1-p)H$$

which indeed has a regular singularity at $p = 0$. The solution is

$$H = Cp^{-5/6}(2-p)^{-5/6}$$

and it can be easily checked that any integral of the form

$$h = \int_0^{\infty e^{i\phi}} e^{-ps}H(p)dp$$

for $\phi \neq 0$ is a solution of (1.67) yielding the expression

$$f = e^{\frac{2}{3}x^{3/2}} \int_0^{\infty e^{i\phi}} e^{-\frac{2}{3}x^{3/2}p}p^{-5/6}(2-p)^{-5/6}dp \tag{1.68}$$

for a solution of the Airy equation. A second solution can be obtained in a similar way, replacing $e^{\frac{2}{3}x^{3/2}}$ by $e^{-\frac{2}{3}x^{3/2}}$, or by taking the difference between two integrals of the form (1.68).

Example 2. By a similar method, we can find a formal solution for the Gamma function $a_{n+1} = na_n$. We look directly for transseries of level at least one, $a_n = e^{f_n}$ and thus $f_{n+1} = \ln n + f_n$. It is clear that $f_{n+1} - f_n \ll f_n$; this suggests to write $f_{n+1} = f_n + f_n' + \frac{1}{2}f_n'' + ...$ and, taking $f' = h$ we get the equation

$$h_n = \ln n - \frac{1}{2}h_n' - \frac{1}{6}h_n'' - ... \tag{1.69}$$

which is contractive in transseries of zero height. We get

$$h = \ln n - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} \cdots$$

and thus

$$f_n = n \ln n - n - \frac{1}{2} \ln n + \frac{1}{12n} - \frac{1}{360n^3} ... + C$$

Expression as Laplace transform. The procedure in (1.69) indicates factorial divergence and suggests taking inverse Laplace transform of $g_n = f_n - (n \ln n - n + \frac{1}{2} \ln n)$.

Inverse Laplace transform is given by the *Bromwich integral* along a vertical contour in the right half plane:

$$(\mathcal{L}^{-1}F)(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xp} F(p) dp$$

The recurrence satisfied by g is

$$g_{n+1} - g_n = q_n = 1 - \Big(\frac{1}{2} + n\Big) \ln \Big(1 + \frac{1}{n}\Big) = \frac{1}{12n^2} - \frac{1}{12n^3} + ...$$

First note that $\mathcal{L}^{-1}q = p^{-2}\mathcal{L}^{-1}q''$ which can be easily evaluated by residues since

$$q'' = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}\Big(\frac{1}{(n+1)^2} + \frac{1}{n^2}\Big)$$

Thus, with $\mathcal{L}^{-1}g_n := G$ we get

$$(e^{-p} - 1)G(p) = \frac{1 - \frac{p}{2} - \Big(\frac{p}{2} + 1\Big)e^{-p}}{p^2}$$

$$g_n = \int_0^\infty \frac{1 - \frac{p}{2} - \Big(\frac{p}{2} + 1\Big)e^{-p}}{p^2} e^{-np} dp$$

(It is easy to check that the integrand is analytic at zero; its Taylor series is $\frac{1}{12} - \frac{1}{720}p^2 + O(p^3)$.)

The integral is well defined, and it easily follows that

$$f_n = C + n(\ln n - 1) - \frac{1}{2} \ln n + \int_0^\infty \frac{1 - \frac{p}{2} - \Big(\frac{p}{2} + 1\Big)e^{-p}}{p^2} e^{-np} dp$$

solves our recurrence. The constant $C = \frac{1}{2} \ln(2\pi)$ is most easily obtained by comparing with Stirling's series (1.45) and we thus get the identity

$$\ln \Gamma(n) = n(\ln n - 1) - \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi) + \int_0^\infty \frac{1 - \frac{p}{2} - \Big(\frac{p}{2} + 1\Big)e^{-p}}{p^2} e^{-np} dp \tag{1.70}$$

which holds with n replaced by $z \in \mathbb{C}$ as well.

Other recurrences can be dealt with in the same way. One can calculate $\sum_{j=1}^{n} j^{-1}$ as a solution of the recurrence

$$s_{n+1} - s_n = \frac{1}{n}$$

Proceeding as in the Gamma function example, we have $f' - \frac{1}{n} = O(n^{-2})$ and the substitution $s_n = \ln n + g_n$ yields

$$g_{n+1} - g_n = \frac{1}{n} + \ln\Big(\frac{n}{n+1}\Big)$$

and in the same way we get

$$f_n = C + \ln n + \int_0^\infty e^{-np}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp$$

where the constant can be obtained from the initial condition, $f_1 = 0$,

$$C = -\int_0^\infty e^{-p}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp$$

which, by comparison with the usual asymptotic expansion of the harmonic sum also gives

$$\gamma = \int_0^\infty e^{-p}\Big(\frac{1}{1-e^{-p}} - \frac{1}{p}\Big)dp$$

Comparison with (1.70) gives

$$\sum_{j=0}^{n-1}\frac{1}{j} - \gamma = \ln n + \int_0^\infty e^{-np}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp = \frac{\Gamma'(n)}{\Gamma(n)} \tag{1.71}$$

Exercise: Zeta function. Use the same strategy to show that

$$n!\zeta(n) = \int_0^\infty p^{n-1}\frac{e^{-p}}{1-e^{-p}}dp = \int_0^1 \frac{\ln^{n-1} s}{1-s}ds \tag{1.72}$$

1.1.10 Steepest descent method

There are instances when there is further analytic structure in a problem involving oscillatory integrals with large parameter which can be used to get sharper estimates on the asymptotic behavior.

Consider as a first example the problem of finding the asymptotic behavior of the integral

$$J(n) = \int_{-\pi}^{\pi} \frac{e^{-int}}{2-e^{it}}dt := \int_{-\pi}^{\pi} F(t)dt$$

as $n \to \infty$. We see by corollary 1.53 that $J = o(x^{-m})$ for any $m \in \mathbb{N}$. In this case the stationary phase method fails to provide show what the leading asymptotic behavior of the integral is (it only shows what it *is not*). We are dealing here with an analytic periodic function, and the Fourier coefficients decay faster than power-like. We can use this analyticity information to understand in fact what the behavior is for large n. Note that F is analytic in $\mathbb{C} \setminus \{-i \ln 2 + 2k\pi i\}_{k\in\mathbb{Z}}$. and meromorphic in $\mathbb{C}$. Furthermore, as $N \to \infty$ we have $F(t - iN) \to 0$ exponentially fast. This allows us to push the contour of integration down, in the following way. Note that

$$\oint_C F(t)dt = -2\pi i \mathrm{Res}\Big(F(t); t = -i \ln 2\Big)$$

where the contour C of integration is a clockwise rectangle with vertices $-\pi, \pi, -iN + \pi, -iN - \pi$ for any N sufficiently large. As $N \to \infty$ the integral over the segment $-iN + \pi, -iN - \pi$ goes to zero exponentially fast, and we find out that

$$\int_{-\pi}^{\pi} F(t)dt = \int_{-\pi}^{-\pi-i\infty} F(t)dt - \int_{\pi}^{\pi-i\infty} F(t)dt + \frac{\pi}{2}2^{-n}$$

