FLOER FIELD THEORY
FOR COPRIME RANK AND DEGREE

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Abstract. We construct partial category-valued field theories in 2+1-dimensions using Lagrangian Floer theory in moduli spaces of central-curvature unitary connections with fixed determinant of rank $r$ and degree $d$ where $r,d$ are coprime positive integers. These theories associate to a closed, connected, oriented surface the Donaldson-Fukaya category of the moduli space, and to a connected bordism between two surfaces a functor between the Donaldson-Fukaya categories. We obtain the latter by combining Cerf theory with holomorphic quilt invariants. In particular, these functors satisfy the natural composition law.

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1. Introduction

Floer’s instanton homology [14] associates to any homology three-sphere a $\mathbb{Z}/8\mathbb{Z}$-graded group, roughly speaking the Morse homology of the Chern-Simons functional on the space of $SU(2)$-connections. Floer’s homology forms a natural receptacle for a relative instanton invariant of four-manifolds with boundary, defined by counting solutions to the anti-self-dual Yang-Mills equations on the four-manifold obtained by attaching a cylindrical end. These satisfy a gluing law for four-manifolds with a common boundary component. In particular, the Donaldson invariants of a four-manifold that splits along a homology three-sphere can be computed from the relative invariants of the two parts [9]. The guiding principle of topological field theory now asks for an invariant of three-manifolds with boundary that satisfies a gluing law such that composition gives rise to the instanton Floer homology for closed three-manifolds. A natural algebraic framework for such an invariant is that of category-valued field theories: functor-valued invariants of bordisms satisfying a composition law. The most well-known example of such a theory is associated with the Wess-Zumino-Witten model of conformal field theory; it associates to a two-dimensional

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bordism a functor between categories of representations of affine Lie algebras. In particular, the functor associated to the pair of pants gives these categories the structure of a tensor product [20]. This theory extends to a 1 + 1 + 1-dimensional theory, which associates to any closed three-manifold the Witten-Reshetikhin-Turaev invariant.

The field theory associated to the anti-self-dual Yang-Mills equations is hoped to be a 2 + 1 + 1-dimensional theory, in particular, one expects a category-valued 2 + 1-dimensional theory. A strategy towards defining such an invariant was proposed by Donaldson and Fukaya [15] who, inspired by the Atiyah-Floer conjecture, suggested that the category associated to the 2-manifold should be a category of Lagrangian submanifolds of the moduli space of flat $SU(2)$-bundles.

These moduli spaces, however, are singular due to unavoidable reducible connections on the necessarily trivial bundles. In order to obtain smooth moduli spaces of flat connections, one can consider nontrivial $SO(3)$-bundles, which also give rise to instanton Floer homology for closed 3-manifolds with $b_1 > 0$. These were some of the motivating examples for the definition of the Fukaya category, and Fukaya also proposed in [15] that a three-manifold with boundary should give rise to an object of the dual of the Fukaya category. Such an instanton Floer homology with Lagrangian boundary conditions was also proposed by Salamon in [42] and is rigorously constructed in Salamon-Wehrheim [43].

More generally, if we fix coprime positive integers $r,d$, then there is a unique isomorphism class of $U(r)$-bundles of degree $d$ on each surface $Σ$, and the moduli space of central-curvature connections with fixed determinant is a smooth symplectic manifold $M(Σ)$, described in more detail in Section 3. To (various types of) symplectic manifolds one can associate a Fukaya category or its cohomological category, the Donaldson category whose objects are Lagrangian submanifolds and morphism spaces are Floer homology groups. We use this as starting point for the construction of a 2 + 1-dimensional connected category-valued field theory, that is, a functor from the category of compact, connected, oriented (2-manifolds, 3-bordisms) to the category of (small categories, isomorphism classes of functors). That is, we associate to each such 2-manifold $Σ$ the Donaldson-Fukaya category $Don^\#(M(Σ))$ of $M(Σ)$ (an extension of the Donaldson category involving Lagrangian correspondences and quilted Floer homology [50]). The main result of this paper (Theorem 4.2.1 below) is the construction of functors

$$Φ(Y) : Don^\#(M(Σ_−)) → Don^\#(M(Σ_+))$$

between these Donaldson-Fukaya categories, which arise from moduli spaces of central-curvature connections over 3-bordisms $Y$ from $Σ_−$ to $Σ_+$, and satisfy the natural composition law

$$Φ(Y ∪ Y′) = Φ(Y) ∘ Φ(Y′).$$

Rather than using anti-self-duality equations, we apply a dimensional reduction. Thus we wish to associate to a bordism $Y$ a Lagrangian correspondence

$$L(Y) ⊂ M(Σ_−)^− \times M(Σ_+)$$

arising from the moduli space of central-curvature fixed-determinant connections on a bundle over $Y$. While such moduli spaces on an arbitrary bordism are not smooth,
the moduli spaces for elementary bordisms (cylinders or handle attachments) are smooth, and hence a Cerf decomposition into elementary pieces provides a sequence of Lagrangian correspondences. Now a categorification functor constructed from pseudoholomorphic quilts in [47] can be used to convert each Lagrangian correspondence to a functor. To show independence of the composition of these functors from the choice of decomposition of the bordism we prove that Cerf moves between the decompositions are reflected by an equivalence (embedded geometric composition) between the associated Lagrangian correspondences. Finally, equivalent sequences of correspondences give rise to isomorphic functors by the “strip-shrinking” analysis developed in [51]. This construction uses only few specific properties of the moduli spaces, so that our strategy provides a blueprint for the construction of Floer Field Theories: \textit{Any field theory with values in the extended symplectic category can be composed with the categorification functor from [47] to obtain a category-valued field theory.}

Precise definitions of the involved categories can be found in Section 2, leading to a rigorous formulation of this construction framework. This strategy has already been applied to obtain various other gauge theoretic 2+1 field theories. For example, a sequel to this paper [49] uses similar $U(r)$ moduli spaces to develop invariants for tangles that conjecturally correspond to Floer homology invariants arising from singular instantons developed by Kronheimer-Mrowka [23]. The analogous field theory using Donaldson-Fukaya categories of nilpotent slices was developed by Rezazadeh [41] to generalize Seidel-Smith theory to tangles. Another field theory of the same type, using symmetric products as symplectic manifolds, is constructed by Lekili-Perutz [27]. Furthermore, a first application of the results of this paper to symplectic mapping class groups of representation varieties was given by Smith [45], and the results are extended to tangles in [49].

Because we need to restrict to situations in which there are no reducible connections to obtain smooth symplectic manifolds, not all the axioms of a topological field theory are satisfied. For instance, the surfaces must be connected, which precludes the product axiom. Moreover, the invariant for closed three-manifolds arising from this Floer field theory is trivial, since the moduli spaces associated to two-spheres are empty. However, via a connect sum construction we associate to a closed three-manifold a $\mathbb{Z}/4\mathbb{Z}$-graded quilted Floer homology group (Definition 4.4.1 below) which, by an unproven version of the Atiyah-Floer conjecture, should agree with a $U(r)$ instanton Floer homology of the connect sum.

2. Field theories via connected Cerf theory

In this section we review a version of the theory of Cerf describing the decomposition of connected bordisms into elementary bordisms between connected surfaces, and the “Cerf moves” between different decompositions. Then we explain how to build a field theory by assigning morphisms to elementary bordisms so that certain “Cerf relations” are satisfied. We begin by fixing the notation for field theories.

2.1. Field theories. Our language for topological field theories for tangles adapts that in, for example, Lurie [29], rephrasing the earlier definition of Atiyah. Let $n$
be a non-negative integer and $X_\pm$ compact oriented $n$-manifolds. Since the theory of connected bordisms is trivial for $n = 0, 1$, we take $n \geq 2$.

**Definition 2.1.1.** (Connected cobordism category)

(a) A bordism from $X_-$ to $X_+$ is a compact, oriented $n+1$-manifold $Y$ with boundary equipped with an orientation-preserving diffeomorphism $\phi : \partial Y \to \overline{X}_- \sqcup X_+$. Here $\overline{X}_-$ denotes the manifold $X_-$ equipped with the opposite orientation.

(b) The **connected cobordism category** $\text{Cob}_{n+1}^0$ is the category whose

(i) objects are compact, connected, oriented $n$-dimensional smooth manifolds $X$;

(ii) morphisms from $X_-$ to $X_+$ are $n+1$-dimensional connected bordisms $(Y, \phi)$ from $X_-$ to $X_+$ modulo the equivalence given by setting two bordisms $(Y, \phi), (Y', \phi')$ from $X_-$ to $X_+$ to be equivalent if there exists an orientation-preserving diffeomorphism $Y \to Y'$ which extends the given diffeomorphism on their boundaries $\partial Y \cong \overline{X}_- \sqcup X_+ \cong \partial Y'$;

(iii) composition of morphisms is given by gluing bordisms together: Given two bordisms $(Y_1, \phi_1)$ from $X_0$ to $X_1$ and $(Y_2, \phi_2)$ from $X_1$ to $X_2$, we may glue them together to a bordism $(Y_1 \cup_{X_1} Y_2, \phi_1 \cup_{X_1} \phi_2)$ from $X_0$ to $X_2$. For an explicit construction choose collar neighborhoods $\kappa_1 : (-\epsilon, 0] \times X_1 \to Y_1$ and $\kappa_2 : [0, \epsilon) \times X_1 \to Y_2$ and define

$$Y_1 \cup_{X_1} Y_2 := Y_1 \sqcup Y_2 \sqcup ((-\epsilon, \epsilon) \times X_1) / \sim$$

where $\sim$ is the obvious equivalence. The resulting bordism $Y_1 \cup_{X_1} Y_2$ is well-defined up to equivalence, since any two choices of collar neighborhoods are isotopic, see [34, Thm.1.4];

(iv) The identity morphism for a manifold $X$ is represented by the trivial bordism $[0, 1] \times X$ with the obvious identifications on the boundary.

**Definition 2.1.2.** (Connected field theories) Let $\mathcal{C}$ be a category. A $\mathcal{C}$-valued connected field theory in $n+1$ dimensions is a functor $\Phi$ from $\text{Cob}_{n+1}^0$ to $\mathcal{C}$.

**Example 2.1.3.** A connected category-valued field theory is a $\text{Cat}$-valued field theory where $\text{Cat}$ is the category of (small categories, isomorphism classes of functors). In Section 3.1 below we will construct a connected category-valued field theory by composing a symplectic-valued field theory and the categorification functor from [47], which associates functors to Lagrangian correspondences.

**Remark 2.1.4.** (Why the connected restriction?) Field theories usually allow for disconnected manifolds and bordisms. In this case one would take $\mathcal{C}$ to be a symmetric monoidal category and require the product axiom $\Phi(X_0 \sqcup X_1) = \Phi(X_0) \otimes \Phi(X_1)$ for disjoint union. However, in our examples the Donaldson-Fukaya categories associated to 2-manifolds are well defined only in the connected case – otherwise the underlying symplectic space, a moduli space of bundles over a disconnected surface, may be singular. Hence we have restricted to the connected cobordism category.
2.2. Cerf decompositions of bordisms. In this section we describe the decomposition of bordisms into elementary pieces in the connected bordism category. In the following, let $X_-, X_+$ be compact, connected, oriented manifolds of dimension $n \geq 1$, and let $(Y, \phi : \partial Y \to \overline{X_- \cup X_+})$ be a compact, connected, oriented bordism from $X_-$ to $X_+$.

Definition 2.2.1. (Elementary and cylindrical bordisms)

(a) A Morse datum for $(Y, \phi)$ consists of a pair $(f, \mathbf{b})$ of a Morse function $f : Y \to \mathbb{R}$ and an ordered tuple $\mathbf{b} = (b_0 < b_1 < \ldots < b_m) \subset \mathbb{R}^{m+1}$ such that

(i) $\phi^{-1}(X_-) = f^{-1}(b_0)$ and $\phi^{-1}(X_+) = f^{-1}(b_m)$ are the sets of minima resp. maxima of $f$;
(ii) each level set $f^{-1}(b)$ for $b \in \mathbb{R}$ is connected;
(iii) $f$ has distinct values at the (isolated) critical points, i.e. it induces a bijection $\text{Crit}_f : f(\text{Crit}_f)$ between critical points and critical values;
(iv) $b_0, \ldots, b_m \in \mathbb{R} \setminus f(\text{Crit}_f)$ are regular values of $f$ such that each interval $(b_{i-1}, b_i)$, $i = 1, \ldots, m$ contains at most one critical value of $f$.

A Morse function $f : Y \to \mathbb{R}$ is adapted to $Y$ if the first condition above holds.

(b) A connected bordism $(Y, \phi)$ is an elementary bordism if $Y$ admits a Morse datum $(f, \mathbf{b} = (\min f, \max f))$, that is $f$ is a Morse function with at most one critical point.

(c) $(Y, \phi)$ is a cylindrical bordism if $Y$ admits a Morse datum $(f, \mathbf{b} = (\min f, \max f))$, where $f$ is a Morse function with no critical point.

(d) A morphism $[(Y, \phi)]$ of $\text{Cob}^0_{n+1}$ is an elementary resp. cylindrical morphism if one (and hence all) of its representatives is an elementary resp. cylindrical bordism.

Definition 2.2.2. (Cerf decompositions)

(a) A Cerf decomposition of the bordism $(Y, \phi)$ is a decomposition

$$Y = Y_1 \cup X_1 Y_2 \cup X_2 \ldots \cup X_{m-1} Y_m$$

into a sequence $(Y_i \subset Y)_{i=1, \ldots, m}$ of elementary bordisms embedded in $Y$ that are disjoint from each other and $\partial Y$ except for $Y_1 \cap \partial Y = \phi^{-1}(X_-)$, $Y_m \cap \partial Y = \phi^{-1}(X_+)$, and the intersections $X_i := Y_i \cap Y_{i+1}$, which are also connected submanifolds in $Y$ of codimension 1. As a consequence we have $\partial Y_i \cong \overline{X_{i-1} \cup X_i}$ for $i = 1, \ldots, m$ with $X_0 = \phi^{-1}(X_-)$ and $X_m = \phi^{-1}(X_+)$. 

(b) A Cerf decomposition of the morphism $[Y]$ in the connected bordism category $\text{Cob}_{n+1}$ is a sequence $([Y_i])_{i=1, \ldots, m}$ of elementary morphisms that compose to $[Y] = [Y_1] \circ [Y_2] \circ \ldots \circ [Y_m]$.

Remark 2.2.3. (Cerf decomposition via Morse datum) Any Morse datum $(f, \mathbf{b})$ for the bordism $(Y, \phi)$ induces a Cerf decomposition $Y = Y_1 \cup X_1 \ldots \cup X_{m-1} Y_m$ into the sequence $(Y_i := f^{-1}([b_{i-1}, b_i]))_{i=1, \ldots, m}$ of elementary bordisms between the connected level sets $X_i = f^{-1}(b_i)$. Moreover, any Cerf decomposition of a representative $Y \in [Y]$ induces a Cerf decomposition of the morphism $[Y]$. On the other hand, any Cerf decomposition of a bordism or morphism arises from a Morse datum.
Remark 2.2.4. (Handle attachments) The pieces $Y_i$ of a Cerf decomposition have simple topological descriptions as follows: If the elementary bordism $Y_i$ contains no critical point then it is in fact a cylindrical bordism. In that case $Y_i$ is diffeomorphic to the cylinder $X_i \times [b_{i-1}, b_i]$, and $X_{i-1}$ is diffeomorphic to $X_i$.

If $Y_i$ contains a single critical point, then it has index $k \in \{1, \ldots, n\}$. In that case $Y_i$ is obtained (up to homotopy) from $X_{i-1}$ by attaching a handle $B^k \times B^{n-k}$, where $B^k$ is a $k$-ball, via an attaching map $S^{k-1} \times B^{n-k} \to X_{i-1}$. The image of $S^{k-1} \times \{0\}$ in $X_{i-1}$ is an attaching cycle and the image of $S^{k-1} \times B^{n-k}$ is viewed as a thickening of the attaching cycle; we often omit the thickenings from the description.

Conversely, $Y_i$ can be obtained from $X_i$ by attaching a handle of opposite index to an attaching cycle in $X_i$. Concrete attaching cycles can be specified by choosing a metric on $Y_i$. Then the attaching cycles in $X_{i-1}$ resp. $X_i$ are given by the intersection with the stable resp. unstable manifold for the upward gradient flow of $f$ from the unique critical point in $Y_i$, see Figure 1. As explained in Milnor [34], the notion of attaching cycle can also be defined without a metric via a gradient-like vector field, that is, a vector field $v \in \text{Vect}(Y_i)$ such that $D_v f \geq 0$ everywhere. This is sometimes useful in order to, for example, show that any Cerf decomposition can be re-arranged so that the indices of the critical points are in increasing order, see [34, Theorem 8.1].

![Figure 1. Stable and unstable manifolds inside an elementary bordism](image)

Lemma 2.2.5. For $n \geq 2$ any bordism $Y$ as above admits a Morse datum.