Note now that the two integrals cancel each-other completely because of periodicity of the integrand and we are left with

$$J(n) = \pi 2^{-n}$$

Note also that the same calculation works if we replace $n \in \mathbb{N}$ with $x \in \mathbb{R}^+$. In this case the integrals will not cancel each-other for all x and we end up with

$$J(x) = i(e^{ix\pi} - e^{-ix\pi}) \int_0^\infty \frac{e^{-xs}}{2+e^s} ds + \pi 2^{-x} = -2 \sin \pi x \int_0^\infty \frac{e^{-xs}}{2+e^s} ds + \pi 2^{-x}$$

which is now in the form of a combination of integrals to which Watson's Lemma applies and gives a power series behavior and a small exponential. By applying Watson's lemma we get

$$J(x) - \sim 2 \sin \pi x \left(\frac{1}{3x} - \frac{2}{9x^2} - \frac{4}{27x^3} + \frac{8}{27x^4} + \frac{80}{81x^5} - \frac{224}{243x^6} + ... \right) + \pi 2^{-x} \tag{1.73}$$

which is a simple example of a transseries. We shall see that (1.73) is the actual transseries of $J(x)$. However the power series diverges factorially and adding to it the exponentially small term makes classical sense only if n is an integer. The divergence follows from the fact that the term of order k of the series is by Watson's lemma $k!$ times the Taylor coefficient of the function $(2+e^s)^{-1}$ at $s = 0$ and this function is not entire. Thus its Taylor coefficients must grow faster than a^k for some a. Thus the power series part cannot be simply subtracted out of J to see "what is left" and on the other hand 2^{-x} is smaller for large enough x than any x^{-m} thus cannot be made part of the scales x^{-m}. In some sense we may say that Poincaré type asymptoticity is restricted to ordinal type ω and our example has higher ordinal length.

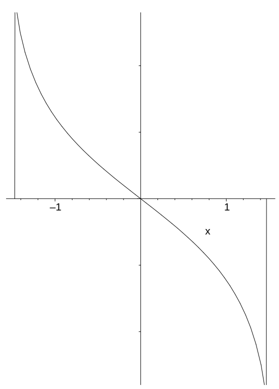
Another example of contour deformation.

Proposition 1.74 *Assume f is periodic of period 2π and is analytic in a strip of width $> R > 0$ sitting in the upper half plane over the real line. Then the Fourier coefficients $(2\pi)^{-1}\int_0^{2\pi} e^{int}f(t)dt$ are (at most) $O(e^{-nR})$ for large n.*

Proof. By analyticity, we have

$$\int_0^{2\pi} e^{int}f(t)dt = \int_0^{iR} e^{int}f(t)dt - \int_{2\pi}^{2\pi+iR} e^{int}f(t)dt + \int_{iR}^{2\pi+iR} e^{int}f(t)dt$$

The first two integrals on the rhs cancel by periodicity while the last one is manifestly $O(e^{-nR})$ for large n.



More general cases of the method of steepest descent

Example The Bessel function $J_0(\xi)$ can be written as $\frac{1}{\pi}\mathrm{Re}\, I$, where

$$I = \int_{-\pi/2}^{\pi/2} e^{i\xi\cos t}dt \tag{1.75}$$

which in turn equals

$$I = \int_{-\pi/2}^{-\pi/2+i\infty} e^{i\xi\cos t}dt + \int_\gamma e^{i\xi\cos t}dt + \int_{\pi/2}^{\pi/2-i\infty} e^{i\xi\cos t}dt \tag{1.76}$$

as shown in the figure. All the curves involved in this decomposition of I are lines of constant imaginary part of the exponent, and the ordinary Laplace method can be applied to find their asymptotic behavior for $\xi \to +\infty$ (note also that the integral along the curve γ, called Sommerfeld contour, is the only one contributing to J_0).

A more general discussion of the method of steepest descent. Let

$$\int_C f(s)e^{xg(s)}ds \tag{1.77}$$

where g is analytic and f is meromorphic in a domain in the complex plane of the contour C and x is a large parameter.

The crucial idea is to use deformation of contours to bring the integral to one which is suitable to the application of the Laplace method. We can assume without loss of generality that x is real and positive.

(A) Let $g = u + iv$. and assume for C' is a curve such that $v = K$ is constant along C'. Then

$$\int_{C'} f(s)e^{xg(s)}ds = e^{xiK}\int_{C'} f(s)e^{xu(s)}ds = e^{xiK}\int_0^1 f(\gamma(t))e^{xu(s)}\gamma'(t)dt$$

is such that the Laplace method may apply. The method of steepest descent consists in using the meromorphicity of f, analyticity of g to deform the contour of integration such that modulo residues, the original integral can be written as a sum of integrals of the type C' mentioned. The name steepest descent comes from the following remark. As a consequence of the Cauchy-Riemann equations we have $\nabla u \cdot \nabla v = 0$ and the lines $v = K$ are lines of steepest variation of $|e^{xg(s)}|$, and that to best control the integral it is convenient to go along the descent direction. If we interpret $\nabla u \cdot \nabla v = 0$ as a PDE for v, $u_x v_x + u_y v_y = 0$, the characteristic curves of this equation solve the system of ODEs

$$\dot{x} = -u_x(x, y); \quad \dot{y} = -u_y(x, y) \tag{1.78}$$

Note that $(x(s), y(s))$; $s > 0$ are the curves of steepest descent of u. Assume for simplicity that g is entire and f is meromorphic. We can let the points on the curve $C = (x_0(\tau), y_0(\tau))$; $\tau \in [0, 1]$ evolve with (1.78) keeping the endpoints fixed. More precisely, at time t consider the curve $(t) = C_1 \cup C_2 \cup C_3$ where $C_1 = (x(s, x_0(0)), y(s, y_0(0))$; $s \in [0, t)$, $C_2 =$