Proof. By Milnor [34, Theorem 8.1], there exists an adapted Morse function $f : Y \to \mathbb{R}$ such that $f$ is self-indexing in the sense that the critical points of index $i$ have critical value $i$, for each $i = 0, \ldots, n+1$, and furthermore there are no critical points of index 0 or $n+1$. After a small perturbation, we may assume that the critical values of $f$ are distinct, by Milnor [34, Chapter 4], but still with the property that if $y, y'$ are critical points with index of $y$ less than that of $y'$, then $f(y) < f(y')$. We claim the fibers of $f$ are connected. Indeed, each level set is obtained by attaching handles to lower level sets; the level sets become disconnected either because of a critical point of index 0, which does not exist by assumption, or by attaching a handle of index $n$ with disconnecting attaching cycle. Once a level set is disconnected, it can be become connected again only by attaching a handle of index one, with the points
of the attaching cycle in different components. But since the Morse function is self-indexing and \( n \geq 2 \), the \( n \)-handles are attached after the 1-handles, so the existence of a disconnecting \( n \)-handle would imply that \( X_+ \) is disconnected, a contradiction. Given such a Morse function \( f \), \( b_0 := \min f \), and \( b_m := \max f \), for \( m \geq \# \text{Crit} f \), there evidently always exists a choice of \( b_1 < \ldots < b_{m-1} \) satisfying condition (iv), hence making \((f, b)\) a Morse datum. □

Note that our definition of a Cerf decomposition differs from the standard handle decomposition in that we allow the elementary bordisms \( Y_i \) to be cylindrical bordisms and we do not keep track of the attaching cycles. This also simplifies the moves between different decompositions: Since we do not fix a metric or require the Smale condition of stable and unstable manifolds intersecting transversally, we need not consider the handle slide move discussed by Kirby [22, p. 40]. On the other hand, we use much finer decompositions than Heegaard splittings, which are commonly used to define topological invariants via Floer theory. These are specific to dimension 3 and for bordisms defined as follows.

**Definition 2.2.6.** (Heegaard splittings of bordisms) Let \( Y \) be a compact connected oriented bordism between compact connected oriented surfaces \( X_{\pm} \).

(a) \( Y \) is a **compression body** if it admits a Morse datum such that all critical points have the same index (namely 1 or 2). Equivalently, \( Y \) is obtained from \( \phi^{-1}(X_-) \) by adding only handles of the same index, or from \( \phi^{-1}(X_+) \) by adding only handles of the opposite index.

(b) A **Heegaard splitting** of \( Y \) is a decomposition \( Y = Y_- \cup_X Y_+ \) into compression bodies \( Y_- \), \( Y_+ \) with a common boundary \( X \), such that \( Y_- \) contains \( \phi^{-1}(X_-) \) and \( Y_+ \) contains \( \phi^{-1}(X_+) \), and both are obtained from these boundary components by adding handles of index 1. We call \( X \) the **Heegaard surface** of the splitting.

**Remark 2.2.7.** (Heegaard splittings via Morse functions) Let \( f : Y \to \mathbb{R} \) be an adapted Morse function such that all critical points of index 1 have values less than those of the critical points of index 2. Pick a value \( c \in \mathbb{R} \) that separates the critical values of index 1 from those of index 2. Then \( Y_- = f^{-1}(-\infty, c] \) and \( Y_+ = f^{-1}(c, \infty) \) form a Heegaard splitting of \( Y \). Note that any such \( f \) also satisfies (ii) in Definition 2.2.1 automatically, and can be perturbed to satisfy (iii). It can then be completed to a Morse datum \((f, b)\), which induces a special Cerf decomposition of \( Y \), whose elementary bordisms are ordered by index.

Recall that in the bordism category, morphisms are equivalence classes of bordisms modulo diffeomorphisms that act trivially on the boundary. In order for Cerf decompositions to make sense in this category, we need to ensure that equivalent bordisms have equivalent decompositions. This is made precise in the following Definition and Remark.

**Definition 2.2.8.** (Equivalence of Cerf decompositions) A **diffeomorphism equivalence** between two Cerf decompositions \( Y = Y_1 \cup \ldots \cup Y_m \) and \( Y' = Y'_1 \cup \ldots \cup Y'_m \) of the same length \( m \) between surfaces \( X_- \) and \( X_+ \) is a collection of diffeomorphisms
implies that Cerf decompositions are unique up to the Cerf moves, which

\[
\psi \quad \text{and} \quad \phi \quad \text{are diffeomorphisms} \quad \text{and the attaching cycles of} \quad Y \quad \text{switch in the following sense:} \quad \text{The attaching cycles of} \quad y_j \quad \text{and} \quad y_j+1 \quad \text{in} \quad X_j \quad \text{are disjoint, while in} \quad X_{j-1} = X'_{j-1} \quad \text{the attaching cycle of} \quad y_j \quad \text{is homotopic to that of} \quad y'_j+1, \quad \text{and the attaching cycle of} \quad y_{j+1} \quad \text{is homotopic to that of} \quad y'_j; \quad \text{and analogously for the intersections of stable manifolds with} \quad X_{j+1} = X'_{j+1}. \quad \text{More precisely, in this situation, critical point switch is the move from} \quad ([Y_i])_{i=1,\ldots,m} \quad \text{to} \quad ([Y'_i])_{i=1,\ldots,m'} \quad \text{with} \quad m' = m - 1, \quad [Y'_i] = [Y_i] \quad \text{for} \quad i < j, \quad [Y'_j] = [Y_j \cup Y_{j+1}], \quad \text{and} \quad [Y'_i] = [Y_{i+1}] \quad \text{for} \quad i > j. \quad \text{A critical point switch is the same move with the roles of} \quad ([Y_i])_{i=1,\ldots,m} \quad \text{and} \quad ([Y'_i])_{i=1,\ldots,m'} \quad \text{interchanged.}
\]

A critical point cancellation is the move

\[
\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y_j \cup Y_{j+1}] \ldots,
\]

where for some \( j \in \{1,\ldots,m-1\} \) the two consecutive elementary bordisms \( Y_j, Y_{j+1} \) compose to a cylindrical bordism \( Y_j \cup Y_{j+1} \subset Y \). More precisely, in this situation, critical point cancellation is the move from \(([Y_i])_{i=1,\ldots,m} \) to \(([Y'_i])_{i=1,\ldots,m'} \) with \( m' = m - 1, [Y'_i] = [Y_i] \) for \( i < j, [Y'_j] = [Y_j \cup Y_{j+1}], \) and \( [Y'_i] = [Y_{i+1}] \) for \( i > j \).

A critical point cancellation is the move

\[
\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y'_j] \circ [Y'_{j+1}] \ldots,
\]

where for some \( j \in \{1,\ldots,m-1\} \) the composition \([Y_j] \circ [Y_{j+1}] \) equals \([Y'_j] \circ [Y'_{j+1}] \), and the two Cerf decompositions \([Y_j] \circ [Y_{j+1}] = [Y'_j] \circ [Y'_{j+1}] \) of the same morphism are given by Morse data \((f, \tilde{b}) \) and \((f', \tilde{b}') \) on representatives with unique critical points \( y_{j(i)+1} \in Y_{j(i)+1} \) and \( y'_{j(i)+1} \in Y'_{j(i)+1} \) in each part, whose attaching cycles (for some choice of a metric) switch in the following sense: The attaching cycles of \( y_j \) and \( y_{j+1} \) in \( X_j \) and those of \( y'_j \) and \( y'_{j+1} \) in \( X'_j \) are disjoint, while in \( X_{j-1} = X'_{j-1} \) the attaching cycle of \( y_j \) is homotopic to that of \( y'_j+1 \), and the attaching cycle of \( y_{j+1} \) is homotopic to that of \( y'_j \); and analogously for the intersections of stable manifolds with \( X_{j+1} = X'_{j+1} \). More precisely, in this situation, critical point switch is the move from \(([Y_i])_{i=1,\ldots,m} \) to \(([Y'_i])_{i=1,\ldots,m'} \) with \( m' = m, [Y'_i] = [Y_i] \) for \( i < j \) and \( i > j + 1, \) and \( [Y'_j] \circ [Y'_{j+1}] = [Y_j] \circ [Y_{j+1}] \) as above.

A cylinder cancellation is the move

\[
\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y_j \cup Y_{j+1}] \ldots,
\]

\[\text{Definition 2.2.10. (Cerf moves)}\]

Let \( Y \) be a bordism and \( [Y] = [Y_1] \circ \ldots \circ [Y_m] \) a Cerf decomposition. A Cerf move is one of the following operations on \([(Y_i)]_{i=1,\ldots,m}\).

(a) A critical point cancellation is the move

\[
\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y_j \cup Y_{j+1}] \ldots,
\]

where for some \( j \in \{1,\ldots,m-1\} \) the two consecutive elementary bordisms \( Y_j, Y_{j+1} \) compose to a cylindrical bordism \( Y_j \cup Y_{j+1} \subset Y \).

(b) A critical point switch is the move

\[
\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y'_j] \circ [Y'_{j+1}] \ldots,
\]

where for some \( j \in \{1,\ldots,m-1\} \) the composition \([Y_j] \circ [Y_{j+1}] \) equals \([Y'_j] \circ [Y'_{j+1}] \), and the two Cerf decompositions \([Y_j] \circ [Y_{j+1}] = [Y'_j] \circ [Y'_{j+1}] \) of the same morphism are given by Morse data \((f, \tilde{b}) \) and \((f', \tilde{b}') \) on representatives with unique critical points \( y_{j(i)+1} \in Y_{j(i)+1} \) and \( y'_{j(i)+1} \in Y'_{j(i)+1} \) in each part, whose attaching cycles (for some choice of a metric) switch in the following sense: The attaching cycles of \( y_j \) and \( y_{j+1} \) in \( X_j \) and those of \( y'_j \) and \( y'_{j+1} \) in \( X'_j \) are disjoint, while in \( X_{j-1} = X'_{j-1} \) the attaching cycle of \( y_j \) is homotopic to that of \( y'_j+1 \), and the attaching cycle of \( y_{j+1} \) is homotopic to that of \( y'_j \); and analogously for the intersections of stable manifolds with \( X_{j+1} = X'_{j+1} \). More precisely, in this situation, critical point switch is the move from \(([Y_i])_{i=1,\ldots,m} \) to \(([Y'_i])_{i=1,\ldots,m'} \) with \( m' = m, [Y'_i] = [Y_i] \) for \( i < j \) and \( i > j + 1, \) and \( [Y'_j] \circ [Y'_{j+1}] = [Y_j] \circ [Y_{j+1}] \) as above.

(c) A cylinder cancellation is the move

\[
\text{from} \quad [Y] = \ldots [Y_j] \circ [Y_{j+1}] \ldots \quad \text{to} \quad [Y] = \ldots [Y_j \cup Y_{j+1}] \ldots,
\]
where for some \( j \in \{1, \ldots, m - 1\} \) one of the two consecutive elementary morphisms \([Y_j], [Y_{j+1}]\) is cylindrical. Then the composition \([Y_j] \circ [Y_{j+1}]\) is an elementary morphism as well. More precisely, in this situation, critical point cancellation is the move from \((Y_i)_{i=1, \ldots, m}\) to \((Y'_i)_{i=1, \ldots, m'}\) with \(m' = m - 1\), \([Y'_i] = [Y_i]\) for \(i < j\), \([Y'_j] = [Y_j] \circ [Y_{j+1}]\), and \([Y'_i] = [Y_{i+1}]\) for \(i > j\). A cylinder creation is the same move with the roles of \((Y_i)_{i=1, \ldots, m}\) and \((Y'_i)_{i=1, \ldots, m'}\) interchanged.

(d) A diffeomorphism equivalence is the move

\[ [Y] = [Y_1] \circ \ldots \circ [Y_m] \quad \text{to} \quad [Y'] = [Y'_1] \circ \ldots \circ [Y'_m] \]

between two diffeomorphism equivalent Cerf decompositions as in Definition 2.2.8. That is, there exist diffeomorphisms \((\varphi_i : Y_i \to Y'_i)_{i=1, \ldots, m}\) satisfying \(\varphi_i|_{X_-} = \text{Id}_{X_-}\), \(\varphi_m|_{X_+} = \text{Id}_{X_+}\), and \(\varphi_i|_{X_i} = \varphi_{i+1}|_{X_i}\) for \(i = 0, \ldots, m - 1\). Here \(X_i\) is the common boundary of \(Y_i\) and \(Y_{i+1}\).

**Remark 2.2.11.** (Stabilizations of Heegaard splittings versus Cerf moves) In dimension \(n = 2\) we can compare critical point creation to the stabilization of Heegaard splittings. A stabilization of a Heegaard splitting is obtained by connect sum with the standard Heegaard splitting of a sphere \(S^3 = H_- \cup H_+\) into solid tori \(H_-, H_+\). More precisely, given a Heegaard splitting \(Y = Y_- \cup_X Y_+\), its stabilization is obtained by pulling the Heegaard splitting \(Y \# S^3 = Y' \cup Y''_+\) into compression bodies \(Y''_\pm = Y'_\pm \# H_\pm\) back to \(Y \cong Y \# S^3\). Equivalently, let \([-1, 1] \times X = Y''_+ \cup Y''_-\) be the decomposition of the cylindrical bordism consisting of two elementary bordisms, each carrying a Morse function with index 1 resp. 2. Then \(Y''_\pm\) is obtained from \(Y_\pm\) by attaching \(Y''_\pm\) at \(X\), since attaching a one-handle is equivalent to connected sum with a torus. Thus if the Heegaard splitting \(Y = Y_- \cup Y_+\) is induced by a Cerf decomposition then the stabilization is obtained from a critical point creation. Conversely any critical point creation can be viewed as a connected sum as above, and so induces a stabilization of the corresponding Heegaard splitting.

\[ \text{Figure 2. Critical point cancellation} \]

**Theorem 2.2.12** (Connected Cerf theory). Let \(Y\) be a bordism as before of dimension \(n + 1 \geq 3\), and fix a Cerf decomposition \([Y] = [Y_1] \circ \ldots \circ [Y_m]\). Then any other Cerf decomposition of \([Y]\) can be obtained from \((Y_i)_{i=1, \ldots, m}\) by a finite sequence of Cerf moves.
Remarks on the Proof of Theorem 2.2.12: The statement without the connectedness conditions can be proved using theorems of Thom and Mather [31], see e.g. [30] which say that a generic homotopy between two Morse functions has only a finite number of fold singularities, corresponding to the critical point cancellations and creations, as in Cerf [8]. However, one then has the problem that a homotopy between Morse functions with connected fibers does not necessarily have connected fibers. There are several approaches to fixing the lack of connectedness. One possible approach to this problem is given by Gay and Kirby, see [16], which modifies a given homotopy with generic fibers and only fold and cusp singularities between any two Morse functions with connected fibers.

Remark 2.2.13. (Alternative approaches in dimension three) Alternatively in dimension three, one may show that given a sequence of Cerf moves with possibly disconnected fibers, one may modify the sequence so that one obtains a sequence of Cerf moves preserving connectedness, but so that the sequence is not necessarily associated to a homotopy. This is the approach taken in Juhasz [21, p. 1434-1437]. Finally, and this essentially amounts to the same thing, one may reduce the theorem to a relative case of the the Reidemeister-Singer theorem (that any two Heegaard splittings are related by a sequence of stabilizations and de-stabilizations). By Motto’s thesis, [35, Chapter 4] any two Heegaard splittings of a connected bordism Y are related by a sequence of stabilizations and de-stabilizations. In order to deduce Theorem 2.2.12, it suffices to modify the given Cerf decomposition to one corresponding to a Heegaard splitting by a sequence of critical point switches. That this is possible follows from Milnor [34, Theorem 4.4.1,4.2 Extension].

The Cerf theorem above implies that for any category C, in order to construct a connected C-valued field theory in the sense of Definition 2.1.2, it suffices to construct the functors on elementary bordisms and check that the Cerf moves correspond to composition identities in C.

Theorem 2.2.14. (Field theories via morphisms for elementary bordisms) Suppose we are given for some $n \geq 2$ a a partial functor $\Phi$ from $\text{Cob}_{n+1}^0$ to $C$ that associates
(a) to each compact oriented $n$-manifold $X$, an object $\Phi(X) \in \text{Obj}(C)$,
(b) to each elementary morphism $[Y]$ from $X_-$ to $X_+$, a morphism $\Phi([Y])$ from $\Phi(X_-)$ to $\Phi(X_+)$,
(c) to the trivial morphism $[[0,1] \times X]$ the identity morphism $1_{\Phi(X)}$ of $\Phi(X)$;
and satisfy the following Cerf relations for any pair of elementary morphisms $[Y_1]$ from $X_0$ to $X_1$ and $[Y_2]$ from $X_1$ to $X_2$ :
(a) If $[Y_1] \circ [Y_2]$ is a cylindrical morphism, then
$$\Phi([Y_1]) \circ \Phi([Y_2]) = \Phi([Y_1 \circ Y_2]).$$
(b) If $[Y_1], [Y_2]$ are related by critical point switch to two other elementary morphisms $[Y'_1], [Y'_2]$ from $X_0$ to $X'_1$ and from $X'_1$ to $X_2$, then
$$\Phi([Y_1]) \circ \Phi([Y_2]) = \Phi([Y'_1]) \circ \Phi([Y'_2]).$$

1We thank T. Perutz for pointing out this reference to us.
(c) If one of \([Y_1], [Y_2]\) is cylindrical, then
\[ \Phi([Y_1]) \circ \Phi([Y_2]) = \Phi([Y_1] \circ [Y_2]). \]

(d) If \([Y] = [Y_1] \circ \ldots \circ [Y_m] \) and \([Y] = [Y'_1] \circ \ldots \circ [Y'_m] \) are diffeomorphism equivalent then \( \Phi([Y_1]) \circ \ldots \circ \Phi([Y_m]) = \Phi([Y'_1]) \circ \ldots \circ \Phi([Y'_m]) \).

Then there exists a unique extension of \( \Phi \) to a \( n+1 \)-dimensional connected \( C \)-valued field theory \( \Phi : \text{Cob}_{n+1} \rightarrow C \).

3. Central curvature connections

In this section we show that assigning to each closed manifold resp. elementary bordism a moduli space of connections, considered as a symplectic manifold resp. Lagrangian correspondence, gives rise to a symplectic-valued field theory. The general idea is well-known to experts, especially in the context of quantum Chern-Simons theory where according to Witten’s suggestion the invariants of three-dimensional bordisms arise by “quantizing” these Lagrangian correspondences [53].