$x(t, x_0(\tau)), y(t, y_0(\tau)),\ \tau \in [0,1]$) and $C_3 = (x(s, x_0(1)), y(s, y_0(1));\ s \in [0, t)$ (see figure). Clearly,

$$\int_C f(s)e^{xg(s)}ds = \int_{C(t)} f(s)e^{xg(s)}ds \tag{1.79}$$

We can see that $z(t, x_0(\tau)) = (x(t, x_0(\tau)), y(t, x_0(\tau)))$ has a limit as $t \to \infty$ on the Riemann surface, since

$$\frac{d}{dt}u\Big(x(t), y(t)\Big) = -u_x^2 - u_y^2 \tag{1.80}$$

Define $\mathcal{S}$ as the smallest forward invariant set with respect to the evolution (1.78) which contains $(x_0(0), y_0(0))$, all the limits in $\mathbb{C}$ of $z(t, x_0(\tau))$ (by (1.80), these limits are saddle points of g, i.e. points where $g' = 0$) and the descent lines originating at these points. The set $\mathcal{S}$ is a union of steepest descent curves of u, $\mathcal{S} = \cup_n C_n$ and, if s_j are poles of f crossed by the curve $C(t)$ we have, under suitable convergence assumptions[1],

$$\int_C f(s)e^{xg(s)}ds = \sum_n \int_{C_n} f(s)e^{xg(s)}ds + 2\pi i \sum_j \mathrm{Res}(f(s)e^{xg(s)})_{s=s_j} \tag{1.81}$$

and the situation described in (A) above has been achieved.

One can allow for branch points of f, each of which adds a contributions of the form

$$\int_C \Delta f(s)e^{xg(s)}ds$$

where C is a cut starting at the branch point of f, along a line of steepest descent of g, and $\Delta f(s)$ is the jump across the cut of f.

(B) It is often more convenient to proceed as follows.

We may assume we are dealing with a simple smooth curve. We assume $g' \neq 0$ at the endpoints (the case of vanishing derivative is illustrated shortly on an example). Then, possibly after an apppropriate small deformation of C we have $g' \neq 0$ along the path of integration C and g is invertible in a small enough neighborhood $\mathcal{D}$ of C. We make the change of variable $g(s) = -\zeta$ and note that the image of C is smooth and has at most finitely many self-intersections. We can break this curve into piecewise smooth, simple curves.

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[1] which are required, as can be seen by applying the described procedure to very simple integral

$$\int_0^i e^{xe^{-z}}dz$$

The procedure described in (B) is better in many respects.

Without loss of generality we then assume that the image , C' of C is simple and take the endpoints of C' to be 0 and i. We deform the contour of integration toward $+\infty$ and end up with a sum of integrals of the form

$$\sum_n \int_{c_n}^{c_n+\infty} f\left(s(\zeta)\right) e^{-x\zeta} \frac{ds}{d\zeta} d\zeta + 2\pi i \sum_j \operatorname{Res}\left(f((s(\zeta))e^{-x\zeta} \frac{ds}{d\zeta} \right)_{s=s_j} + \sum_j \int_{d_j}^{d_j+\infty} \Delta \left[f(s(\zeta)) \frac{ds}{d\zeta} \right] e^{-x\zeta} d\zeta \quad (1.82)$$

If more convenient, one can alternatively subdivide C such that g' is nonzero on the (open) subintervals.
Example In the integral (1.83) we have, using the substitution $\cos(t) = it$,

$$I = \int_{-\pi/2}^{\pi/2} e^{i\xi \cos t} dt = \int_{-\pi/2}^{0} e^{i\xi \cos t} dt + \int_{0}^{\pi/2} e^{i\xi \cos t} dt = 2 \int_{0}^{\pi/2} e^{i\xi \cos t} dt = \int_0^\infty \frac{e^{-\xi t}}{\sqrt{1+t^2}} dt + \int_i^{i+\infty} \frac{e^{-\xi t}}{\sqrt{1+t^2}} dt \quad (1.83)$$

which can be immediately brought to a combination of Laplace transforms of functions having convergent Puiseux series at the origin.
Other examples (1) To find the behavior of the integral

$$\int_{-1}^{1} \frac{e^{ixs}}{s^2+1} ds$$

for large positive x, we deform the contour of integration in the upper half plane towards $i\infty$ where lines of steepest descent "end," collect the residue at the pole and write

$$\int_{-1}^{1} \frac{e^{ixs}}{s^2+1} ds = -\int_{-1}^{-1+i\infty} \frac{e^{ixs}}{s^2+1} ds + \int_{1}^{1+i\infty} \frac{e^{ixs}}{s^2+1} ds + \pi e^{-x}$$
$$= -ie^{-ix} \int_0^\infty \frac{e^{-xt}}{1+(it-1)^2} dt + ie^{ix} \int_0^\infty \frac{e^{-xt}}{1+(it+1)^2} dt + \pi e^{-x}$$
$$\sim e^{-ix} \sum_{k=1}^{\infty} c_k x^{-k} + e^{ix} \sum_{k=1}^{\infty} d_k x^{-k} + \pi e^{-x} \quad (1.84)$$

by Watson's Lemma, where we kept the exponentially small term since it turns out that this is also the complete transseries of our function.

(2) Similarly, one can check that the contour in

$$\int_{-1}^{1} e^{ixs^2} ds$$

can be deformed in such a way that the integral becomes

$$2e^{ix}\int_0^\infty \frac{e^{-xu}}{2\sqrt{u+i}}du + \int_{-\sqrt{i}\infty}^{\sqrt{i}\infty} e^{ixs^2}ds = 2e^{ix}\int_0^\infty \frac{e^{-xu}}{2\sqrt{u+i}}du + \frac{\sqrt{2\pi i}}{\sqrt{x}}$$
$$\sim \frac{\sqrt{2\pi i}}{\sqrt{x}} + e^{ix}\sum_{k=1}^{\infty} c_k x^{-k} \quad (1.85)$$

(and where one of the integrals now passes through the saddle point $s = 0$); the last expression is the transseries of our integral.

1.1.11 Asymptotics of Taylor coefficients

There is dual relation between the (trans)asymptotic behavior of the Taylor coefficients of an analytic function and the structure of its singularities in the complex plane. We will study a few examples in which this relationship is exhibited.

Proposition 1.86 *Assume f is analytic in the open unit disk and on the boundary of the disk it has an isolated singularity, say at $z = 1$, in a neighborhood of which the function is described by a convergent Puiseux series. In other words f is analytic in a disk of radius $1+\epsilon$ with a cut on $(1, 1+\epsilon)$ and near $x = 1$ we have*

$$f(z) = (1-z)^{\beta_1}A_1(z) + ... + (1-z)^{\beta_m}A_m(z)$$

where $A_1, ..., A_m$ are analytic in a neighborhood of $z = 1$. Then we have

$$c_k \sim k^{-\beta_1 - 1}\sum_{j=0\infty}\frac{c_{j;1}}{k^j} + ... + k^{-\beta_n - 1}\sum_{j=0\infty}\frac{c_{j;m}}{k^j}$$

where the coefficients $c_{j;m}$ can be calculated from the Taylor coefficients of the function A_m, and conversely, this asymptotic expansion determines the functions A_m. The theorem has a straightforward generalization to the case when there are finitely many isolated singularities on the unit circle.

Proof. We have

$$c_{k-1} = \frac{1}{2\pi i}\oint \frac{f(s)}{s^k}ds$$

where the contour is a small circle around the origin. This contour can be deformed, by assumption, to the union between the $C_{1+\epsilon}$, the circle of radius $(1+\epsilon)$, an integral along and below the lower side of the cut, avoiding $z = 1$

and then moving forward along and above the upper side of the cut, and whose sum we denote as $\int_C$. The integral along $C_{1+\epsilon}$ can be estimated by

$$\frac{1}{2\pi}\left|\oint_{C_{1+\epsilon}}\frac{f(s)}{s^k}ds\right| \le \|f\|_\infty(1+\epsilon)^{-k} = O((1+\epsilon)^{-k})$$

and will not participate in the asymptotic expansion. By checking the branches of the roots we can write the integral along C as a sum of integrals of the form

$$\frac{1}{2\pi i}\int_C (1-s)^\beta A(s)s^{-k}ds \tag{1.87}$$

we can restrict ourselves to the case when β is not an integer, the other case being trivial. By performing an appropriate number of integrations we can make $\Re(\beta) > 0$. We then have

$$\frac{1}{2\pi i}\int_C (1-s)^\beta A(s)s^{-k}ds = -\frac{\sin(\pi\beta)}{\pi}\int_1^{1+\epsilon}(s-1)^\beta A(s)s^{-k}ds$$
$$= -\frac{\sin(\pi\beta)}{\pi}\int_0^\epsilon t^\beta A(t)(1+t)^{-k}dt = -\frac{\sin(\pi\beta)}{\pi}\int_0^\epsilon t^\beta A(t)e^{-k\ln(1+t)}dt \tag{1.88}$$

where it is convenient to change variables to $t+1 = e^u$. This is a regular change of variables, and noting that $e^u - 1 = u\phi(u)$ where $\phi(0) = 1$ we have

$$-\frac{\sin(\pi\beta)}{\pi}\int_0^\epsilon t^\beta A(t)e^{-k\ln(1+t)}dt = -\frac{\sin(\pi\beta)}{\pi}\int_0^{\ln(1+\epsilon)} u^\beta\phi(u)A(e^u)e^{-ku}du$$
$$= -\frac{\sin(\pi\beta)}{\pi}\int_0^{\ln(1+\epsilon)} u^\beta G(u)e^{-ku}du = \tag{1.89}$$

where the assumptions of Watson's lemma are satisfied and we thus have

$$c_{k-1} \sim k^{-\beta-1}\sum_{j=0}^\infty \frac{d_j}{k^j}$$

where the d_j can be calculated straightforwardly from the Taylor coefficients of A.

To generalize to the case when there are n isolated singularities on the unit circle note that the same path deformation argument would work, avoiding each isolated singularity separately, and we end up with a sum of integrals of the type studied. Each can be reduced to the $z = 1$ case by taking $g(z) = f(ze^{i\phi}$ and noting that $g^{(n)} = e^{in\phi}f^{(n)}$.

1.2 Singularities of differential equations

We will first review briefly some basic notions about singularities of linear differential equations.

1.2.1 Linear meromorphic differential equations. Regular and irregular singularities

A linear meromorphic m-th order differential equation has the canonical form

$$y^{(m)} + D_{n-1}(x)y^{(n-1)} + ... + D_0(x)y = D(x) \tag{1.90}$$

where the coefficients $D_j(x)$ are meromorphic near x_0. We note first that any equation of the form (1.90) can be brought to a homogeneous meromorphic of order $n = m + 1$

$$y^{(n)} + C_{n-1}(x)y^{(n-1)} + ... + C_0(x)y = 0 \tag{1.91}$$

by applying the operator $D(x)\frac{d}{dx}\frac{1}{D(x)}$ to (1.90). We want to look at the possible singularities of the solutions of this equation, $y(x)$. Note first that by the general theory of linear differential equations if all coefficients are analytic at a point x_0 then the solution is also analytic. Such a point is called regular point. In the linear case singularities can only arise because of singularities in the coefficients.

The main distinction is made with respect to the type of local solutions, whether they can be expressed as convergent asymptotic series (regular singularity) or not (irregular one).

Theorem 1.92 (Frobenius) *If near the point $x = x_0$ the coefficients C_{n-j}, $j = 1...n$ can be written as $(x-x_0)^{-j}A_{n-j}(x)$ where A_{n-j} are analytic, then there is a fundamental system of solutions in the form*

$$y_m(x) = (x - x_0)^{r_m} \sum_{j=0}^{N_m} (\ln(x - x_0))^j B_{j;m}(x) \tag{1.93}$$

where $B_{j;m}$ are analytic in an open disk centered at x_0 with radius equal to the distance from x_0 to the first singularity of A_j. The powers r_m are solutions of the indicial *equation*

$r(r-1)\cdots(r-n+1)+A_{n-1}(x_0)r(r-1)\cdots(r-n+2)+\ldots+A_0(x_0) = 0$
Furthermore, logs appear only in the resonant case, when two (or more) characteristic roots differ by an integer.

A straightforward way to prove the theorem is by induction on n. A transformation of the type $y = x^{r_m}z$ reduces the equation (1.91) to an equation of the same type, but where one of the characteristic roots is zero. One can show there is an analytic solution z_0 of this equation by inserting a power series, identifying the coefficients and estimating the growth of the coefficients. The substitution $z = z_0 \int g(s)ds$ gives an equation for g which is of the same type as (1.91) but of order $n-1$. We will not go into the details of the general case but instead we will illustrate the approach on the simple equation

$$x(x-1)y'' + y = 0$$

around $x = 0$ whose indicial equation is $r(r-1) = 0$ (a resonant case). Substituting $y_0 = \sum_{k=0}^{\infty} c_k x^k$ in the equation and identifying the powers of x yields the recurrence

$$c_{k+1} = \frac{k^2 - k + 1}{k(k+1)} c_k$$

with $c_0 = 0$ and c_1 arbitrary. By linearity we may take $c_1 = 1$ and by induction we see that $c_k < 1$. Thus the power series has radius of convergence 1, and it converges up to the nearest singularity of the equation which is at $x = 1$. We let $y_0 = y_0 \int g(s)ds$ and get for h the equation

$$g' + c(x)g = 0$$

where $c(x) = 2\frac{y_0'}{y_0} = \frac{2}{x} + A(x)$ with $A(x)$ is analytic. The point $x = 0$ is a regular singular point and in fact we see that $g(x) = C_1 x^{-2} B(x)$ where C_1 is an arbitrary constant and $B(x)$ is some function analytic at $x = 0$. Thus $\int g(s)ds = C_1(\frac{a}{x} + b\ln(x) + A_1(x)) + C_2$ where $A_1(x)$ is analytic at $x = 0$. Going back to the original variables we see that we have a fundamental set of solutions in the form $y_0(x), B_1(x) + \ln x B_2(x)$ where B_1 and B_2 are analytic.