We begin by reviewing the construction of a symplectic category in Section 3.1. Section 3.2 provides general background on moduli spaces of central curvature connections with fixed determinant. Then Sections 3.3 and 3.4 construct a partial symplectic-valued field theory in the sense of Theorem 2.2.14.

3.1. The symplectic category. The moduli spaces of connections over compact oriented surfaces will be symplectic manifolds, whose dimension varies with the genus of the surfaces. So the morphisms between these symplectic manifolds, associated to elementary bordisms, can not in general be symplectomorphisms. However, elementary bordisms define the more general morphisms between symplectic manifolds introduced by Weinstein [52] as follows.

**Definition 3.1.1.** (Geometric composition of Lagrangian correspondences) For \( j = 0, 1, 2 \) let \( M_j \) be a symplectic manifold with symplectic form \( \omega_{M_j} \).

(a) A Lagrangian correspondence from \( M_0 \) to \( M_1 \) is a Lagrangian submanifold \( L \subset M_0^- \times M_1 \) with respect to the symplectic structure \(-\omega_{M_0} \oplus \omega_{M_1}\).

(b) The geometric composition of Lagrangian correspondences \( L_1 \subset M_0^- \times M_1 \) and \( L_2 \subset M_1^- \times M_2 \) is the point set
\[ L_1 \circ L_2 := \pi_{M_0 \times M_2}((L_1 \times L_2) \cap (M_0 \times \Delta_{M_1} \times M_2)) \subset M_0 \times M_2. \]

(c) A geometric composition is called transverse if the intersection in (1) is transverse (and hence smooth) and embedded if the projection \( \pi_{M_0 \times M_2} \) restricts to an injection of the smooth intersection. In that case the image of the intersection is a smooth Lagrangian correspondence \( L_1 \circ L_2 \subset M_0^- \times M_2 \).

In general, even after a generic perturbation, the geometric composition of Lagrangian correspondences is at most immersed. In order to be able to work with holomorphic curves, however, we wish to have embedded Lagrangians. In [47] we thus constructed a symplectic category for Lagrangian correspondences using a more algebraic composition, as follows.

**Definition 3.1.2.** (Algebraic composition of Lagrangian correspondences)
(a) A generalized Lagrangian correspondence \( L \) from \( M_- \) to \( M_+ \) (both symplectic manifolds) of length \( m \geq 0 \) consists of

(i) a sequence \( N_0, \ldots, N_m \) of symplectic manifolds with \( N_0 = M_- \) and \( N_m = M_+ \); and

(ii) a sequence \( L = (L_1, \ldots, L_m) \) of compact Lagrangian correspondences with \( L_j \subset N_{j-1}^- \times N_j \) for \( j = 1, \ldots, m \).

Here we allow empty symplectic manifolds or Lagrangian correspondences.

(b) The algebraic composition of generalized Lagrangian correspondences \( L \) and \( L' \) is given by concatenation \( LL' = (L_1, \ldots, L_m, L'_1, \ldots, L'_{m'}) \).

Definition 3.1.3. (Symplectic category) Let \( \text{Symp}^* \) be the category whose

(a) objects are symplectic manifolds (in particular, we include \( \emptyset \));

(b) morphisms \( \text{Hom}(M_-, M_+) \) of \( \text{Symp}^* \) are generalized Lagrangian correspondences from \( M_- \) to \( M_+ \) modulo the composition equivalence relation \( \sim \) generated by

\[
(\ldots, L_j, L_{j+1}, \ldots) \sim (\ldots, L_j \circ L_{j+1}, \ldots)
\]

for all sequences and \( j \) such that \( L_j \circ L_{j+1} \) is transverse and embedded;

(c) composition of morphisms \( [L] \in \text{Hom}(M, M') \) and \( [L'] \in \text{Hom}(M', M'') \) is defined by

\[
[L] \circ [L'] := \left[ L \# L' \right] \in \text{Hom}(M, M'').
\]

An identity \( 1_M \in \text{Hom}(M, M) \) is given by the equivalence class of the empty sequence of length zero (since concatenating with the empty set is the identity on sequences) or equivalently, the equivalence class \( 1_M := [\Delta_M] \) of the diagonal \( \Delta_M \subset M^- \times M \).

Remark 3.1.4. Let \( \text{Symp} \) be the category of (symplectic manifolds, symplectomorphisms). We have a functor

\[
\text{Symp} \to \text{Symp}^*, \quad \left( \begin{array}{c} M \\ \phi \end{array} \mapsto \begin{array}{c} M \\ \{\text{graph}(\phi)\} \end{array} \right).
\]

That is, the identity maps to the diagonal correspondence and the composition of graphs is the graph of the composition of two symplectomorphisms. Indeed, the geometric composition of graphs of symplectomorphisms is always transverse and embedded.

Definition 3.1.5. (Symplectomorphism equivalences) Let \( L, L' \) be generalized Lagrangian correspondences from \( M_- \) to \( M_+ \) of the same length \( m \). A symplectomorphism equivalence from \( L \) to \( L' \) is a collection of symplectomorphisms \( (\varphi_j : N_j \to N'_j)_{j=0, \ldots, m} \) such that \( \varphi_0 = \text{Id}_{M_-}, \varphi_m = \text{Id}_{M_+} \), and \( L_j = (\varphi_j \circ \varphi_j^{-1})(L_j) \) for \( j = 1, \ldots, m \).

Lemma 3.1.6. (Composition equivalences via symplectomorphism equivalences) Any symplectomorphism equivalence from \( L \) to \( L' \) induces a composition equivalence \( L \sim L' \).
Proof. Consider the commutative diagram

\[
\begin{array}{cccccc}
M^{-} = N_0 & \xrightarrow{L_1} & N_1 & \xrightarrow{L_2} & \cdots & \xrightarrow{L_m} N_m = M^{+} \\
\varphi_0 = \text{Id} & & \varphi_1 & & \varphi_{m-1} & \varphi_m = \text{Id} \\
M^{-} = N'_0 & \xrightarrow{L'_1} & N'_1 & \xrightarrow{L'_2} & \cdots & \xrightarrow{L'_m} N'_m = M^{+} \\
\end{array}
\]

Each diagonal morphism can be written in two ways as the composition of a Lagrangian correspondence with the graph of a symplectomorphism,

\[L_j \circ \text{graph}(\varphi_j) = \text{graph}(\varphi_{j-1}) \circ L'_j.\]

So we obtain a composition equivalence from \(L \) to \(L' \) by first replacing \(L_m\) with \((\text{graph}(\varphi_{m-1}), L'_m)\), then iteratively replacing \((L_j, \text{graph}(\varphi_j))\) with \((\text{graph}(\varphi_{j-1}), L'_j)\) for \(j = m - 1\) to \(j = 2\), and eventually replacing \((L_1, \text{graph}(\varphi_1))\) with \(L'_1\).

\[\square\]

For the purposes of Floer theory, we will need our symplectic manifolds and correspondences to carry additional structures and satisfy additional hypotheses.

**Definition 3.1.7.** (Monotone symplectic manifolds) For any monotonicity constant \(\tau > 0\) we introduce the following admissible classes of symplectic manifolds and generalized Lagrangian correspondences.

(a) A symplectic manifold \((M, \omega)\) is **monotone** with monotonicity constant \(\tau\) if \(\tau c_1(M) = [\omega]\) in \(H^2(M)\).

(b) A symplectic manifold is **\(\tau\)-admissible** if it is compact, monotone with monotonicity constant \(\tau\), and has even minimal Chern number.

(c) A generalized Lagrangian correspondence from \(M^{-}\) to \(M^{+}\) is **admissible** if each Lagrangian correspondence in the sequence is simply-connected, compact, oriented, and spin.

Let \(\text{Symp}_{\tau}^\#\) denote the category whose

(i) objects are \(\tau\)-admissible symplectic manifolds;

(ii) morphisms are equivalence classes of admissible generalized Lagrangian correspondences, where the composition equivalence relation \(\sim\) is that generated by (2) restricted to all admissible sequences; and

(iii) composition of morphisms is defined as concatenation as before.

**Remark 3.1.8.**

(a) (Other possible assumptions on correspondences) The condition in (c) can be replaced with other conditions which guarantee monotonicity, or just requiring monotonicity itself, but in practice in this paper we just check simply-connectedness.

(b) (Inclusion of monotone symplectic categories in the symplectic category) There is a canonical functor \(\text{Symp}_{\tau}^\# \to \text{Symp}^\#\) induced by inclusion of objects and mapping equivalence classes of generalized admissible Lagrangian correspondences to equivalence classes of generalized Lagrangian correspondences.
However, the map on morphisms may not be an inclusion since two admissible generalized Lagrangian correspondences may be equivalent through a non-admissible generalized Lagrangian correspondence.

3.2. Moduli spaces of central curvature connections with fixed determinant. This section introduces the moduli spaces of central curvature connections with fixed determinant on compact manifolds of any dimension, for a general class of structure groups.

3.2.1. Central curvature connections and gauge transformations. We first introduce the moduli spaces of central curvature connections with fixed determinant on compact manifolds of any dimension, for a general class of structure groups.

**Definition 3.2.1.**

(a) (Associated vector bundles) For any finite-dimensional real $G$-representation $V$ we denote by $P(V) = (P \times V)/G$ the associated vector bundle, where $G$ acts on $P \times V$ by $g(p,v) = (pg^{-1},gv)$. We denote by

$$\Omega(X, P(V)) := \bigoplus_{n=0}^{\infty} \Omega^n(X, P(V))$$

the space of forms with values in $P(V)$.

(b) (Adjoint bundles) In particular, $P(g) = (P \times g)/G$ is the adjoint bundle associated to the adjoint representation of $G$ on $g$.

(c) (Splittings of the adjoint bundles) Any invariant subspace $h \subset g$ induces an inclusion $P(h) \subset P(g)$. In particular, from the splitting $g = [g, g] \oplus \mathfrak{z}$ into the semisimple and central parts of $g$ we obtain a splitting of the adjoint bundle,

$$P(g) = P([g, g]) \oplus P(\mathfrak{z}).$$

In particular, in the case $G = U(r)$ to which we will specialize later, the center is given by the diagonal matrices $Z = U(1)\text{Id}$, and hence the splitting is $u(r) = su(r) \oplus u(1)\text{Id}$.

(d) (Affine space of connections) Let $\mathcal{A}(P)$ be the space of connections on $P$,

$$\mathcal{A}(P) = \left\{ \alpha \in \Omega^1(P, g) \left| \begin{array}{l} \alpha(\xi_P) = \xi \\
\alpha(vg) = \text{Ad}(g^{-1})\alpha(v) \end{array} \right\} \forall \xi \in g, \forall v \in TP, g \in G \right\}.$$ 

Here $\xi_P \in \text{Vect}(P)$ denotes the vector field generated by the action of $\xi \in g$.

(e) (Basic forms) For any non-negative integer $k$ the space $\Omega^k(X, P(g))$ of $k$-forms with values in $P(g)$ is isomorphic via $\pi^*$ to the space $\Omega^k(P(g))$ of basic (that is, equivariant and horizontal) $k$-forms. With this notation $\mathcal{A}(P)$ is an affine space over $\Omega^1(X, P(g))$ in the sense that $\Omega^1(X, P(g))$ acts on $\mathcal{A}(P)$ faithfully transitively by $\alpha \mapsto \alpha + \pi^*b$ for $\alpha \in \mathcal{A}(P)$, $b \in \Omega^1(X, P(g))$.

(f) (Curvature) The curvature of $\alpha \in \mathcal{A}(P)$ is the two form $F_\alpha \in \Omega^2(X, P(g))$ defined by

$$\pi^*F_\alpha = d\alpha + \frac{1}{2}[\alpha \wedge \alpha] \in \Omega^2(P, g)_{\text{basic}}.$$
(g) (Covariant derivative) The covariant derivative in the adjoint representation is
\[ d_\alpha : \Omega^s(X, P(g)) \rightarrow \Omega^{s+1}(X, P(g)), \quad \pi^s d_\alpha \beta = d\pi^s \beta + [\alpha \wedge \pi^s \beta]. \]

(h) The Bianchi identity is \( d_\alpha F_\alpha = 0 \).

(i) (Central curvature connections) A connection \( \alpha \) is central curvature if \( F_\alpha \) takes values in \( P(\mathfrak{z}) \subset P(g) \), that is, \( F_\alpha^{[\mathfrak{z}, \mathfrak{g}]} = 0 \).

(j) (Group of gauge transformations) Let \( G(P) = \{ \phi : P \rightarrow P \mid \pi \circ \phi = \pi, \phi(pg) = \phi(p)g \ \forall \ p \in P, \ g \in G \} \)
denote the group of gauge transformations, that is, \( G \)-equivariant automorphisms of \( P \).

(k) (Action of gauge transformations on connections) The group \( G(P) \) acts on the left on \( A(P) \) by
\[ G(P) \times A(P) \rightarrow A(P), \quad (\phi, \alpha) \mapsto \phi^{-1} \alpha. \]

(l) (Infinitesimal gauge transformations) The Lie algebra of \( G(P) \) can be identified with \( \Omega^0(X, P(g)) \) by associating the vector field \( p \mapsto \frac{d}{dt}|_{t=0} p \exp(t\xi) \) to \( \xi \in \Omega^1(P, g)_{\text{basic}} = \Omega^1(X, P(g)). \) With this notation, the infinitesimal action of \( G(P) \) on \( A(P) \) is given by
\[ \Omega^0(X, P(g)) \times A(P) \rightarrow \Omega^1(X, P(g)), \quad (\xi, \alpha) \mapsto d_\alpha \xi. \]

(m) (Action of gauge transformations on covariant derivatives) The action of \( \phi \in G(P) \) on \( P \) induces an action on \( P(V) \) for any \( G \)-representation \( V \), denoted by \( \phi_V \). The covariant derivative \( d_\alpha \) and curvature \( F_\alpha \) transform as
\[ d_{\phi \alpha} = \phi g d_\alpha \phi g^{-1}, \quad F_{\phi \alpha} = \phi g F_\alpha. \]

Hence the subset of central curvature connections is invariant under gauge transformations.

Remark 3.2.2. (a) (Splitting of the covariant derivative) Using the splitting (3) we write
\[ d_\alpha = d_\alpha^1 \oplus d_\alpha^{[\mathfrak{z}, \mathfrak{g}]} \quad \text{and} \quad F_\alpha = F_\alpha^1 \oplus F_\alpha^{[\mathfrak{z}, \mathfrak{g}]}.

(b) (Basic inner product) Let \( \langle \cdot, \cdot \rangle \) denote the basic invariant inner product on \( g \), see [38, p.49]. For \( G \) simple, this is the unique inner product such that the norm-square of the highest coroot is 2. For example, in the case that \( G = U(r) \), the basic inner product is \( \langle \xi, \zeta \rangle = -\text{Tr}(\xi \zeta) \) for \( \xi, \zeta \in \mathfrak{u}(r) \).

(c) (Adjoint of the covariant derivative) A choice of metric on \( X \) induces a Hodge star operator
\[ * : \Omega^s(X, P(g)) \rightarrow \Omega^{n-s}(X, P(g)), \]
and together with the inner product on \( g \) this induces a metric on \( \Omega^s(X, P(g)) \). The formal adjoint of the covariant derivative is
\[ d^*_\alpha : \Omega^s(X, P(g)) \rightarrow \Omega^{s-1}(X, P(g)), \quad \beta \mapsto -(-1)^{(s-1)(n-s)} * d_\alpha * \beta. \]
3.2.2. Linear theory. In this subsection we discuss the cohomology of the covariant derivative associated to a central curvature connection. We review the well-known fact that in certain dimensions the cohomology of a compact oriented manifold with boundary $X$ restricts to a Lagrangian subspace in the middle dimensional cohomology on the boundary $\partial X$.

**Definition 3.2.3.** Let $P \to X$ be a principal $G$-bundle.

(a) (Cohomology of a central curvature connection) Let $\alpha \in \mathcal{A}(P)$ be a central curvature connection. Since its curvature $F_\alpha$ is central we have $d^2 \alpha = 0$ and can define cohomology groups

$$H(X; d_\alpha) = \bigoplus_{* = 0}^n H^*(X; d_\alpha), \quad H^*(X; d_\alpha) = \frac{\ker(d_\alpha|\Omega^*(X; P(g)))}{\text{im}(d_\alpha|\Omega^{*-1}(X; P(g)))}.$$

(b) (Relative cohomology of a central curvature connection) We denote restriction to the boundary by $\rho : H(X, P(g)) \to H(\partial X, P(g)|\partial X)$, and will also consider the relative cohomology groups consisting of forms whose tangential part to the boundary vanishes,

$$H(X, \partial X; d_\alpha) = \ker(d_\alpha|\ker \rho)/\text{im}(d_\alpha|\ker \rho).$$

(c) (Hodge isomorphisms) The Hodge isomorphisms on manifolds with boundary (see e.g. [17, Section 4.1]) give

$$H(X; d_\alpha) \cong \ker(d_\alpha) \cap \text{im}(d_\alpha) = \ker(d_\alpha \oplus d_\alpha^* \oplus \rho \star),$$

$$H(X, \partial X; d_\alpha) \cong \ker(d_\alpha \oplus \rho) \cap \text{im}(d_\alpha|\ker \rho) = \ker(d_\alpha \oplus d_\alpha^* \oplus \rho).$$

Using the first identification we have a Hodge decomposition

$$\Omega(X, P(g)) = \text{im}(d_\alpha) \oplus H(X; d_\alpha) \oplus \text{im}(d_\alpha|\ker \rho).$$

Here the direct sum holds in any $L^2$-Sobolev completion for $s \geq 0$ (that is we use the norm $H^s$ of [19] measuring $s$ fractional derivatives in $L^2$), as a consequence of the following elliptic estimate from e.g. [19, Section 20.1]. Let $\Pi_{H^s}$ denote the $L^2$-orthogonal projection onto $H^k(X; d_\alpha) \subset \Omega^k(X, P(g))$. There is a constant $C$ depending on $\alpha$ such that for all $\eta \in \Omega^k(X, P(g))$

$$\|\eta\|_{L^{s+1}_2(X)} \leq C\left(\|(d_\alpha + d_\alpha^*)\eta\|_{L^2_2(X)} + \|\rho \ast \eta\|_{L^{s+1/2}_2(\partial X)} + \|\Pi_{H^s}\eta\|_{L^2_2(X)}\right).$$

(d) (Non-degeneracy of the duality pairings) Using these Hodge isomorphisms, the Hodge star induces a duality isomorphism

$$H^*(X; d_\alpha) \to H^{n-*}(X, \partial X; d_\alpha), \quad \beta \mapsto \star \beta,$$

which implies non-degeneracy of the pairing

$$H^*(X; d_\alpha) \times H^{n-*}(X, \partial X; d_\alpha) \to \mathbb{R}, \quad (\gamma, \beta) \mapsto \int_X \langle \gamma \wedge \beta \rangle.$$

Replacing $X$ with $\partial X$ we have a non-degenerate pairing on $H(\partial X; d_\alpha|\partial X)$. 
If \( \dim(X) = 2k+1 \) is odd, the pairing in the last item restricts to a non-degenerate pairing on the middle dimensional homology \( H^k(\partial X; d_{\alpha|\partial X}) \), which is symplectic if \( k \) is odd. In that case, the following Lemma shows that the restriction of \( H(X; d_{\alpha}) \) to the boundary gives rise to a Lagrangian subspace.