A converse of this theorem also holds, namely

Theorem 1.94 (Fuchs) *If a meromorphic linear differential equation has, at $x = x_0$, a fundamental system of solutions in the form (1.93), then x_0 is a regular singular point of the equation.*

In fact, for irregular singularities the general formal solution of the equation may contain divergent power series and exponentially small (large) terms, which lead naturally to the concept of transseries, studied later.

Example. Consider the equation

$$y' + \frac{1}{x^2} y = 1 \tag{1.95}$$

which has an irregular singularity at $x = 0$ since the order of the pole in $C_0 = x^{-2}$ exceeds the order of the equation. Substituting $y = \sum_{k=0}^{\infty} c_k x^k$ we get $c_0 = c_1 = 0$, $c_2 = 1$ and in general the recurrence

$$c_{k+1} = -kc_k$$

whence $c_k = (-1)^k (k-1)!$ and the series has zero radius of convergence. The associated homogeneous equation $y' + \frac{1}{x^2} y = 0$ has the general solution $y = Ce^{1/x}$ with an exponential singularity at $x = 0$.

1.2.2 Singularities of nonlinear differential equations; formal Painlevé test

For nonlinear differential equations, the solutions may be singular at points x where the equation is regular. Indeed, for example, the equation

$$y' = y^2 + 1$$

has a one parameter family of solutions $y(x) = \tan(x + C)$; each solution has infinitely many poles. Since the location of these poles depends on C, thus on the solution itself, these singularities are called *movable* or *spontaneous*. Painlevé studied the problem of finding differential equations whose only movable singularities are poles. These equations were interpreted as giving "nice" functions, with reasonable behavior in the complex plane. We can think of this property as guaranteeing some form of integrability of the equation, in the following sense. If we denote by $Y(x; x_0; C_1, C_2)$ the solution of the differential equation $y'' = F(x, y, y')$ with initial conditions $y(x_0) = C_1, y'(x_0) = C_1$ we see that $y(x_1) = Y(x_1; x; y(x), y'(x))$ is formally constant along trajectories and so is $y'(x_1) = Y'(x; x_0; y(x), y'(x))$. This gives thus two "integrals of motion" in $\mathbb{C}$ provided the solution Y is well defined almost everywhere in $\mathbb{C}$, i.e., if Y is meromorphic.

On the contrary, movable branch-points, if bad enough, may make the inversion process badly multi-valued, and one may expect in such circumstances that any integral of motion, which is necessarily a function of the C_i, is badly behaved. Since the Painlevé test is not invariant under changes of coordinates, failure of the Painlevé test does not imply nonintegrability. M. Kruskal introduced a test of nonintegrability, the *poly-Painlevé test* which measures whether branching is "dense" in which case one does expect absence of integrals of motion.

Example Painlevé's equation P1.

$$y'' = y^2 + x \tag{1.96}$$

We will look at the local behavior of a solution that blows up, and will find solutions that are meromorphic but not analytic. In a neighborhood of a point where y is large the dominant equation is $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we could have looked for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ and obtained, to leading order, the equation $Ap(p-1)x^{p-2} = A^2p^2$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). We are next looking for actual solutions which are of the form $6(x - x_0)^{-2}(1 + o(1))$ and apply the "squeezing" method. Substituting $y(x) = 6(x - x_0)^{-2} + \delta(x)$ and taking $x = x_0 + z$ leads to the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{1.97}$$

Note now that our assumption $\delta = o(z)^{-2}$ makes $\delta^2 = o(\frac{12}{z^2}\delta)$ which shows that the nonlinear term in (1.97) is negligible. (Thus, to leading order, our equation is linear! This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and an equation becomes approximately linear with this procedure.) It is then natural to apply the general strategy in asymptotics, separating out the large terms from the small terms and setting an iteration scheme accordingly (or, equivalently, writing a fixed point equation for the solution based on this separation). We take $\delta(z)^2 = r(z)$ and solve the remaining (linear) equation as if r was known (by variation of constants), to get, after substituting r for its value an identity as an integral equation which by construction is supposed to be contractive.

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{D}{z^3} - \frac{1}{7z^3}\int_0^z s^4 r(s)ds + \frac{z^4}{7}\int_0^z s^{-3}r(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \tag{1.98}$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$. To find δ formally, we would simply iterate (1.98) in the following way: We take $r = 0$ first and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then take $r = \delta_0^2$ and compute δ_1 from (1.98) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + ... \tag{1.99}$$

This series is actually convergent. To see that, we scale out the leading power of z in δ, z^2 and write $\delta = z^2 u$. The equation for u is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z s u^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \tag{1.100}$$

It is straightforward to check that, given C_1 large enough (compared to $x_0/10$ etc.) there is an ϵ such that this is a contractive equation for u in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \epsilon$. Our conclusion is that δ is analytic and that y is meromorphic near $x = x_0$.

Thus the equation P_I passes the local Painlevé test.

Note. The true Painlevé test requires that y is globally meromorphic, and we did *not* prove this. That indeed y is globally meromorphic is also true, but the proof is more delicate (see e.g. [1]).