**Lemma 3.2.4.** (Lagrangian restriction of odd cohomology) Let \( \alpha \in \mathcal{A}(P) \) be a central curvature connection. The image \( \rho(H(X; d_{\alpha})) \) of \( H(X; d_{\alpha}) \) in \( H(\partial X; d_{\alpha|\partial X}) \) is maximally isotropic. In particular, if \( \dim(X) = 2k+1 \) for some integer \( k \) is odd then \( \rho(H^k(X; d_{\alpha})) \subset H^k(\partial X; d_{\alpha|\partial X}) \) is maximally isotropic, and if \( k \) is odd,

\[
\dim(\rho(H^k(X; d_{\alpha}))) = \frac{1}{2} \dim(H^k(\partial X; d_{\alpha|\partial X})).
\]

**Proof.** Stokes’ theorem implies that the image \( \rho(H(X; d_{\alpha})) \) is isotropic. To see that it is maximal isotropic, first note that restriction to the boundary induces a long exact sequence

\[
\ldots \rightarrow H^*(X, \partial X; d_{\alpha}) \rightarrow H^*(X; d_{\alpha}) \rightarrow H^*(\partial X; d_{\alpha|\partial X}) \rightarrow \ldots.
\]

Now consider the commutative diagram formed from the long exact sequence (11) and its dual:

\[
\begin{array}{ccc}
H^*(X; d_{\alpha}) & \longrightarrow & H^*(\partial X; d_{\alpha|\partial X}) \\
\downarrow & & \downarrow \\
H^{n-*}(X, \partial X; d_{\alpha})^\vee & \longrightarrow & H^{n-1-*}(\partial X; d_{\alpha|\partial X})^\vee
\end{array}
\]

Here the vertical arrows are given by (8) and the pairing (9). To check commutativity, use Stokes’ theorem and the fact that the connecting morphism \( c \) above is given by \( \beta \mapsto d_{\alpha} \tilde{\beta} \) for any extension \( \tilde{\beta}|_{\partial X} = \beta \).

Now suppose \( \beta \in H^*(\partial X; d_{\alpha|\partial X}) \) lies in the annihilator of the image of \( H^{n-1-*}(X; d_{\alpha}) \). By definition this means that \( \beta \in H^*(\partial X; d_{\alpha|\partial X}) \) maps to 0 in \( H^{n-1-*}(X; d_{\alpha})^\vee \).

Since the vertical maps are isomorphisms and the diagram is commutative, the image of \( \beta \) in \( H^{n-1-*}(X, \partial X; d_{\alpha}) \) vanishes, so by exactness of the top sequence \( \beta \) lies in the image of \( H^*(X; d_{\alpha}) \). The last claim follows from the fact that maximally isotropic subspaces of symplectic vector spaces are half-dimensional. For analogous results on Dirac operators on manifolds with boundary see [4, Theorem 22.24]. □

**Corollary 3.2.5.** If \( X \) has dimension 3 then \( \rho(H^1(X; d_{\alpha})) \subset H^1(\partial X; d_{\alpha|\partial X}) \) is a Lagrangian subspace. Furthermore, if \( H^1(X, \partial X; d_{\alpha}) = 0 \) then \( \rho : H^1(X; d_{\alpha}) \rightarrow H^1(\partial X; d_{\alpha|\partial X}) \) is a Lagrangian embedding.

**Proof.** The first statement follows immediately from Lemma 3.2.4 with \( k = 1 \). The injectivity of \( \rho \) in the second statement follows from the long exact sequence (11). □

### 3.2.3. Moduli spaces.
This subsection introduces the moduli space of connections with central curvature and fixed determinant. As before, \( G \) is a compact connected group and \( X \) a compact oriented manifold with (possibly empty) boundary.
Definition 3.2.6. (a) (Commutator subgroup) Let
\[ G_0 := [G, G] = \{ [g_1, g_2] \mid g_1, g_2 \in G \} \]
denote the commutator subgroup of \( G \). In particular, \([U(r), U(r)] = SU(r)\).

More generally, \( G_0 \) is the subgroup whose Lie algebra equals to the semisimple part \([\mathfrak{g}, \mathfrak{g}]\) of the Lie algebra \( \mathfrak{g} \).

(b) (Determinant homomorphism) The group homomorphism \( G \to G/G_0 \) induces for any principal \( G \)-bundle \( P \) a principal \( G/G_0 \)-bundle
\[
\text{det}(P) := P/G_0 \cong P \times_G (G/G_0) \xrightarrow{\pi'} X.
\]
In the special case \( G = U(r) \) the bundle \( \text{det}(P) \) is the principal \( U(1) \)-bundle induced by the determinant \( \text{det}: U(r) \to U(1) \).

(c) (Determinant of a connection) The homomorphism of Lie algebras \( \pi_1: \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \mathfrak{z} \) induces a map \( \text{det}: \mathcal{A}(P) \to \mathcal{A}(\text{det}(P)) \), since for any connection \( \alpha \in \mathcal{A}(P) \) the form \( \pi_1 \circ \alpha \in \Omega^1(P, \mathfrak{z}) \) is a basic form for the \( G_0 \)-bundle \( P \to \text{det}(P) \), hence descends to a connection form on \( \text{det}(P) \). Given a connection \( \delta \in \mathcal{A}(\text{det}(P)) \) we denote by
\[
\mathcal{A}_\delta(P) = \{ \alpha \in \mathcal{A}(P) \mid F^\alpha_{[\mathfrak{g}, \mathfrak{g}]} = 0, \text{det}(\alpha) = \delta \}
\]
its inverse image in the space of central curvature connections.

(d) (Gauge transformations fixing the determinant) The group of gauge transformations with trivial determinant \( \mathcal{G}_0(P) \) is defined to be the kernel of the homomorphism \( \mathcal{G}(P) \to \mathcal{G}(\text{det}(P)) \). It acts on \( \mathcal{A}_\delta(P) \) for any \( \delta \in \mathcal{A}(\text{det}(P)) \).

(e) (Moduli spaces with fixed determinant) Denote by
\[
M_\delta(P) = \mathcal{A}_\delta(P)/\mathcal{G}_0(P)
\]
its quotient, the moduli space of connections with central curvature and fixed determinant.

Remark 3.2.7. (Independence of the moduli spaces from the choice of determinant) We may identify the spaces \( M_\delta(P) \) as \( \delta \) ranges over connections on \( \text{det}(P) \) as follows. The vector space \( \Omega^1(X, \mathfrak{z}) \) acts on \( \mathcal{A}(P) \) by \( \alpha \mapsto \alpha + \pi^*a \) for \( a \in \Omega^1(X, \mathfrak{z}) \). This action is gauge-equivariant, the curvature transforms as \( F_{\alpha + \pi^*a} = F_\alpha + da \), and the determinant transforms as \( \text{det}(\alpha + \pi^*a) = \text{det}(\alpha) + \pi^*a \). Indeed, the defining equation is
\[
\pi^*_{\mathcal{G}_0} \text{det}(\alpha + \pi^*a) = \pi_3 \circ (\alpha + \pi^*a) = \pi_3 \circ \alpha + \pi^*_{\mathcal{G}_0} \pi^*a = \pi^*_{\mathcal{G}_0} (\text{det}(\alpha) + \pi^*a).
\]
So each \( a \in \Omega^1(X, \mathfrak{z}) \) induces an identification of moduli spaces \( M_\delta(P) \to M_{\delta + \pi^*a}(P) \). This provides unique identifications for all pairs in \( \mathcal{A}(\text{det}(P)) \) since the latter is an affine space over \( \Omega^1(X, \mathfrak{z}) \) via \( \delta \mapsto \delta + \pi^*a \) for \( a \in \Omega^1(X, \mathfrak{z}) \). We will hence from now on refer to
\[
M(P) := M_\delta(P) = \mathcal{A}_\delta(P)/\mathcal{G}_0(P)
\]
as the moduli spaces of central curvature connections with fixed determinant with only minor abuse of language.
**Proposition 3.2.8.** (Condition for smoothness of the moduli space) Let \( \alpha \in A(P) \) be a central curvature connection. If \( H^0(X; d_\alpha^{[\delta,g]}) = H^2(X; \bar{d}_\alpha^{[\delta,g]}) = 0 \) then \( M(P) \) is a finite-dimensional orbifold at \( [\alpha] \) with tangent space isomorphic to \( H^1(X; d_\alpha^{[\delta,g]}) \). If in addition \( G = U(r) \), then \( M(P) \) is a finite-dimensional manifold in a neighbourhood of \( [\alpha] \).

**Proof.** Fix \( \delta \in A(\det(P)) \). Any \( \alpha \in A_\delta(P) \) splits into semi-simple and central part \( \alpha = \alpha_1 + \pi^*_G \delta \), where \( \alpha_1 \in \Omega^1(P, [\mathfrak{g}, \mathfrak{g}]) \) satisfies \( \pi^* F^{[\delta,\mathfrak{g}]}_\alpha = d\alpha_1 + \frac{1}{2}[\alpha_1, \alpha_1] = 0 \). By standard arguments (as for flat connections), any solution of Sobolev class \( L^s_2 \) with \( 2s > n \) is gauge equivalent to a smooth solution. Hence we can think of \( M(P) \) as the quotient of solutions of class \( L^s_2 \) by the \( L^{s+1}_2 \)-closure of \( \mathcal{G}_0(P) \). In the first step, we will show that the equation \( d\alpha_1 + \frac{1}{2}[\alpha_1, \alpha_1] = 0 \) for \( \alpha_1 \in \Omega^1(P, [\mathfrak{g}, \mathfrak{g}])_s \) cuts out a smooth Banach submanifold whose tangent space at \( \alpha_1 \) is the kernel of \( d^{[\delta,\mathfrak{g}]}_\alpha = d\alpha_1 : \Omega^1(X, P([\mathfrak{g}, \mathfrak{g}])_s \to \Omega^2(X, P([\mathfrak{g}, \mathfrak{g}]))_s = 0 \). Here we denote by subscripts such as \( \Omega(X, P([\mathfrak{g}, \mathfrak{g}]))_s \) the \( L^s_2 \)-completion of spaces of smooth forms such as \( \Omega(X, P([\mathfrak{g}, \mathfrak{g}]))_s \), and moreover choose \( s \geq 2 \). Then by the vanishing of \( H^2(X; d^{[\delta,\mathfrak{g}]}_\alpha) \), the Hodge estimate (7) becomes

\[
\|b\|_{L^{s-1}_2(X)} \leq C(\|d\alpha_1 b\|_{L^{s-2}_2(X)} + \|d^{*}_{\alpha_1} b\|_{L^{s-2}_2(X)} + \|\ast b \|_{L^{s-3/2}_2(\partial X)}).
\]

for all \( b \in \Omega^2(X, P([\mathfrak{g}, \mathfrak{g}]))_{s-1} \). The same estimate holds with \( d\alpha_1 \) replaced by \( d\alpha_1 + \pi^*_\alpha \) for sufficiently small \( \|a\|_{L^2_2(X)} \). Hence, using the Bianchi identity, \( E^{[\delta,\mathfrak{g}]}_{\alpha_1 + \pi^*_\alpha} = 0 \) is equivalent to \( (d^{\ast}_{\alpha_1} + \rho^\ast) F^{[\delta,\mathfrak{g}]}_{\alpha_1 - \pi^*_\alpha} = 0 \). It follows that \( A_\delta(P) \) near \( \alpha \) is the set of sums \( \alpha + \pi^*_\alpha \) where \( \alpha \) is a zero of the map

\[
(12) \quad \Omega^1(X, P([\mathfrak{g}, \mathfrak{g}]))_s \to \text{im}(d^{\ast}_{\alpha_1} + \rho^\ast), \quad a \mapsto (d^{\ast}_{\alpha_1} + \rho^\ast) F^{[\delta,\mathfrak{g}]}_{\alpha_1 + \pi^*_\alpha}.
\]

Here the target

\[
(d^{\ast}_{\alpha_1} + \rho^\ast) \Omega^2(X, P([\mathfrak{g}, \mathfrak{g}]))_{s-1} \subset \Omega^1(X, P([\mathfrak{g}, \mathfrak{g}]))_s \times \Omega^{n-2}(\partial X, P([\mathfrak{g}, \mathfrak{g}]))_{s-\frac{n}{2}}
\]

is closed. Indeed, since \( \text{im}(d_{\alpha_1}) = \ker(d^{\ast}_{\alpha_1} + \rho^\ast) \), we have

\[
\text{im}(d^{\ast}_{\alpha_1} + \rho^\ast) = \text{im}((d^{\ast}_{\alpha_1} + \rho^\ast) d_{\alpha_1}) \cong \text{im}((d_{\alpha_1} + d^{\ast}_{\alpha_1} + \rho^\ast) d_{\alpha_1}),
\]

where the latter isomorphism holds since \( d_{\alpha_1}^{\ast} = 0 \). Now \( \text{im}(d_{\alpha_1}) \) is a closed subspace of \( \Omega^2(X, P([\mathfrak{g}, \mathfrak{g}]))_{s-1} \), by (6), and the Hodge estimate implies that the image of \( \text{im}(d_{\alpha_1}) \) under \( d_{\alpha_1} + d^{\ast}_{\alpha_1} + \rho^\ast \) is also closed, by [32, Lemma A.1.1].

With this setup, the linearized operator \((d^{\ast}_{\alpha_1} + \rho^\ast) d_{\alpha_1} \) of (12) is surjective and has kernel that of \( d_{\alpha_1} \) since \( \ker(d^{\ast}_{\alpha_1} + \rho^\ast) = \text{im}(d_{\alpha_1}) \). Hence the implicit function theorem provides a smooth map from the formal tangent space \( \ker(d_{\alpha_1}) \) to its complement in (6)

\[
\ker(d_{\alpha_1}) \to \text{im}(d_{\alpha_1} | \ker \rho^\ast), \quad a \mapsto b(a)
\]

such that the map

\[
\ker(d_{\alpha_1}) \to A_\delta(P), \quad a \mapsto \alpha + \pi^*(a + b(a))
\]

is a local chart for \( A_\delta(P) \).
To construct the orbifold structure on the quotient, we show that $\alpha + \ker(d^*_\alpha \oplus \rho^\#)$ is a local slice for the action of $G_0(P)$. The assumption $H^0(d_{[\xi]}^*) = 0$ ensures that the local slice conditions $d^*_\alpha((a + b(a)) = 0, *(a + b(a))|_X = 0$ are transverse to $\ker(d_{\alpha_1})$. To see that the stabilizers are finite, note that any automorphism of a bundle with connection is determined by its restriction to a point, and so the stabilizer embeds into $G$. Since it is discrete by vanishing of $H^0$, it is a finite subgroup of $G$. Standard arguments (e.g. [10, Lemma 4.2.4]) show that $M(P) = A_\delta(P)/G_0(P)$ is Hausdorff, hence $M(P)$ is a smooth orbifold.

Note that the subgroup $G^0_{\text{central}}(P) \subset G_0(P)$ given by the central automorphisms of the form $g(p) = pz(p)$ for some $z : P \rightarrow Z$ acts trivially on $A_\delta(P)$. On the other hand, $G^0_{\text{central}}(P)$ is finite since (by connectedness of $X$ and $G$) the map $Z \equiv Z^{ss} \subset Z^{ss} = Z \cap G_0$ is constant. Hence we can also realize $M(P)$ as the quotient $A_\delta(P)/(G_0(P)/G^0_{\text{central}}(P))$. As a consequence, if all stabilizers of $\alpha$ are contained in $G^0_{\text{central}}(P)$, then $M(P)$ is a smooth manifold near $[\alpha]$.

In the case $G = U(r)$, this condition is satisfied since the vanishing of $H^0(X; d_{\alpha}^{[\xi, g]})$ implies that there are no non-central automorphisms of $\alpha$ in $G_0(P)$. Indeed, suppose $g \in G_0(P) \setminus G^0_{\text{central}}(P)$ is in the stabilizer of $\alpha$. Then the induced connection $\alpha$ on $P C^r$ has an automorphism $g$ whose action on some (and hence any) fiber has at least two different eigenvalues. Thus we obtain a splitting $P C^r = E_1 \oplus E_2$ into unitary bundles of ranks $r_1 + r_2 = r$, which is preserved by $\alpha$. Hence there is a one-parameter family $(e^{it/r_1}1_{E_1} \oplus e^{-it/r_2}1_{E_2})_{t \in \mathbb{R}}$ of automorphisms of $\alpha$ on $P C^r$. Since $P$ is the frame bundle of $P C^r$, this would imply a one-parameter family of non-central automorphisms of $\alpha$ on $P$, contradicting $H^0(X; d_{\alpha}^{[\xi, g]}) = 0$.