An equation taken "at random" will generically fail even the local Painlevé test. For instance, for the simpler, autonomous, equation

$$f'' + f' + f^2 = 0 \tag{1.101}$$

the same analysis yields a local behavior starting with a double pole, $f \sim -6z^{-2}$. Taking $f = -6z^{-2} + \delta(z)$ with $\delta = o(z^{-2})$ again leads to a nearly linear equation for δ which can be solved by convergent iteration, using arguments similar to the ones above. The iteration is now (for some $a \neq 0$)

$$\delta = \frac{6}{5z} + Cz^4 - \frac{1}{7z^3}\int_0^z s^4 r(s)ds + \frac{z^4}{7}\int_a^z s^{-3}r(s)ds \tag{1.102}$$

but now the leading behavior of δ is larger, $\delta = \frac{6}{5z}$. Iterating in the same way as before, we see that this will eventually produce logs in the expansion for δ (it first appears in the second integral, thus in the form $z^4 \ln z$). We get

$$\delta = \frac{6}{5z} + \frac{1}{50} + \frac{z}{250} + \frac{7z^2}{5000} + \frac{79}{75000}z^3 - \frac{117}{2187500}z^4\ln(z) + Cz^4 + ... \tag{1.103}$$

where later terms will contain higher and higher powers of $\ln(z)$. This is effectively a series in powers of z and $\ln z$ a simple *transseries*, which here is classically convergent, as can be straightforwardly shown using the contractive mapping method, as above.

1.3 Singular perturbations

In problems depending analytically on a small parameter, internal or external, the dependence of the solution on this parameter may be analytic or not; the dependence on the parameter may be regular or singular. In differential equations, singular perturbations are usually those in which the small perturbation is such that the highest derivative does not participate in the dominant balance. An example is the Schrödinger equation

$$-\epsilon^2\psi'' + V(x)\psi - E\psi = 0 \tag{1.104}$$

for small ϵ, which will be studied in more detail later. Setting $\epsilon = 0$ removes the second derivative from the equation. Similarly, in the problem

$$x^2 f' + f = x^2 \tag{1.105}$$

the presence of x^2 in front of f' makes it subdominant if $f \sim x^p$ for some p. In this sense the Airy equation that we looked at in §1.1.8 is singularly perturbed at $x = \infty$, as can be seen by taking $x = 1/z$. It turns out that in many of these problems the behavior of solutions is exponential in the parameter, generically yielding level one transseries, of the form Qe^P where P and Q have algebraic behavior in the parameter. An exponential substitution of the form $f = e^w$ may be used in order to make the leading behavior algebraic. This is the first step in the method known as WKB.

We have already used this substitution in §1.1.8 to determine the asymptotic behavior of the Airy functions and of the factorial. We will first illustrate the idea in some further instances.

Consider the heat equation

$$\psi_t = \psi_{xx} \tag{1.106}$$

This is a degenerate (parabolic) PDE. The effect of this degeneracy is similar to that of a singular perturbation. If we attempt to solve the PDE in the spirit of Cauchy-Kowalewsky's method by a power series

$$\psi = \sum_{k=0}^{\infty} t^k F_k(x) \tag{1.107}$$

this series will generically have zero radius of convergence. Indeed, introducing this expansion in the equation and identifying the powers of t we get a recurrence relation for the coefficients $F_k = F''_{k-1}/k$ whose solution, $F_k = F_0^{(2k)}/k!$ behaves like $F_k \sim k!$ for large k, if F is analytic but not entire.

If we take $\psi = e^w$ in (1.106) we get

$$w_t = w_x^2 + w_{xx} \tag{1.108}$$

where the assumption of algebraic behavior of w is expected to make $w_x^2 \gg w_{xx}$ and so the leading equation is approximately

$$w_t = w_x^2 \tag{1.109}$$

which can be solved by characteristics, e.g. in the following way. We take $w_x = u$ and get for u the quasilinear equation

$$u_t = 2uu_x \tag{1.110}$$

with a particular solution $u = -x/(2t)$, giving $w = -x^2/(4t)$. We thus take $w = -x^2/(4t) + \delta$ and get for δ the equation

$$\delta_t + \frac{x}{t}\delta_x + \frac{1}{2t} = \delta_x^2 + \delta_{xx} \tag{1.111}$$

where we have separated the relatively small terms to the rhs. We would normally solve the leading equation (the lhs of (1.111)) and continue the process, but for this equation we note that $\delta = -\frac{1}{2}\ln t$ solves not only the leading equation, but the full equation (1.111). Thus

$$w = -\frac{x^2}{4t} - \frac{1}{2}\ln t \tag{1.112}$$

which gives the classical heat kernel

$$\psi = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \tag{1.113}$$

This exact solvability is of course rather accidental, but a perturbation approach formally works in a more PDE general context.

1.3.1 Singularly perturbed Schrödinger equation (1.104). Setting and heuristics

We look at (1.104) under the assumption that $V \in C^\infty(\mathbb{R})$ and would like to understand the behavior of solutions for small ϵ. Applying the WKB transformation $\psi = e^w$ we get

$$-\epsilon^2 w'^2 - \epsilon^2 w'' + V(x) - E = 0 \tag{1.114}$$

where, near an x_0 where $V(x_0) - E \neq 0$ the only consistent balance is between $-\epsilon^2 w'^2$ and $V(x) - E$ with $\epsilon^2 w''$ much smaller than both. We then write the equation in the iterative form

$$w'^2_{n+1} = \epsilon^{-2} U - w''_n \tag{1.115}$$

or

$$w' = \pm\sqrt{\epsilon^{-2} U - w''_n} = \pm\frac{\sqrt{U}}{\epsilon}\sqrt{1 - \frac{\epsilon^2 w''}{U}} \tag{1.116}$$

and solve it formally, taking first $w''_0 = 0$. To first order we thus have

$$w' = \pm\epsilon^{-1} U^{1/2} \tag{1.117}$$

Using this to approximate w'' we get

$$w' \approx \pm\epsilon^{-1} U^{1/2} - \frac{1}{4}\frac{U'}{U} \tag{1.118}$$

and thus

$$w \approx \pm\epsilon^{-1}\int U^{1/2}(s)ds - \frac{1}{4}\ln U \tag{1.119}$$

and thus

$$\psi \sim e^{\pm\epsilon^{-1}\int U^{1/2}(s)ds} U^{-1/4} \tag{1.120}$$

If we proceed formally we would get an expansion of the form

$$\psi \sim \exp\left(\pm\epsilon^{-1}\int U^{1/2}(s)ds\right)U^{-1/4}\left(1+\epsilon F_1(x)+\epsilon^2 F_2(x)+...\right) \quad (1.121)$$

There are two possibilities compatible with our assumption about x_0, namely $V(x_0) > E$ and $V(x_0) < E$. In the first case there is (formally) an exponentially small solution and an exponentially large one, in the latter two rapidly oscillating ones.