The moduli spaces $M(P)$ can be described in terms of spaces of representations of the fundamental group, up to a twist which is determined by the determinant bundle:

**Definition 3.2.9.** (Adjoint moduli spaces) Let $\text{Ad}(G) = G/Z$ and $Z^{ss} = Z \cap G_0$. (In case $G = U(r)$ this means $\text{Ad}(G) = PSU(r) = SU(r)/e^{2\pi iZ/r}1_{\text{Id}}$ and $Z^{ss} = e^{2\pi iZ/r}1_{\text{Id}}$.) We denote by $M_{\text{Ad}}(X) := \text{Hom}(\pi_1(X), \text{Ad}(G))/\text{Ad}(G)$.

The moduli space of representations of $\pi_1(X)$ in $\text{Ad}(G)$, up to conjugacy. As explained in e.g. Atiyah-Bott [2], the space $M_{\text{Ad}}(X)$ is the union of the moduli spaces $M_{\text{Ad}}(P)$ of flat connections on $P$ as $P$ ranges over $\text{Ad}(G)$-bundles.

**Lemma 3.2.10.** (Relation to moduli of flat bundles) Let $P \rightarrow X$ be a principal $G$-bundle. The moduli space $M(P)$ has the structure of a topological principal $\text{Hom}(\pi_1(X), Z^{ss})$-bundle over a component $M_{\text{Ad}}(P)$ of $M_{\text{Ad}}(X)$. In particular $M(P)$ is compact.

**Proof.** We can express the space of central curvature connections with fixed determinant as

$$A_\delta(P) = \{\alpha \in \Omega^1(P, g)^G | \alpha(\xi|_P) = \xi \forall \xi \in g, \det(\alpha) = \delta, F_{\alpha}^{[\xi, g]} = 0\},$$
where $\Omega^1(\ldots)^G$ denotes the equivariant forms and $\xi_P \in \text{Vect}(P)$ is the vector field generated by $\xi \in \mathfrak{g}$. The exact sequence $Z \to G \to G/Z =: \text{Ad}(G)$ induces a splitting of Lie algebras $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$. The $\mathfrak{z}$-component of any $\alpha \in A_\delta(P)$ is uniquely determined by $\det(\alpha) = \delta$, hence the projection to the $[\mathfrak{g}, \mathfrak{g}]$-component induces a homeomorphism between $A_\delta(P)$ and the space

$$\{ \alpha \in \Omega^1(P, [\mathfrak{g}, \mathfrak{g}])^G | \alpha(\xi_P) = \xi \forall \xi \in [\mathfrak{g}, \mathfrak{g}], \alpha(\xi_P) = 0 \forall \xi \in \mathfrak{z}, da + \frac{1}{2}[\alpha \wedge \alpha] = 0 \}$$

of flat, equivariant $[\mathfrak{g}, \mathfrak{g}]$-forms on $P$ that are horizontal with respect to $\mathfrak{z}$. The latter descends to $\Omega^1(P/Z, [\mathfrak{g}, \mathfrak{g}])$, hence we obtain a homeomorphism between $A_\delta(P)$ and the space of flat $\text{Ad}(G)$-connections on $P/Z$,

$$A_{\text{flat}}(P/Z) = \{ \alpha \in \Omega^1(P/Z, [\mathfrak{g}, \mathfrak{g}])^{G/Z} | \alpha(\xi_{P/Z}) = \xi \forall \xi \in [\mathfrak{g}, \mathfrak{g}], F_\alpha = 0 \}.$$ 

While $G_0(P)$ acts on both these spaces, the larger group of gauge transformations $G(P/Z)$ acts on $A_{\text{flat}}(P/Z)$. Since $[G, G] \to G/Z$ is a finite cover with fiber $Z^{ss}$, the quotient $G(P/Z)/G_0(P)$ can be identified with $\text{Hom}(\pi_1(X), Z^{ss})$ by viewing the group of gauge transformations as sections of the bundle $P \times_G (G/Z)$ resp. $P \times_G [G, G]$. On the other hand, any gauge transformation in $G(P/Z)$ that fixes a flat connection in $A_{\text{flat}}(P/Z)$ automatically lifts to an element of $G_0(P)$. Hence the projection $M(P) \to M_{\text{Ad}}(P) := A_{\text{flat}}(P/Z)/G(P/Z)$ is a $\text{Hom}(\pi_1(X), Z^{ss})$-principal bundle. Finally, $M_{\text{Ad}}(P)$ is homeomorphic to a component of the representation space $M_{\text{Ad}}(X)$, given by those representations which lift to connections on $P$. Compactness follows since $\text{Ad}(G), Z^{ss}$ are compact and $\pi_1(X)$ is finitely generated. \qed

### 3.2.4. Moduli spaces for compositions of bordisms.

In this section we study moduli spaces for bordisms of bundles, and the associated gluing law. Suppose that $(Y, \phi)$ is a compact oriented connected bordism between compact oriented connected manifolds $X_\pm$. Let $(Q, \psi)$ be a bundle bordism between bundles $P_\pm \to X_\pm$, that is, a bordism from $P_-$ to $P_+$ with the structure of a $G$-bundle over $Y$ such that the identification $\psi: \partial Q \to \overline{P}_- \cup P_+$ is an isomorphism of $G$-bundles.

**Definition 3.2.11.** (a) (Restriction to the boundary) Denote the pullback map on connections by $\rho: A(Q) \to A(P_-) \times A(P_+)$. The map $\rho$ preserves the central curvature and fixed determinant conditions (with respect to appropriate restrictions) and is gauge equivariant, hence it induces a map $\rho: M(Q) \to M(P_-) \times M(P_+)$.

(b) (Correspondences associated to bordisms) We denote by $L(Q)$ the image of $M(Q)$ in $M(P_-) \times M(P_+)$,

$$L(Q) = \rho(M(Q)) \subset M(P_-) \times M(P_+).$$

Thus $L(Q)$ is a topological correspondence from $M(P_-)$ to $M(P_+)$, that is, a subspace of the product. In our applications, $L(Q)$ will be a Lagrangian correspondence between symplectic manifolds $M(P_-), M(P_+)$.\n
**Remark 3.2.12.** (Compatibility with maps to representation varieties) If $x_\pm \in X_\pm, y \in Y$ are base points, then the of the boundary $X_\pm \to Y$ induces a map $\pi_1(X_\pm, x_\pm) \to \pi_1(Y, y)$ depending on a choice of path from $x_\pm$ to $y$ up to conjugacy. The map $\rho$ is
compatible with the bundle structure described in Lemma 3.2.10 in the sense that the diagram

\[
\begin{array}{ccc}
M(Q) & \rightarrow & M(P_-) \times M(P_+) \\
\downarrow & & \downarrow \\
M_{Ad}(Y) & \rightarrow & M_{Ad}(X_-) \times M_{Ad}(X_+) 
\end{array}
\]

commutes and the top arrow is \(\text{Hom}(\pi_1(Y), Z^{ss})\)-equivariant, where \(\text{Hom}(\pi_1(Y), Z^{ss})\) acts via the restriction homomorphism to \(\text{Hom}(\pi_1(X_-), Z^{ss}) \times \text{Hom}(\pi_1(X_+), Z^{ss})\) on the right side of the diagram. Hence \(L(Q)\) is contained in the image of the bottom arrow.

**Proposition 3.2.13.** (Sufficient condition for the correspondence associated to a bordism to be embedded) Suppose that one of \(\pi_1(X_+) \rightarrow \pi_1(Y)\), \(\pi_1(X_-) \rightarrow \pi_1(Y)\) is surjective. Then \(\rho : M(Q) \rightarrow L(Q)\) is a bijection. Furthermore, suppose that

(a) for any \([\alpha] \in M(P_\pm)\) we have \(H^0(X_\pm; d_\alpha^{[\alpha, \emptyset]}) = 0\), \(H^2(X_\pm; d_\alpha^{[\alpha, \emptyset]}) = 0\);
(b) for any \([\alpha] \in M(Q)\) we have \(H^0(Y; d_\alpha^{[\emptyset, \emptyset]}) = 0\), \(H^2(Y; d_\alpha^{[\emptyset, \emptyset]}) = 0\), and \(H^1(Y, \partial Y; d_\alpha^{[\emptyset, \emptyset]}) = 0\).

Then \(\rho : M(Q) \rightarrow L(Q) \subset M(P_-) \times M(P_+)\) is an embedding.

**Proof.** Suppose that one of \(\pi_1(X_\pm) \rightarrow \pi_1(Y)\) is surjective. Then the maps on the corresponding representation varieties for \(\text{Ad}(G)\) and \(Z^{ss}\) in the diagram above are injective. Since the maps on the fiber and the base are injective, \(\rho\) is injective. To prove the second statement, note that the assumptions imply that the moduli spaces \(M(Q), M(P_-), M(P_+)\) are smooth by Lemma 3.2.8. The linear map for \(M(Q) \rightarrow M(P_-) \times M(P_+)\) is then \(H^1(Y; d_\alpha^{[\emptyset, \emptyset]}) \rightarrow H^1(X_-; d_\alpha^{[\emptyset, \emptyset]}) \times H^1(X_+; d_\alpha^{[\emptyset, \emptyset]})\). This is injective by the assumption \(H^1(Y, \partial Y; d_\alpha^{[\emptyset, \emptyset]}) = 0\). Since any injective immersion of a compact space is an embedding, this completes the proof.

**Definition 3.2.14.** (Compositions of bundle bordisms) For bundle bordisms \(Q_0 \rightarrow Y_0\) from \(P_0 \rightarrow X_0\) to \(P_1 \rightarrow X_1\) and \(Q_1 \rightarrow Y_1\) from \(P_1 \rightarrow X_1\) to \(P_2 \rightarrow X_2\), the composition \(Q_0 \cup_{P_1} Q_1\) is defined using equivariant collar neighborhoods of \(P_1\) in \(Q_0\) and \(Q_1\), and is independent up to bundle isomorphism of the choices.

**Remark 3.2.15.** (Correspondences for a composition of bundle bordisms) Pullback of connections under \(\pi : Q_0 \cup_{P_1} Q_1 \rightarrow Q_0 \cup_{P_1} Q_1\) induces a map

\[
\pi^* : M(Q_0 \cup_{P_1} Q_1) \rightarrow M(Q_0) \times_{M(P_1)} M(Q_1)
\]
where the fiber product is defined using the restriction maps \( M(Q_j) \to M(P_1), j = 0, 1 \). The map \( \pi^* \) fits into a commutative diagram

\[
\begin{array}{ccc}
M(Q_0 \cup_{P_1} Q_1) & \xrightarrow{\pi^*} & M(Q_0) \times_{M(P_1)} M(Q_1) \\
\rho \downarrow & & \downarrow \rho \circ \pi M(P_0) \times M(P_2) \\
L(Q_0 \cup_{P_1} Q_1) & \xrightarrow{id_{M(P_0)} \times M(P_2)} & L(Q_0) \circ L(Q_1)
\end{array}
\]

The bottom map has image in \( L(Q_0) \circ L(Q_1) \) by the following argument: Each choice of lift \([\alpha] \in M(Q_0 \cup_{P_1} Q_1)\) of a given \(([\eta_0], [\eta_2]) \in L(Q_0 \cup_{P_1} Q_1) \subset M(P_0) \times M(P_2)\) provides an element \([\eta_1] = [\alpha|_{P_1}] \in M(P_1)\) such that \(([\eta_0], [\eta_1]) = \rho([\alpha|_{Q_0}]) \in L(Q_0)\) and \(([\eta_1], [\eta_2]) = \rho([\alpha|_{Q_1}]) \in L(Q_1)\). Hence \(([\eta_0], [\eta_2])\) lies in

\[ L(Q_0) \circ L(Q_1) = \{ (\zeta_1, \zeta_2) \mid \exists \zeta_1 \in M(P_1) : (\zeta_0, \zeta_1) \in L(Q_0), (\zeta_1, \zeta_2) \in L(Q_1) \} \]

This ends the remark.

Proposition 3.2.13 gives conditions for the left arrow in the commutative diagram above to be a diffeomorphism; the following Proposition does the same for the top arrow; and the right arrow will be discussed in more specific cases in Section 3.4.

The following composition property of the moduli spaces is a nonabelian version of the Mayer-Vietoris principle, which applies directly to the abelian, trivial determinant case, when the moduli spaces are cohomology groups. For simplicity of notation we restrict to the case of gluing along two boundary components, although the same discussion holds e.g. for gluing along two boundary components of the same connected bundle bordism.

**Proposition 3.2.16.** (Conditions for composition of bordisms to give compositions of correspondences) For bundle bordisms \( Q_0 \to Y_0 \) from \( P_0 \to X_0 \) to \( P_1 \to X_1 \) and \( Q_1 \to Y_1 \) from \( P_1 \to X_1 \) to \( P_2 \to X_2 \), suppose that the following conditions hold:

(a) for any \([\alpha] \in M(P_1)\) we have \( H^0(X_1; d_\alpha^{[\theta, \vartheta]}) = 0 \), \( H^2(X_1; d_\alpha^{[\theta, \vartheta]}) = 0 \);

(b) for any \([\alpha] \in M(Q_j)\) for \( j = 0 \) or \( j = 1 \) we have

\[ H^0(Y_j; d_\alpha^{[\theta, \vartheta]}) = 0, \quad H^2(Y_j; d_\alpha^{[\theta, \vartheta]}) = 0, \]

and the restriction map on stabilizers

\[ \{ g \in G_0(Q_j) \mid g^* \alpha = \alpha \} \to \{ g \in G_0(P_1) \mid g^* \alpha|_{P_1} = \alpha|_{P_1} \} \]

is surjective for \( j = 0 \) or \( j = 1 \);

(c) for any \([\alpha] \in M(Q_0 \cup_{P_1} Q_1)\) we have

\[ H^0(Y_0 \cup_{X_1} Y_1; d_\alpha^{[\theta, \vartheta]}) = 0, \quad H^2(Y_0 \cup_{X_1} Y_1; d_\alpha^{[\theta, \vartheta]}) = 0, \]
and the difference of the restriction maps on homology

\[ H^1(Y_0; d_0^{\phi \theta}) \times H^1(Y_1; d_0^{\phi \theta}) \to H^1(X_1; d_{\alpha|X_1}), (\eta_0, \eta_1) \mapsto \eta_0|_{X_1} - \eta_1|_{X_1} \]

is surjective;

(d) the restriction map \( r : G_0(Q_j) \to G_0(P_1) \) is surjective for \( j = 0 \) or \( j = 1 \).

Then all the spaces in (13) are smooth and the map \( \pi^* \) is a diffeomorphism.

Proof. First we check that \( \pi^* \) is a bijection. To see that \( \pi^* \) is injective suppose that \( \alpha, \alpha' \) are connections on \( Q_0 \cup_{P_1} Q_1 \) mapping to the same element of \( M(Q_0) \times M(Q_1) \) under \( \pi^* \). Then there exists \( g = (g_0, g_1) \in G_0(Q_0) \times G_0(Q_1) \) such that \( g^*(\pi^* \alpha) = \pi^* \alpha' \). Then \( (g_1|_{P_1})^* \alpha|_{P_1} = (g_0|_{P_1})^* \alpha|_{P_1} \), and hence \( (g_1|_{P_1})^{-1}(g_0|_{P_1}) \) is an automorphism of \( \alpha|_{P_1} \). By assumption (b), we may assume without loss of generality that it extends to an automorphism \( h \) of \( \pi^* \alpha|_{Q_1} \). Then \( g_0|_{P_1} = hg_1|_{P_1} \) and so \( g_0 \) and \( hq_1 \) glue together to a gauge transformation \( \tilde{g} \) of \( Q_0 \cup_{P_1} Q_1 \) of class \( W^{1,p} \) for any \( p < \infty \). Since \( \tilde{g}^* \alpha = \alpha' \), and \( \alpha, \alpha' \) are smooth, \( \tilde{g} \) is also smooth.

To see that \( \pi^* \) is surjective, let \((\alpha_0, \alpha_1)\) represent an element of \( M(Q_0) \times_{M(P_1)} M(Q_1) \), and hence the restrictions \( \alpha|_{P_1}, \alpha|_{P_1} \) are gauge equivalent by some \( g_1 \in G(P_1) \). By assumption (d), we may assume without loss of generality that \( g_1 \) extends over \( Q_1 \), and so after gauge transformation of \((\alpha_0, \alpha_1)\) that \( \alpha|_{P_1} = \alpha|_{P_1} \). We may also suppose that the determinant connection \((\delta_0, \delta_1)\) is the pull-back \( \pi^* \delta \) of a smooth determinant connection \( \delta \) on \( Q_0 \cup_{P_1} Q_1 \). Then, after another gauge transformation \( Q_1 \), we may assume that the normal components of \( \alpha_0 \) and \( \alpha_1 \) agree on \( P_1 \). Then the curvature equation implies that \((\alpha_0, \alpha_1) = \pi^* \alpha \) for some smooth connection \( \alpha \) on \( Q_0 \cup_{P_1} Q_1 \).