The points where $V(x_0) = E$ are special in this analysis and are called *turning points*. In applying our approximation procedure we needed the quantity $\epsilon^2 w'' U^{-1}$ to be small. To leading order, $w' = \epsilon^{-1}U^{1/2}$. The formal condition of validity of the iteration is then

$$\epsilon U' U^{-3/2} \ll 1 \quad (1.122)$$

which typically rules out small neighborhoods of points where $U = 0$. For instance if U has a simple root at $x = 0$, the only one that we will consider here (but multiple roots are not substantially more difficult) then condition (1.122) reads $x \gg \epsilon^{2/3}$. The region where this condition holds is called *outer* region. In a small region where (1.122) fails, called *inner* region, a different approximation will be sought. We see that $V(x) - E = V'(0)x + x^2h(x) =: \alpha x + x^2 h(x)$ where $h(x) \in C^\infty(\mathbb{R})$. We then write

$$-\epsilon^2\psi'' + \alpha x = -x^2 h(x)\psi \quad (1.123)$$

and treat the rhs of (1.123) as a small perturbation. The substitution $x = \epsilon^{2/3}t$ makes the leading equation an Airy equation:

$$-\psi'' + \alpha t\psi = -\epsilon^{2/3}t^2 h(\epsilon^{2/3}t)\psi \quad (1.124)$$

which is a regularly perturbed equation! For a perturbation method to apply, we merely need that $x^2h(x)\psi$ in (1.123) is much smaller than the lhs, roughly requiring $x \ll 1$. This shows that the inner and outer regions overlap, there is a subregion of both *the matching region* where both expansions apply, and where, by equating them, the free constants in each of them can be linked.

1.3.2 Outer region. Rigorous analysis

We first look at a region where $U(x)$ is bounded away from zero. We will write $U = F^2$.

Proposition 1.125 *Let $F \in C^\infty(\mathbb{R})$, $F^2 \in \mathbb{R}$, and assume $F(x) \neq 0$ in $[a,b]$. Then for small enough ϵ there exists a fundamental set of solutions of (1.104) in the form*

$$\psi_\pm = \Phi_\pm(x;\epsilon)\exp\left[\pm\epsilon^{-1}\int F(s)ds\right] \quad (1.126)$$

where $\Phi_\pm(x;\epsilon)$ are C^∞ in $\epsilon > 0$.

Proof. We show that there exists a fundamental set of solutions in the form

$$\psi_{\pm} = \exp\left[\pm\epsilon^{-1}R_{\pm}(x;\epsilon)\right] \tag{1.127}$$

where $R_{\pm}(x;\epsilon)$ are C^{∞} in ϵ. The proof is by rigorous WKB.

Note first that linear independence is immediate, since for small enough ϵ the ratio of the two solutions cannot be a constant, given their ϵ behavior.

We take $\psi = e^{w/\epsilon}$ and get, as before, to leading order $w' = \pm F$. We look at the plus sign case, the other case being similar. It is then natural to substitute $w = F + \delta$; we get

$$\delta' + 2\epsilon^{-1}F\delta = -F' - \epsilon^{-1}\delta^2 \tag{1.128}$$

which we transform into an integral equation by treating the rhs as if it was known and integrating the resulting linear inhomogeneous differential equation. Setting $H = \int F$ the result is

$$\delta = -e^{-\frac{2H}{\epsilon}}\int_a^x F'(s)e^{\frac{2H(s)}{\epsilon}}ds - \frac{1}{\epsilon}e^{-\frac{2H}{\epsilon}}\int_a^x \delta^2(s)e^{\frac{2H(s)}{\epsilon}}ds =: J(\delta) =: \delta_0 + N(\delta) \tag{1.129}$$

We assume that $F > 0$ on (a,b), the case $F < 0$ being very similar. The case $F \in i\mathbb{R}$ is not too different either, as we will explain at the end.

Let now $\|F'\|_{\infty} = A$ in (a,b) and assume also that $\min_{s\in(a,b)}|U(s)| > B > 0$.

Lemma 1.130 *For small ϵ, the operator J is contractive in a ball $\mathcal{B} := \{\delta : \|\delta\|_{\infty} \le 2AB^{-1}\epsilon\}$*

Proof. i) Preservation of $\mathcal{B}$. We have

$$|\delta_0(x)| \le Ae^{-\frac{2}{\epsilon}H(x)}\int_a^x e^{\frac{2}{\epsilon}H(s)}ds$$

By assumption, H is increasing on (a,b) and $H' \neq 0$ and thus, by the Laplace method, cf. Proposition 1.21, for small ϵ we have

$$|\delta_0(x)| \le 2Ae^{-\frac{2}{\epsilon}H(s)}\frac{e^{\frac{2}{\epsilon}H}}{\frac{2}{\epsilon}H'} \le \epsilon AB^{-1}$$

Note We need this type of estimates to be uniform in $x \in [a,b]$ as $\epsilon \to 0$. To see that this is the case, we write

$$\int_a^x e^{\frac{2}{\epsilon}H(s)}ds = \int_a^x e^{\frac{2}{\epsilon}H(s)}\frac{2F(s)}{\epsilon}\frac{\epsilon}{2F(s)}ds \le \frac{\epsilon}{2B}e^{\frac{2}{\epsilon}H(s)}\Bigg|_a^x \le \frac{\epsilon}{2B}e^{\frac{2}{\epsilon}H(x)} \tag{1.131}$$

Similarly,

$$\left|\frac{1}{\epsilon}e^{-\frac{2H}{\epsilon}}\int_a^x \delta^2(s)e^{\frac{2H(s)}{\epsilon}}ds\right| \le 2\epsilon^2A^2B^{-3}$$

and thus, for small ϵ and $\delta \in \mathcal{B}$ we have

$$J(\delta) \le \epsilon^{-1}AB^{-1} + 2\epsilon^2A^2B^{-3} \le 2\epsilon AB^{-1}$$

ii) *Contractivity.* We have, with $\delta_1, \delta_2 \in \mathcal{B}$, using similarly Laplace's method,

$$|J(\delta_2) - J(\delta_1)| \le \frac{1}{\epsilon}e^{-\frac{2H}{\epsilon}}\int_a^x |\delta_2(s) - \delta_1(s)||\delta_2(s) + \delta_1(s)|e^{\frac{2H(s)}{\epsilon}}ds$$
$$\le \frac{\epsilon^2A^2}{B^3}\|\delta_2 - \delta_1\| \quad (1.132)$$

and thus the map is contractive for small enough ϵ.