Finally we check that \( \pi^* \) is a diffeomorphism. Let \( \alpha \) represent an element of \( M(Q_0 \cup_{P_1} Q_1) \) and \((\alpha_0, \alpha_1) = \pi^* \alpha \). By vanishing of \( H^0 \) and \( H^2 \) for \( \alpha_0, \alpha_1, \alpha \), and \( \alpha|_{P_1} \), the moduli spaces \( M(Q_0), M(Q_1), M(Q_0 \cup_{P_1} Q_1), \) and \( M(P_1) \) are smooth at the given points. The fiber product \( M(Q_0) \times_{M(P_1)} M(Q_1) \) is smooth because the surjectivity of the difference map in assumption (c) implies transversality. To see that the linearization of \( \pi^* \) is an isomorphism, note that the Mayer-Vietoris long exact sequence gives a short exact sequence in first cohomology (since \( H^0(X_1; d_{\alpha|_{P_1}}) = 0 \) and \( H^2(Y_0 \cup_{X_1} Y_1; d_\alpha) = 0 \))

\[ 0 \to H^1(Y_0 \cup_{X_1} Y_1; d_\alpha) \to H^1(Y_0; d_{\alpha_0}) \oplus H^1(Y_1; d_{\alpha_1}) \to H^1(X_1; d_{\alpha|_{P_1}}) \to 0. \]

The induced isomorphism

\[ H^1(Y_0 \cup_{X_1} Y_1; d_\alpha) = H^1(Y_0; d_{\alpha_0}) \times_{H^1(X_1; d_{\alpha|_{P_1}})} H^1(Y_1; d_{\alpha_1}) \]

is equal to the linearization of (13), since the latter is the tangent space to the fiber product. It follows that (13) is a diffeomorphism. \( \square \)

3.3. Moduli spaces for surfaces. In this Section we make the first step towards constructing a symplectic-valued field theory via Theorem 2.2.14 by associating to any bundle any compact connected oriented surface a moduli space of constant curvature connections.
Remark 3.3.1. Let $X$ be a compact, connected, oriented surface without boundary and $P \to X$ a $G$-bundle.

(a) (Symplectic structure on the affine space of connections) The affine space $\mathcal{A}(P)$ carries a canonical weakly symplectic $\mathcal{G}(P)$-invariant two-form $\omega$ which induces the Hodge pairing (9). At a point $\alpha \in \mathcal{A}(P)$ it is given by

$$\omega_\alpha : \Omega^1(X, P(g)) \times \Omega^1(X, P(g)) \to \mathbb{R}, \quad (a_1, a_2) \mapsto \int_X \langle a_1 \wedge a_2 \rangle.$$  

Here $\langle a_1 \wedge a_2 \rangle$ denotes the form in $\Omega^2(X)$ obtained by combining wedge product and the inner product on $P(g)$. Weakly symplectic means weakly non-degenerate and closed, where

(i) weakly non-degenerate means that for every non-zero $a_1 \in \Omega^1(X, P(g))$ there exists an $a_2 \in \Omega^1(X, P(g))$ so that $\omega_\alpha(a_1, a_2) \neq 0$;

(ii) closed means $d\omega = 0$ where the de Rham operator is defined on a Sobolev completion as in [25, Section 5.3].

The action of $\mathcal{G}_0(P)$ is Hamiltonian with moment map

$$\mathcal{A}(P) \hookrightarrow \Omega^0(X, P([g, g]))^\vee, \quad \alpha \mapsto \int_X \langle F_\alpha^{[g,g]} \wedge \cdot \rangle$$

in the sense that

(i) $\omega$ is $\mathcal{G}_0(P)$-invariant,

(ii) the map (15) is equivariant with respect to the coadjoint action of $\mathcal{G}_0(P)$ on $\Omega^0(X, P([g, g]))^\vee$, and

(iii) the infinitesimal action (4) of $\xi \in \Omega^0(X, P([g, g]))$ is the Hamiltonian vector field of the function obtained from (15) by pairing: For all $\eta \in \Omega^1(X, P(g))$, we have

$$\omega(d_\alpha \xi, \eta) = \int_X \langle d_\alpha \xi \wedge \eta \rangle = \int_X \langle \xi \wedge -d_\alpha \eta \rangle = -\frac{d}{dt}\bigg|_{t=0} \int_X \langle F_\alpha^{[g,g]} \wedge \eta \wedge \xi \rangle.$$  

(b) (Moduli of connections as a symplectic quotient [2]) The connections with fixed determinant $\det(\alpha) = \delta$ form a symplectic submanifold $\det^{-1}(\delta)$ of $\mathcal{A}(P)$. Indeed, the tangent space $\Omega^1(X, P([g, g]))$ of $\det^{-1}(\delta)$ is a symplectic subspace of $\Omega^1(X, P(g))$. The action of $\mathcal{G}_0(P)$ preserves $\det^{-1}(\delta)$ and hence the moduli space of fixed determinant central curvature connections

$$M(P) = \{ \alpha \in \mathcal{A}(P) \mid \det(\alpha) = \delta, \ F_\delta^{[g,g]} = 0 \}/\mathcal{G}_0(P) = \mathcal{A}_\delta(P)/\mathcal{G}_0(P)$$

can be viewed as a symplectic quotient.

Using this point of view we establish the existence of a symplectic structure on $M(P)$. Recall, see e.g. [38], that the dual Coxeter number $c$ associated to simple $G$ is the positive integer $c := \langle \rho, \alpha_0 \rangle + 1$ where $\alpha_0$ is the highest root and $\rho$ the half-sum of positive roots.

Proposition 3.3.2. (Symplectic nature of the moduli space) Let $X$ be a compact, connected, oriented surface without boundary and $P \to X$ a $G$-bundle. Suppose that for any $[\alpha] \in M(P)$ we have $H^0(X; d^{[g,g]}_\delta) = 0$. Then $M(P)$ is a compact symplectic orbifold of dimension $(2g(X) - 2) \dim G_0$, where $g(X)$ is the genus of $X$. 
If furthermore \([G, G]\) is simple, then \(M(P)\) is monotone with monotonicity constant 1/2c.

Proof. Lemma 3.2.10 proves compactness, and the smoothness assertions were proven below, that the first Chern class

\[ 16 \]

3.2.8

33

3.3.3

since

26

24

3.3.7

3.2.8

Remark

Moreover,

L

of class

L

Sobolev class

T o establish the symplectic structure, note that after Sobolev completion (taking Sobolev class \(L^s_2\) with \(s > 1\)) we may take \(\mathcal{A}(P)\) to be an affine Banach space, equipped with the form \(\omega\) above. The action of \(G_0(P)\) extends to the Sobolev completion of \(G_0(P)\) of class \(L^{s+1}_2\) on connections of class \(L^s_2\). Hence the moment map equation (16) holds for \(\xi \in \Omega^0(X, P([g, g]))\) of class \(L^{s+1}_2\) and \(\eta \in \Omega^1(X, P([g, g]))\) of class \(L^s_2\). Recall that \(\omega\) is translationally invariant, hence closed in the sense of forms on Banach manifolds as in [25, Section 5.3]. Next, we restrict \(\omega\) to \(\mathcal{A}_\delta(P) \subset \mathcal{A}(P)\), which is a Banach submanifold for \(s \geq 2\) by Proposition 3.2.8. Since the de Rham operator commutes with pull-back, \(\omega|_{\mathcal{A}_\delta(P)}\) is again a closed two-form. Finally, the fact that \(\omega|_{\mathcal{A}_\delta(P)}\) is \(G_0(P)\)-invariant and vanishes on the vertical vectors \(d_a\xi, \xi \in \Omega^0(X, P([g, g]))\) by (16) implies that it is the pull-back of a two-form on the quotient \(M(P)\). The form on the quotient is closed because the de Rham operator commutes with pull-back and the map \(\mathcal{A}_\delta(P) \to M(P)\) is a submersion. Non-degeneracy follows from the identity

\[
\int_X \langle a \wedge *a \rangle = \|a\|^2_{L^2}, \quad \forall a \in H^1(X; d_a) \cong T_{[a]}M(P).
\]

The assertion on monotonicity is a variation on the Drezet-Narasimhan theorem [12, Theorem F], see also [26] and [24]. A symplectic proof is given in [33], where it is shown, using the description in Remark 3.3.7 below, that the first Chern class \(c_1(M(P))\) is 2c times the symplectic class \(\omega\). \(\square\)

Now restricting to \(G = U(r)\) we have the following main result of this section.

**Theorem 3.3.3.** (Properties of moduli spaces of connections on a surface) Let \(P\) be a principal \(U(r)\)-bundle of degree \(d\) coprime to \(r\) over a compact, connected, oriented surface \(X\) without boundary.

(a) If \(X\) has genus \(g(X) \geq 1\), then the moduli space \(M(P)\) is a nonempty compact symplectic manifold of dimension \((2g(X) - 2)(r^2 - 1)\), with even minimal Chern number and monotonicity constant 1/2r.

(b) If \(X\) has genus \(g(X) = 0\), then \(M(P) = \emptyset\).

Moreover, \(M(P)\) is always connected and simply-connected; in particular if \(g(X) = 1\) then \(M(P) = \pt\) is a point.

**Remark 3.3.4.** (Non-coprime case) Suppose we are in the situation of Theorem 3.3.3, except that \(r, d\) do have a common divisor. Then for \(g(X) \geq 1\) there exist connections \(\alpha \in \mathcal{A}_\delta(P)\) with non-discrete automorphism groups, i.e. \(H^0(X; d_a^{[g, g]}) \neq 0\). For \(g(X) = 0\) we have \(M(P) = \emptyset\) iff \(d/r \in \mathbb{Z}\), and otherwise \(M(P)\) is a point.
Proof of Theorem 3.3.3 and Remark 3.3.4. The smoothness assertions are standard, see for example [36], but we give a proof for convenience. The smooth manifold structure follows from Proposition 3.2.8 if we can show vanishing of $H^0(X; d_\alpha^0)$ (and hence of $H^2(X; d_\alpha^0)$) by Poincaré duality for all $\alpha \in A_0(P)$.

Suppose that $\alpha \in A_0(P)$ is fixed by an infinitesimal gauge transformation with trivial determinant, defining a splitting of the corresponding vector bundle $E := P(C') = E_1 \oplus E_2$ into unitary bundles of ranks $k, r - k \geq 1$. Then the adjoint bundle $E^\vee \otimes E$ contains the direct summands $E_j^\vee \otimes E_j$ and the identity section $Id_E = Id_{E_1} \oplus Id_{E_2}$ is the sum of identities in the summands. The curvature also splits $F_\alpha = F_{\alpha|E_1} \oplus F_{\alpha|E_2}$ and is a multiple of the identity $F_\alpha = Id_E \eta$ for some $\eta \in \Omega^2(X)$. This implies $F_{\alpha|E_1} = Id_{E_1} \eta$ and hence the first Chern number of $E_1$ equals

$$\langle c_1(E_1), [X] \rangle = \frac{r \text{rk}(E_1)}{r} \langle c_1(E), [X] \rangle = \frac{kd}{r}.$$ 

Since the first Chern number of $E_1$ should be an integer and $k < r$, this implies that $r$ and $d$ cannot be coprime. That is, if $r, d$ are coprime then no such splitting and hence no infinitesimal automorphism can occur.

Now the symplectic structure and monotonicity follow from Proposition 3.3.2 together with dual Coxeter number $c = r$ for $U(r)$. The assertion on the minimal Chern number is proved in [12]. The claims on (non)emptiness, connectedness and simply-connectedness follow from the stratification of $A(P)$ established in [2], see especially [2, Theorem 9.12] with $G_0 = SU(r)$ simply-connected. So if $X \cong T^2$ has genus $g = 1$, then $\text{dim } M(P) = 0$ by the dimension formula and connectedness implies that $M(P)$ is a point. See also [44, 5] where moduli spaces of bundles on elliptic curves are investigated more extensively.

Alternatively, the (non)emptiness for general $r, d$ can be seen from the description in Remark 3.3.7 below. If $X \cong S^2$, then $d/r \notin \mathbb{Z}$ implies $M(P) = \emptyset$ since $e = \exp(2\pi id/r)Id_{U(r)} \neq Id_{U(r)}$. On the other hand, $M(P)$ is a single point for $d/r \in \mathbb{Z}$. If $X$ has positive genus $g \geq 1$, then $M(P)$ is nonempty by e.g. Goto’s theorem [18], which states that the commutator mapping $(a_1, b_1, \ldots, a_g, b_g) \rightarrow \prod_{j=1}^g [a_j, b_j]$ is surjective for the semisimple, compact, connected group $G_0$.

Finally, if $r = r'm$ and $d = d'm$ have a common factor $m > 1$, and $g(X) \geq 1$, then by the above the moduli space $M(P')$ for a $U(r')$-bundle $P'$ of degree $d'$ is nonempty. Any representative of $[\alpha'] \in M(P')$ induces a connection on the corresponding vector bundle $E' = P'(C'r')$. Then $\alpha'$ also induces a central curvature connection on $E = \oplus_{i=1}^m E^i$ and hence on the corresponding principal $U(r)$-bundle, which is isomorphic to $P$. The splitting of the bundle implies that the connection has a non-discrete automorphism group.

\[\square\]

Remark 3.3.5. (Holomorphic description of the moduli spaces) The moduli spaces $M(P)$ have a holomorphic description, for $G = U(r)$ due to a famous theorem of Narasimhan-Seshadri [36], generalized to arbitrary groups in Ramanathan’s thesis [39], [40]. However, we never use the holomorphic description.
Lemma 3.3.6. (Symplectomorphisms of moduli spaces induced by bundle isomorphisms) Let $r, d$ be coprime integers.

(a) For any two $U(r)$-bundles $P_0 \to X_0, P_1 \to X_1$ of degree $d$, any bundle isomorphism $\psi : P_0 \to P_1$ covering an orientation-preserving diffeomorphism from $X_0$ to $X_1$ induces a symplectomorphism $\psi^* : M(P_1) \to M(P_0)$ given by pull-back of representatives.

(b) Any two such bundle isomorphisms from $P_0$ to $P_1$ covering the same diffeomorphism from $X_0$ to $X_1$ induce the same symplectomorphism.

(c) If $P_0 \to X, P_1 \to X$ are $U(r)$-bundles of degree $d$ coprime over the same surface, then the moduli spaces $M(P_0), M(P_1)$ are canonically symplectomorphic.

Proof. For manifolds of dimension at most three, bundles are classified up to isomorphism by their first Chern class. For connected surfaces, the Chern class is determined by the degree. Hence if $\pi_j : P_j \to X, j = 0, 1$ are bundles of the same degree then there exists an isomorphism $\phi : P_0 \to P_1$ with $\pi_0 \circ \phi = \pi_0$. This induces a map from $\mathcal{A}(P_1)$ to $\mathcal{A}(P_0)$, given on the level of tangent spaces by the pull-back of basic forms $\phi^* : \Omega^1(P_1, g)_{\text{basic}} \to \Omega^1(P_0, g)_{\text{basic}}$. This pull-back acts symplectically because for $a, b \in \Omega^1(P_1, g)_{\text{basic}}$ we have

$$\int_{X_1} \pi_{1,*}(a \wedge b) = \int_{X_0} \pi_{0,*}(\phi^* a \wedge \phi^* b)$$

since $\phi$ covers an orientation-preserving diffeomorphism from $X_0$ to $X_1$. Moreover pull-back by $\phi$ maps $\mathcal{A}_0(P_1)$ to $\mathcal{A}_0(P_0)$ and is equivariant with respect to the gauge actions of $G_0(P_1)$ and $G_0(P_0) = \phi^* G_0(P_1)$. Hence $\phi^*$ descends to a symplectomorphism $M(P_1) \to M(P_0)$. Any two such isomorphisms covering the same diffeomorphism differ by a gauge transformation of $P_1$. Since any gauge transformation of $P_1$ induces the identity on $M(P_1)$ and $M(P_1)$ is independent of the choice of determinant connection (see Remark 3.2.7), the identification $M(P_1) \to M(P_0)$ depends only on the choice of diffeomorphism from $X_0$ to $X_1$. \(\square\)

Remark 3.3.7. (Moduli spaces as representations of fundamental group of the punctured surface) Atiyah-Bott [2] provide a description of $M(P)$ in terms of representations, which makes Lemma 3.2.10 more precise in the case of bundles over surfaces. In that case, $M(P)$ can be identified with the moduli space of $G_0$-representations of the fundamental group of the punctured surface $X - \{x\}$, whose value on a small loop around $x$ is equal to a certain central element $z \in G_0$. Here $z$ is determined by the choice of the bundle $P \to X$, for example in the case that $G = U(r)$ and $\langle c_1(P), [X] \rangle = d$ we have $z = \exp(2\pi i d/r) \text{Id}_{U(r)}$. More explicitly, the fundamental group of $X - \{x\}$ can be described as the free group on $2g$ generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ with the product of commutators $\prod_{j=1}^g [\alpha_j, \beta_j]$ being the class of the loop around the base point $x$. Thus we have a homeomorphism

$$M(P) \cong \left\{ (a_1, b_1, \ldots, a_g, b_g) \in G_0^{2g} \mid \prod_{j=1}^g [a_j, b_j] = z \right\}/G_0.$$

In particular, for $X \cong S^2$ we have
The map is given by choosing a determinant connection whose curvature is concentrated near a base point, and mapping each \([\alpha]\) to the representation of \(\pi_1(X - \{x\})\) given by the holonomies.

3.4. Moduli spaces for three-dimensional elementary bordisms. In this Section we make the second step towards constructing a symplectic-valued field theory in dimension \(2 + 1\) via Theorem 2.2.14, by defining a partial functor on elementary bordisms.

**Definition 3.4.1.** (Correspondences for bordisms) Fix \(G = U(r)\) for some positive integer \(r\) and let \(P_{\pm} \to X_{\pm}\) be principal \(U(r)\)-bundles of the same degree \(d\) coprime to \(r\) over surfaces \(X_{\pm}\) as in Section 3.3. Let \((Q, \psi)\) be a bundle bordism from \(P_- \to X_-\) to \(P_+ \to X_+\) as in Section 3.2.4; in particular \(Y\) is a 3-dimensional bordism from \(X_-\) to \(X_+\). The image of \((Q)\) under pullback as in Definition 3.2.11 is

\[
L(Q) := \rho(M(Q)) \subset M(P_-)^- \times M(P_+).
\]

**Remark 3.4.2.** (Low genus cases)

(a) If either \(X_+\) or \(X_-\) has genus 0, then the corresponding moduli space \(M(P_{\pm})\) is empty by Theorem 3.3.3, and therefore so is \(L(Q)\).

(b) If either \(X_+\) or \(X_- = \mathbb{T}^2\) has genus 1, then the corresponding moduli space \(M(P_{\pm})\) is a point. Furthermore, by a special case of the following Theorem, \(L(Q)\) is a Lagrangian correspondence from a point to \(M(P_{\pm})\) and can also be viewed as Lagrangian submanifold of \(M(P_{\pm})\).

**Theorem 3.4.3.** (Lagrangians for elementary bordisms) If \(Y\) is a compression body as in Definition 2.2.6 and \(Q \to Y\) is a principal \(G\)-bundle then \(\rho : M(Q) \to L(Q) \subset M(P_-)^- \times M(P_+)\) is a Lagrangian embedding. If moreover \(Y\) is an elementary bordism as in Definition 2.2.1 (b), then \(L(Q)\) is simply-connected and spin.

**Proof.** By Theorem 3.3.3, \(H^0(X_{\pm}; d_{\alpha}[\alpha, \overline{\alpha}]) \cong H^2(X_{\pm}; d_{\alpha}[\alpha, \overline{\alpha}]) = 0\) for any \([\alpha] \in M(P_{\pm})\).

Lemma 3.4.4 (a) below shows that the assumptions of Proposition 3.2.13 are satisfied, which ensures that \(\rho\) is an embedding. It is Lagrangian since the image of \(T_{[\alpha]}M(Q) \cong H^1(Y; d_{\alpha})\) in \(T_{([\alpha]_{P_-}, [\alpha]_{P_+})}(M(P_-)^- \times M(P_+)) \cong H^1(\partial Y; d_{\alpha}|_{\partial Y})\) is a Lagrangian subspace by Corollary 3.2.5.

Next, suppose that \(Y\) is elementary and without loss of generality \(X_-\) is the surface of lower genus. By Lemma 3.4.5 below, \(L(Q)\) is a principal \(G_0\)-bundle over the moduli space \(M(P_-)\), which is simply-connected by Theorem 3.3.3. Since the fiber and base are simply-connected, so is \(L(Q)\). Now let \(\pi\) denote the projection of \(L(Q)\) onto \(M(P_-)\). By Lemma 3.4.5 the tangent bundle of \(L(Q)\) is the sum of the pull-back of \(TM(P_-)\) with a vector bundle with fiber \(g_0\) and structure group \(G_0\).

By Theorem 3.3.2 the canonical bundle of \(M(P_-)\) admits a square root. Since \(G_0\) is simply-connected, the representation \(g_0\) is spin and hence so is the vertical part of \(TL(Q)\). Hence \(L(Q)\) is also spin. \(\square\)
Lemma 3.4.4. Suppose that $Y$ is a compression body as in Definition 2.2.6 between surfaces $X_-, X_+$ of positive genus, and the genus of $X_+$ is larger than or equal to the genus of $X_-$. Then the following holds.

(a) The map of fundamental groups $\pi_1(X_+) \to \pi_1(Y)$ is a surjection and the map $\pi_1(X_-) \to \pi_1(Y)$ is an injection.

(b1) For any connection $\alpha \in A_{\delta}(Q)$, the map induced by restriction $H^0(Y, d_\alpha^{[0,\delta]}) \to H^0(X_+, d_{\alpha|_P}^{[0,\delta]})$ is an isomorphism and the map $H^0(Y, d_\alpha^{[0,\delta]}) \to H^0(X_-, d_{\alpha|_{P_-}}^{[0,\delta]})$ is an injection;

(b2) the map $H^1(Y, d_\alpha^{[0,\delta]}) \to H^1(X_+, d_{\alpha|_P}^{[0,\delta]})$ is injective and the map $H^1(Y, d_\alpha^{[0,\delta]}) \to H^1(X_-, d_{\alpha|_{P_-}}^{[0,\delta]})$ is surjective;

(b3) the map $H^2(Y, d_\alpha^{[0,\delta]}) \to H^2(X_+, d_{\alpha|_P}^{[0,\delta]})$ is an isomorphism and the map $H^2(Y, d_\alpha^{[0,\delta]}) \to H^2(X_-, d_{\alpha|_{P_-}}^{[0,\delta]})$ is a surjection.

Proof. For simplicity of notation assume that $\partial Y + \overline{X_-} \sqcup X_+$ so that the boundary identification $\phi$ is the identity. By assumption, $Y$ supports a Morse function that is maximal on $X_+$, minimal on $X_-$, and only has critical points of index 1. Then $Y$ deformation retracts onto the union of $X_-$ and the stable manifolds of the critical points, which are intervals. Since $X_-$ is connected, the attaching points of these intervals may be homotoped so that they are equal, so that $Y$ is homotopic to the wedge product of $X_-$ and a collection of circles. By Seifert-van Kampen, $\pi_1(Y)$ is the free product of $\pi_1(X_-)$ and a copy of $\mathbb{Z}$ for each critical point. Hence $\pi_1(X_-) \to \pi_1(Y)$ is an injection. Similarly $Y$ is obtained from $X_+$ by attaching 2-handles, and Seifert-van Kampen in this case presents $\pi_1(Y)$ as the quotient of $\pi_1(X_+)$ by actions of $\mathbb{Z}$ for each critical point, given by the image of the attaching cycle in $\pi_1(X_+)$.

The assertions on zeroth cohomology follow from its interpretation as infinitesimal automorphisms of a flat connection, which are the Lie algebras of the centralizers of its holonomy groups, and part (a). The remaining assertions follow from the Mayer-Vietoris principle for the cohomology of $d_\alpha$. To apply it, we consider $Y$ as the the union $U \sqcup V$ where $U$ is a neighborhood of $X_\pm$ and $V$ is a neighborhood of the union of stable manifolds of the critical points, so that $U \cap V$ is a neighborhood of the attaching cycles. Then $H^0(V; d_\alpha^{[0,\delta]}) \to H^0(X_+ \cap V; d_\alpha^{[0,\delta]})$ is surjective, since any infinitesimal automorphism of a flat connection over the boundary of a disk extends over the interior. By the long exact sequence $H^1(Y; d_\alpha^{[0,\delta]}) \to H^1(X_+; d_{\alpha|_P}^{[0,\delta]})$ is injective. On the other hand $H^1(X_- \cap V; d_\alpha^{[0,\delta]})$ vanishes, since $X_- \cap V$ is homotopic to a finite set of points and the Poincaré lemma holds for cohomology with twisted coefficients. Hence $H^1(Y; d_\alpha^{[0,\delta]}; d_{\alpha|_P}^{[0,\delta]})$ surjects onto $H^1(X_-; d_{\alpha|_{P_-}}^{[0,\delta]})$. Since $H^1(X_- \cap V; d_{\alpha|_P}^{[0,\delta]})$ and $H^2(X_- \cap V; d_{\alpha|_{P_-}}^{[0,\delta]})$ vanish, $H^2(Y; d_\alpha^{[0,\delta]})$ is isomorphic to $H^2(X_+; d_{\alpha|_P}^{[0,\delta]}) \cong H^0(X_-; d_{\alpha|_{P_-}}^{[0,\delta]})$. Finally surjectivity onto $H^2(X_+; d_{\alpha|_P}^{[0,\delta]})$ follows from the exact sequence. \[\square\]
In order to understand the topology of $L(Q)$ we use an explicit description in terms of representations of the extended fundamental group as in Remark 3.3.7.

**Lemma 3.4.5.** Suppose that $Y$ is an elementary bundle bordism from a surface $X_-$ of genus $g$ to a surface $X_+$ of genus $g + 1$. Let $Q$ be a $G$-bundle over $Y$. Then in the description of Remark 3.3.7, the Lagrangian $L(Q) \subset M(P_-)^{-} \times M(P_+)$ is the set of pairs of equivalence classes

$$(\{a_1, b_1, \ldots, a_g, b_j\}, [a_1, b_1, \ldots, a_g, b_j, \text{Id}]) \in G_0^{2g} / G_0 \times G_0^{2g+2} / G_0$$

satisfying the relation $\prod_{j=1}^g [a_j, b_j] = z$. In particular, the projection $\pi : L(Q) \rightarrow M(P_-)$ gives $L(Q)$ the structure of a $G_0$-bundle over $M(P_-)$, whose vertical tangent space is the associated bundle

$$\ker(D\pi) \cong \left\{ [a_1, b_1, \ldots, a_g, b_j, \xi] \in G^{2g}_0 \times g | \prod_{j=1}^g [a_j, b_j] = z, \xi \in g \right\} / G_0.$$

**Proof.** By Lemma 3.2.10, the moduli space $M(Q)$ is a $\text{Hom}(\pi_1(Y), Z^w)$ fiber bundle over a component of $\text{Hom}(\pi_1(Y), \text{Ad}(G)) / \text{Ad}(G)$. We may choose generators $\alpha_1, \beta_1, \ldots, \alpha_{g+1}, \beta_{g+1}$ of $\pi_1(X_+)$ with the single relation $\prod_{j=1}^{g+1} [\alpha_j, \beta_j] = \text{Id}_{\pi_1(X_+)}$, such that

(a) the maps of fundamental groups in Lemma 3.4.4 (a) realize $\pi_1(Y)$ as quotient of $\pi_1(X_+)$ by the subgroup generated by $\beta_{g+1}$, and

(b) $\pi_1(X_-)$ as the subgroup of $Y$ generated by $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$.

Now $\text{Hom}(\pi_1(Y), \text{Ad}(G)) / \text{Ad}(G)$ and $\text{Hom}(\pi_1(Y), Z)$ are the subsets of the spaces of representations of $\pi_1(X_+)$ given by requiring that the representation vanishes on $\beta_{g+1}$. It follows that $L(Q)$ is equal to the space of representations mapping $\beta_{g+1}$ to some element $h$ of $Z$.

Now we assume that $Q$ is equipped with a determinant connection whose curvature vanishes except on a small neighborhood of a path connecting base points in $X_\pm$, disjoint from the stable and unstable manifolds. In particular, the curvature vanishes on a disk with boundary $\beta_{g+1}$. Since $h$ is equal to the holonomy around the vanishing cycle $\beta_{g+1}$, $h$ must equal the identity. \qed

This finishes the proof of Theorem 3.4.3, which together with the subsequent Lemma allows to define the partial symplectic-valued field theory for elementary bordisms for any fixed coprime $r, d$.

**Lemma 3.4.6.** (Existence and uniqueness of Lagrangians associated to elementary bordisms) For $d, r$ coprime positive integers:

(a) For any elementary bordism $Y$ from $X_-$ to $X_+$ there exists a $U(r)$-bundle $Q \rightarrow Y$ such that both $Q|_{X_-}$ and $Q|_{X_+}$ have degree $d$.

(b) The Lagrangian correspondence $L(Q) \subset M(P_-)^{-} \times M(P_+)$ is independent of the choice of $Q$ under the canonical symplectomorphisms of Lemma 3.3.6.

(c) The Lagrangian correspondence associated to the trivial bordism $[0, 1] \times X$ is the diagonal $L([0, 1] \times P) = \Delta_{M(P)} \subset M(P)^{-} \times M(P)$. 
Proof. Let $Y$ be an elementary bordism from $X_-$ to $X_+$. A simple computation using e.g. cellular homology shows that since $Y$ is elementary, $H^2(Y) \cong H^2(X_+) \cong \mathbb{Z}$. Taking a $U(r)$ bundle induced from a $U(1)$-bundle with first Chern class $d$ shows that the bundle $Q$ above exists. Since the homotopy groups $\pi_1(SU(r))$ and $\pi_2(SU(r))$ are trivial, the bundle $Q$ is unique up to isomorphism. These arguments imply that $M(Q)$ is well-defined and $L(Q)$ is independent of the choice of $Q$. In the case of a product bordism $Q = [0,1] \times P$, we take $\delta$ to be the pull-back of a determinant connection on $P_{\pm}$. Since any connection is gauge-equivalent to one vanishing on the fibers of $[0,1] \times P \to P$, pull-back defines an isomorphism $M(Q) \cong M(P_{\pm})$ which is embedded in $M(P_-) \times M(P_+)$ via the diagonal. \hfill \square 

3.5. Cerf moves for moduli spaces. This Section provides the final steps in the construction of a symplectic-valued field theory in dimension $2+1$ via Theorem 2.2.14. As before, we fix $G = U(r)$ for some $r \in \mathbb{N}$ and fix a degree $d \in \mathbb{Z}$ coprime to $r$.

Theorem 3.5.1. (Field theory via moduli spaces) For $d, r$ coprime positive integers there exists a unique $2+1$-dimensional connected symplectic-valued field theory $\Phi : \text{Cob}_{2+1}^0 \to \text{Symp}^{\#}_{1/2}$ that associates

(a) to each surface $X$ the moduli space $\Phi(X) = M(P) =: M(X)$ constructed in Section 3.3 using a choice of $U(r)$-bundle $P \to X$ of degree $d$,

(b) to each elementary bordism $[Y]$ from $X_-$ to $X_+$, the equivalence class $\Phi(Y) = [L(Y)]$, $L(Y) := L(Q) \subset M(X_-)^- \times M(X_+)$ constructed in Section 3.4 using a choice of $U(r)$-bundle $Q \to Y$ pulling back to the given bundles over $X_{\pm}$.

Remark 3.5.2. (Generalized correspondences for bordisms) In particular Theorem 3.5.1 associates to any morphism $[Y]$ from $X_-$ to $X_+$ in $\text{Cob}_{2+1}^0$ an equivalence class of generalized Lagrangian correspondences as follows. Let

$$Y = Y_1 \cup X_1 \cup Y_2 \cup X_2 \ldots \cup X_{r-1} \cup Y_r$$

be a Cerf decomposition into elementary bordisms. Associated to each piece $Y_k$ is a Lagrangian correspondence $L(Y_k)$ from $M(X_{k-1})$ to $M(X_k)$, where by convention $M(X_0) = M(X_-)$ and $M(X_r) = M(X_+)$. Hence our construction associates a generalized Lagrangian correspondence from $M(X_-)$ to $M(X_+)$ to the Cerf decomposition of $Y$,

$$L((Y_k)_{k=1,\ldots,r}) := (L(Y_1), \ldots, L(Y_r)).$$

Theorem 3.5.1 implies that the equivalence class $\Phi(Y) = [L((Y_k)_{k=1,\ldots,r})] = [(L(Y_1), \ldots, L(Y_r))] = \Phi(Y_1) \circ \ldots \circ \Phi(Y_r)$ as an element in $\text{Hom}_{\text{Symp}^{\#}_{1/2}, \text{M}}(M(X_-), M(X_+))$ is independent of the choice of Cerf decomposition. This ends the remark.
We already proved in Theorem 3.4.3 that the Lagrangian correspondences involved are simply-connected and spin. To complete the proof of the theorem it suffices to show that the Cerf relations of Theorem 2.2.14 are satisfied for the Lagrangian correspondences constructed in the previous section. This follows from two lemmas:

**Lemma 3.5.3.** Any diffeomorphism equivalence of Cerf decompositions as in Definition 2.2.8 induces a symplectomorphism equivalence of generalized Lagrangian correspondences, see Definition 3.1.5.

**Proof.** For every orientation-preserving diffeomorphism of elementary bordisms $f : Y \to Y'$ we denote by $f_{\pm} : X_{\pm} \to X'_{\pm}$ their restrictions to the boundary. These lift to the bundles $f_{\pm} : P_{\pm} \to P'_{\pm}$ (uniquely up to gauge transformation) and so induce symplectomorphisms $\Psi(f_{\pm}) : M(X_{\pm}) \to M(X'_{\pm})$. Similarly any bundle isomorphism $Q \to Q'$ of bundles over $Y, Y'$ extending $f_{\pm}$ induces an identification $L(Y) \to L(Y')$, commuting with the inclusions into $M(X_{\pm}) \times M(X_{\pm})$ resp. $M(X'_{\pm}) \times M(X'_{\pm})$. Hence these correspondences fit into a commutative diagram

$$
\begin{array}{ccc}
M(X_{-}) & \xrightarrow{\Psi(f_{-})} & M(X'_{-}) \\
L(Y) \downarrow & & \downarrow L(Y') \\
M(X_{+}) & \xrightarrow{\Psi(f_{+})} & M(X'_{+})
\end{array}
$$

**Lemma 3.5.4.** For each of the Cerf moves in Theorem 2.2.12, the conditions in Proposition 3.2.16 hold.

**Proof.** Note that by assumption $H^0$ and $H^2$ of $d^{(g, \theta)}_\alpha$ vanish on $Y_i, Y_{i+1}, X_{i-1}, X_i, X_{i+1}$. Recall $G_0 = [G, G] = SU(r)$ and $G_0(P)$ is the corresponding group of gauge transformations. The restriction map $G_0(Q) \to G_0(P_{-}) \times G_0(P_{+})$ is surjective, since $\pi_1(G_0), \pi_2(G_0)$, vanish. It follows that there is no obstruction to extending over the 1, 2 and 3-dimensional cells of the bordism. It remains to check the surjectivity of the difference of the restriction maps $H^1(Y_i \cup Y_{i+1}; d^{(g, \theta)}_\alpha) \to H^1(X_i; d^{(g, \theta)}_\alpha)$. By Mayer-Vietoris this is equivalent to injectivity of $H^2(Y_i \cup Y_{i+1}, d^{(g, \theta)}_\alpha) \to H^2(Y_i \cup Y_{i+1}, d^{(g, \theta)}_\alpha)$, and in particular holds if $H^2(Y_i \cup Y_{i+1}, d^{(g, \theta)}_\alpha) = 0$. We check this in each case.