Note. We see that the conditions of preservation of $\mathcal{B}$ and contractivity allow for a dependence of (a, b) on ϵ. Assume for instance $a, b > 0$, $V(x) = E$ has no root in $[a, b + \gamma)$ with $\gamma > 0$, and that a is small. Assume further that $V(0) = E$ is a simple root, $|V'(0)| = m \ne 0$. Then for some $C > 0$ we have $B \ge Cm^2a^2$ and the condition of contractivity reads

$$\frac{\epsilon^2 m}{m^3a^3} < 1$$

i.e. $a > \epsilon^{2/3}$ and for small enough ϵ this is also enough to ensure preservation of $\mathcal{B}$ which allows for matching with the inner region expansions.
We thus find that the equation $\delta = J(\delta)$ has a unique solution and that, furthermore, $\|\delta\| \le const.\epsilon$. Using this information and (1.132) which implies

$$\|J(\delta)\| \le \frac{\epsilon A}{B^2}2AB^{-1}\epsilon$$

we easily get that, for some constants $C_i > 0$ independent on ϵ,

$$|\delta - \delta_0| \le C_1\epsilon|\delta| \le C_1\epsilon|\delta_0| + C_1\epsilon|\delta - \delta_0|$$

and thus

$$|\delta - \delta_0| \le C_2\epsilon|\delta_0|$$

and thus, applying again Laplace's method we get

$$\delta \sim \frac{-\epsilon F'}{2F} \quad (1.133)$$

which gives

$$\psi \sim \exp\left(\pm\epsilon^{-1}\int U^{1/2}(s)ds\right)U^{-1/4}$$

The proof of the C^∞ dependence on ϵ can be done by induction, using (1.133) to estimate δ^2 in the fixed point equation, to get an improved estimate on δ, etc.
In the case $F \in i\mathbb{R}$, the proof is the same, by using the Stationary Phase method instead of the Laplace Method.

1.3.3 Inner region. Rigorous analysis

By rescaling the independent variable we may assume without loss of generality that $\alpha = 1$ in (1.124) which we rewrite as

$$-\psi'' + t\psi = -\epsilon^{2/3}t^2 h_1(\epsilon^{2/3}t)\psi := f(t) \tag{1.134}$$

which can be transformed in an integral equation in the usual way,

$$\psi(t) = -\mathrm{Ai}(t)\int^t f(s)\mathrm{Bi}(s)ds + \mathrm{Bi}(t)\int^t f(s)\mathrm{Ai}(s)ds + C_1\mathrm{Ai}(t) + C_2\mathrm{Bi}(t) \tag{1.135}$$

where Ai, Bi are the usual Airy functions, with the asymptotic behavior

$$\mathrm{Ai}(t) \sim \frac{1}{\sqrt{\pi}}t^{-1/4}e^{-\frac{2}{3}t^{\frac{3}{2}}};\quad \mathrm{Bi}(t) \sim \frac{1}{\sqrt{\pi}}t^{-1/4}e^{\frac{2}{3}t^{\frac{3}{2}}} \tag{1.136}$$

and

$$|t^{-1/4}\mathrm{Ai}(t)| < const.,\quad |t^{-1/4}\mathrm{Bi}(t)| < const. \tag{1.137}$$

as $t \to -\infty$. In view of (1.136) we must be careful in choosing the limits of integration in (1.135). It is convenient to ensure that the second term does not have a fast growth as $t \to \infty$, and for this purpose we need to integrate from t towards infinity in the associated integral. The rule of thumb is to ensure that the maximum of the integrand is achieved near the endpoint of integration. We choose to look at an interval in the original variable $x \in I_M = [-M, M]$ where we shall allow for ϵ-dependence of M. We then write the integral equation with concrete limits in the form below, which we analyze in I_M.

$$\psi(t) = -\mathrm{Ai}(t)\int_0^t f(s)\mathrm{Bi}(s)ds + \\ \mathrm{Bi}(t)\int_M^t f(s)\mathrm{Ai}(s)ds + C_1\mathrm{Ai}(t) + C_2\mathrm{Bi}(t) = J\psi + \psi_0 \tag{1.138}$$

Proposition 1.139 *For some positive const., if ϵ is small enough (1.138) is contractive in the sup norm if $M \le const.\epsilon^{2/5}$.*

Proof. Using the Laplace method we see that for $t > 0$ we have

$$t^{-1/4} e^{-\frac{2}{3} t^{\frac{3}{2}}} \int_0^t s^{-1/4} e^{\frac{2}{3} s^{\frac{3}{2}}} ds \le const.(|t| + 1)^{-1}$$

and also

$$t^{-1/4} e^{\frac{2}{3} t^{\frac{3}{2}}} \int_t^M s^{-1/4} e^{-\frac{2}{3} s^{\frac{3}{2}}} ds \le t^{-1/4} e^{\frac{2}{3} t^{\frac{3}{2}}} \int_t^\infty s^{-1/4} e^{-\frac{2}{3} s^{\frac{3}{2}}} ds \le const.(|t| + 1)^{-1} \quad (1.140)$$

and thus for a constant independent of ϵ, using (1.136) we get

$$|J\psi(t)| \le const.\epsilon^{2/3}(|t| + 1)^{-1} \sup_{s \in [0,t]} |\psi(s)|$$

for $t > 0$. For $t < 0$ we use (1.137) and get

$$\left| \mathrm{Ai}(t) \int_M^t f(s) \mathrm{Bi}(s) ds \right| \le (1 + |t|)^{-1/4} \sup_{s \in [-t,0]} |f(s)| (const. + \int_t^0 s^{-1/4} ds$$

and get for a constant independent of ϵ

$$|J\psi(t)| \le const.\epsilon^{2/3}(1 + |t|)^{5/2} \le const.\epsilon^{2/3}(\epsilon^{-2/3} M)^{5/2} < 1$$

We see that for small enough ϵ, the regions where the outer and inner equations are contractive overlap. This allows for performing asymptotic matching in order to relate these two solutions. For instance, from the contractivity argument it follows that

$$\psi = (1 - J)^{-1} \psi_0 = \sum_{k=0}^{\infty} J^k \psi_0$$

giving a power series asymptotics in powers of $\epsilon^{2/3}$ for ψ.

1.3.4 Matching

We may choose for instance $x = const.\epsilon^{1/2}$ for which the inner expansion (in powers of $\epsilon^{2/3}$) and the outer expansion (in powers of ϵ) are valid at the same time. We assume that x lies in the oscillatory region for the Airy functions (the other case is slightly more complicated).

We note that in this region of x the coefficient of ϵ^k of the outer expansion will be large, of order $(U'U^{-3/2})^k \sim \epsilon^{-3k/4}$. A similar estimate holds for the terms of the inner expansion. Both expansions will thus effectively be expansions in $\epsilon^{-1/4}$. Since they represent the same solution, they must agree and thus the coefficients of the two expansions are linked. This enables fixing the constants C_1 and C_2 once the outer solution is prescribed.