**Critical point cancellation:** Suppose that two adjacent pieces $Y_i, Y_{i+1}$ are replaced by a single piece $Y_i \cup Y_{i+1}$ diffeomorphic to a cylinder $[-1, 1] \times X_{i-1}$. Then every point in $M(Y_i \cup Y_{i+1}) \cong M(X_{i-1})$ has vanishing $H^2$, as required.

**Gluing in a cylinder:** An elementary bordism $Y_i$ and cylindrical bordism $Y_{i+1}$ is replaced with another elementary bordism diffeomorphic to $Y_i$. Then every point in $M(Y_i \cup Y_{i+1}) \cong M(Y_i)$ has vanishing $H^2$, as required.

**Reversing order of critical points:** This move can be broken down into stages, where in the first stage two elementary bordisms are replaced by a compression body, and in the second stage the compression body is replaced by two other elementary
bordisms. If the two critical points have the same index, then $H^2$ vanishes by Lemma 3.4.4. In the case of differing index, suppose that the two-handle is attached first. Then the surjectivity claim holds by Lemma 3.4.4, since $X_i$ has larger genus than $X_{i+1}$ and $Y_i$ is a compression body. Thus $H^2$ vanishes. By Mayer-Vietoris the surjectivity property in Proposition 3.2.16 holds for the decomposition corresponding to attaching a one-handle first, as well. 

Remark 3.5.5. (Holonomy description of Cerf moves) We suppose that $X_{i-1}$ is a surface of genus $g$ and $X_i$ is a surface of genus $g + 1$.

(a) (Critical point cancellation) By Remark 3.4.5 $L(Y_i)$ resp. the transpose of $L(Y_{i+1})$ may be identified with the set of pairs of orbits

$$([a_1, b_1, \ldots, a_g, b_g], [a'_1, b'_1, \ldots, a'_{g+1}, b'_{g+1}])$$

such that

$$b'_{g+1} = 1, \quad a_j = a'_j, \quad b_j = b'_j, \quad j = 1, \ldots, g$$

resp.

$$a'_{g+1} = 1, \quad a_j = a'_j, \quad b_j = b'_j, \quad j = 1, \ldots, g.$$ 

Thus the composition $L(Y_i) \circ L(Y_{i+1})$ is the diagonal.

(b) (Critical point switch) We suppose that initial decomposition corresponds to attaching a one-handle and then attaching a two-handle. The surface $X_i$ is obtained from $X'_{i+1}$ by attaching two one-handles, so that the attaching one-cycles in $X_i$ correspond to disjoint generators of $\pi_1(X_i)$ which we may take to equal $\alpha_1$ resp. $\alpha_{g+1}$. Then

$$L(Y_i) = \left\{ ([a_2, b_2, \ldots, a_g, b_g], [a'_1, b'_1, \ldots, a'_{g+1}, b'_{g+1}]) \quad a_1 = 1, \quad a_j = a'_j, \quad b_j = b'_j, \quad j = 2, \ldots, g \right\}.$$ 

Furthermore

$$L(Y_{i+1}) = \left\{ ([a'_1, b'_1, \ldots, a'_{g+1}, b'_{g+1}], [a''_1, b''_1, \ldots, a''_g, b''_g]) \quad a'_{g+1} = 1, \quad a'_j = a''_j, \quad b'_j = b''_j, \quad j = 1, \ldots, g \right\}.$$ 

Thus $L(Y_i \cup Y_{i+1})$ is embedded into $M(X_{i-1})^{-} \times M(X_{i+1})$, the composition $L(Y_i) \circ L(Y_{i+1})$ is transversal and equal to $L(Y_i \cup Y_{i+1})$. The arguments for the other order of attaching, and the other indices of critical points, is similar.

Applying Proposition 3.2.16 and Theorem 2.2.14 yields Theorem 3.5.1.

4. Functors for bordisms via quilts

In this section, we associate to any three-dimensional bordism as above a functor between the Donaldson-Fukaya categories of the moduli spaces of connections associated to the incoming and outgoing surfaces, using quilted Floer theory.
4.1. Quilted Floer theory. In the paper [50] we generalized Lagrangian Floer theory to the setting of Lagrangian correspondences.

**Definition 4.1.1.** (Quilted Floer homology) Suppose that
(a) \( M = (M_0, M_1, \ldots, M_m) \) is a sequence of compact simply-connected monotone symplectic manifolds with the same monotonicity constant \( \tau \) and even minimal Chern number;
(b) \( L = (L_{01}, L_{12}, \ldots, L_{(m-1)m}) \) is a sequence of compact simply-connected oriented spin Lagrangian correspondences from \( M_0 \) to \( M_1, M_1 \) to \( M_2 \) etc.
(c) \( L_0 \subset M_0, L_m \subset M_m \) are compact simply-connected oriented spin Lagrangian submanifolds.

The *quilted Floer cohomology* of a generalized Lagrangian correspondence \( L = (L_1, L_2, \ldots, L_m) \) is defined by

\[
HF(L_0, L_{01}, \ldots, L_m) := \left\{ \begin{array}{ll}
HF(L_0 \times L_{12} \times \ldots, L_{01} \times L_{23} \times \ldots \times L_m), & \text{m even} \\
HF(L_0 \times L_{12} \times \ldots, L_m, L_{01} \times L_{23} \times \ldots \times L_{(m-1)m}), & \text{m odd}
\end{array} \right.
\]

The differential can be viewed as a count of quilted holomorphic strips with boundary in \( L_0, L_m \) and seams in \( L_{(j-1)j}, j = 1, \ldots, m \).

**Remark 4.1.2.** (Effect of geometric composition on quilted Floer cohomology) We proved in [51] that the quilted Floer cohomology groups behave well under composition: if for some \( j \), the composition \( L_{(j-1)(j+1)} := L_{(j-1)j} \circ L_{j(j+1)} \) is smooth and embedded into \( M_{j-1}^c \times M_{j+1} \) then the quilted Floer cohomology group is unchanged up to isomorphism by replacing the pair \( L_{(j-1)j}, L_{j(j+1)} \) with \( L_{(j-1)(j+1)} \).

An example is shown in Figure 3, for the case \( m = 2 \).

![Figure 3. Shrinking the middle strip](image)

In [47] we associated to any Lagrangian correspondence \( L_{01} \) from \( M_0 \) to \( M_1 \) a functor \( \Phi(L_{01}) \) from a version of the Donaldson-Fukaya category of \( M_0 \) to that of \( M_1 \):

**Definition 4.1.3.** Let \( M \) be a \( \tau \)-admissible symplectic manifold as in Definition 3.1.7.

(a) (Generalized Lagrangian submanifolds) A *generalized Lagrangian submanifold* of \( M \) is a generalized Lagrangian correspondence from a point to \( M \), that is, a sequence \( L_{-s(-s+1)}, \ldots, L_{(-1)0} \) of correspondences from \( M_{-s} = \text{pt} \)
that $M_0 = M$. We say that a generalized Lagrangian correspondence satisfies a certain property (simply-connected, compact, etc.) if each correspondence in the sequence satisfies that property.

(b) (extended Donaldson-Fukaya category) Let $\text{Don}^\#(M)$ denote the category whose

(i) objects are compact, oriented, simply-connected generalized Lagrangian submanifolds of $M$;

(ii) morphisms from an object $L_0$ to an object $L_1$ are quilted Floer homology classes:

$$\text{Hom}(L_0, L_1) = HF(\alpha, L_1)$$

the Lagrangian Floer homology of $\alpha, L_1$ in $M$ constructed in [50];

(iii) composition and identities are defined by counting holomorphic quilts with strip-like ends and Lagrangian boundary and seam conditions as in [47].

Note that we dropped the gradings included in the definition of the Donaldson-Fukaya category in [47].

(c) (Functors for Lagrangian correspondences) Let $M_0, M_1$ be $r$-admissible symplectic manifolds. For any compact, oriented, simply-connected correspondence $L_{01} \subset M_0^{-} \times M_1$ the functor

$$\Phi(L_{01}) : \text{Don}^\#(M_0) \to \text{Don}^\#(M_1)$$

is defined on objects by

$$(L_{-(s+1)}, \ldots, L_{(-1)0}) \mapsto (L_{-(s+1)}, \ldots, L_{(-1)0}, L_{01}).$$

On morphisms $\Phi(L_{01})$ is defined by counting holomorphic quilts of the form in Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{The Lagrangian correspondence functor $\Phi(L_{01})$ on morphisms}
\end{figure}

Remark 4.1.4. (Functors for geometric compositions) The composition result on quilted Floer homology generalizes to the categorical setting as follows [51]: Suppose that $L_{01}$ and $L_{12}$ are Lagrangian correspondences as above such that $L_{01} \circ L_{12}$ is smooth and embedded into $M_0^{-} \times M_2$, and simply-connected with spin structure.
induced from that of $L_{01}, L_{12}$ and $M_1$. Then $\Phi(L_{01}) \circ \Phi(L_{12})$ is isomorphic to $\Phi(L_{01} \circ L_{12})$.

Recall that Cat is the category whose objects are small categories and morphisms are isomorphism classes of functors. Remark 4.1.4 implies the following:

**Theorem 4.1.5.** For any $\tau > 0$ the maps $M \mapsto \text{Don}^\#(M), \quad [L_{01}] \mapsto [\Phi(L_{01})]$ define a categorification functor $\text{Don}^\# : \text{Symp}^\# \to \text{Cat}$.

**Proof.** In [47] we constructed the functors $\Phi(L_{01})$ and proved that if $L_{01}, L_{01}'$ are related by a composition equivalence (\ref{composition-equivalence}) then the functors $\Phi(L_{01}), \Phi(L_{01}')$ are isomorphic. It follows that the equivalence class $[\Phi(L_{01})]$ is independent of the choice of representative $L_{01}$. $\square$

### 4.2. Floer field theory

We can now prove the main result of the paper:

**Theorem 4.2.1.** (Floer field theory) For any coprime positive integers $r, d$, the maps for surfaces $X$ given by $X \mapsto \Phi(X) := \text{Don}^\#(M(X))$ and for elementary morphisms $[Y]$ given by $[Y] \mapsto \Phi([Y]) := [\Phi(L(Y))]$ extend to a connected field theory $\Phi : \text{Cob}^\#_{0,1} \to \text{Cat}$ with values in the category $\text{Cat}$ of (small categories, isomorphism classes of functors).

**Proof.** Compose the functors $\Phi$ of Theorem 3.5.1 with the functor $\text{Don}^\#$ of Theorem 4.1.5 with $\tau = 1/2r$. $\square$

**Remark 4.2.2.**

(a) (Group-valued invariants for bordisms) Let $L_\pm \subset M(X_\pm)$ be simply-connected compact oriented spin Lagrangian submanifolds. For a bordism $Y$ from $X_-$ to $X_+$ we denote by $[L(Y)]$ the corresponding equivalence class of generalized Lagrangian correspondences from $M(X_-)$ to $M(X_+)$. The quilted Floer cohomology group

$$HF(Y; L_-, L_+) := \text{Hom}(\Phi([L(Y)]), L_-, L_+) = HF(L_-, L(Y), L_+)$$

is a topological invariant of the bordism $Y$ up to isomorphism. These Floer cohomology groups are relatively $\mathbb{Z}_4$-graded, since the minimal Chern number of the moduli space $M(X)$ of bundles is 2 by Proposition 3.3.2.

(b) (Excision property) The functors $\Phi(Y)$ satisfy a gluing property for gluing: if $Y_{02} = Y_{01} \circ Y_{12}$ then

$$\Phi([L(Y_{02})]) = \Phi([L(Y_{01})]) \circ \Phi([L(Y_{12})]).$$

This can be regarded as a generalization of Floer’s excision property [6]. Namely if $X_1$ has genus zero resp. one then $\Phi(Y)$ is the trivial functor resp. ignoring torsion we have

$$\text{Hom}(\Phi(Y_{02})L_-, L_+) \cong \text{Hom}(\Phi(Y_{01})L_-, \text{pt}) \otimes \text{Hom}(\Phi(Y_{12})\text{pt}, L_+).$$

Equation (18) follows immediately from Theorem 3.3.3 since in this case $M(X_1)$ is empty resp. a point and so the Floer complex is the product of Floer complexes for $\Phi(Y_{01})L_-, \text{pt}$ and $\Phi(Y_{12}), L_+$. 
4.3. **Invariants for three-manifolds with circle-valued Morse functions.** In this section we define invariants for three-manifolds which admit circle-valued Morse functions. Let \( Y \) be a compact connected oriented three-manifold containing a non-separating embedded compact connected oriented surface \( X \). By replacing \( X \) with two copies of itself we obtain a bordism \( Y \) from \( X \) to itself. Any adapted Morse function on \( Y \) gives rise to a circle-valued Morse function on \( Y \).

**Definition 4.3.1.** The cyclic Floer homology of \((Y, X)\) is the quilted Floer cohomology \( \text{HF}(Y, X) := \text{HF}(L(Y), \Delta_{M(X)}) \).

**Remark 4.3.2.** (a) The results of Gay-Kirby [16] show that any two decompositions of \( Y \) into elementary bordisms are related by Cerf moves. It follows that the equivalence class of the generalized correspondence \((\Delta_{M(X)}, L(Y))\) depends only on the homotopy class of the circle-valued Morse function. Furthermore, any homotopy class of maps to the circle inducing a surjection of fundamental groups has a representative which is a Morse function with connected fibers, and so gives rise to such a presentation of \( Y \).

(b) The homotopy class of the circle-valued Morse functions is the first cohomology class on \( Y \), via the identification of \( S^1 \) with the first Eilenberg-Maclane space; thus the invariant \( \text{HF}(Y, X) \) if defined, depends only on the choice of first cohomology class.

(c) If \( Y = S^1 \times X \) is the trivial fiber bundle with fiber \( X \) then \( \text{HF}(Y, X) \) is isomorphic to the quantum cohomology \( \text{HF}(\Delta_{M(X)}) \cong \text{QH}(M(X)) \) of the moduli space \( M(X) \) of central curvature bundles with fixed determinant on \( X \). More generally, for any fiber bundle we may take for \( L(Y) \) the graph of a symplectomorphism, in which case \( \text{HF}(Y, X) \) is equivalent to the periodic Floer cohomology by proof of the Atiyah-Floer conjecture for fibrations by Dostoglou-Salamon [11]. Partial results towards a correspondence with instanton Floer homology in the circle-valued Morse case are given in Duncan [13] and Lipyanskiy [28].

4.4. **Invariants for closed 3-manifolds.** The invariant associated to a bundle over a closed three-manifold, considered as a bordism from the empty surface to itself is trivial. Indeed, such a three-manifold admits a Cerf decomposition where one of the surfaces is a sphere and the moduli space of bundles of coprime rank and degree is empty in genus zero. The following device suggested by Kronheimer and Mrowka [23] gives non-trivial invariants of closed three-manifolds, possibly with trivial first Betti number.

**Definition 4.4.1.** (a) (Torus-summed three-manifolds) Given a closed oriented three-manifold \( Y \) let \( \overline{Y} := Y \#([-1, 1] \times T^2) \) denote the torus-summed bordism obtained as connected sum of \( Y \) with a product bordism \([-1, 1] \times T^2 \) between tori \( T^2 \).

(b) (Torus-summed Floer homology) The torus-summed Lagrangian Floer homology for rank \( r \) and coprime degree \( d \) of a closed three-manifold \( Y \) is

\[
\overline{\text{HF}}(Y) := \text{Hom}(\Phi(\overline{Y}) \text{pt}, \text{pt}) = \text{HF}(L(Y_1), \ldots, L(Y_r))
\]

for some choice of Cerf decomposition \( \overline{Y} = Y_1 \cup X_1 \cup \ldots \cup X_{r-1} Y_r \).
Example 4.4.2. (Torus-summed Lagrangian Floer homology for connect sums of $S^2 \times S^1$) Suppose that $r = 2$, and for some positive integer $n$ the three-manifold $Y$ is the connected sum of $n$ copies of $S^2 \times S^1$. Then $\overline{\mathcal{HF}}(Y) = H(S^3)^n$. Indeed, consider a Morse function on $Y$ which splits $Y \#([-1, 1] \times T^2)$ into compression bodies $Y_\pm$, each with $n$ critical points of index one resp. two. Each compression body $Y_\pm$ is a bordism between a surface $X$ of genus $n + 1$ and a surface of genus 1. The Lagrangians $L(Y_\pm) \cong SU(2) \cong (S^3)^n$ are identical and given in the holonomy description by

$$L(Y_\pm) = \{a_1 = \ldots = a_n = 1\} \subset \{\prod_{i=1}^{n+1} [a_i, b_i] = -1\}.$$ 

By definition $\overline{\mathcal{HF}}(Y) = HF(L(Y_-), L(Y_+))$. As in Biran and Cornea [3, Proposition 6.1.4], if $L$ is a Lagrangian product of spheres, then $HF(L, L)$ is either 0 or isomorphic to the ordinary cohomology $H(L)$; this is a consequence of the spectral sequence computing Floer cohomology starting from the singular cohomology, as in Oh [37], [7]. By Albers [1, Corollary 2.11], $HF(L, L; \mathbb{Z}_2)$ is non-trivial if the class $[L]_{\mathbb{Z}_2} \in H(M(X); \mathbb{Z}_2)$ of $L$ is non-zero. The class of $L(Y_\pm) \cong (S^3)^n$ is non-zero in $H(M(X); \mathbb{Z}_2)$. Indeed, $L$ intersects the Lagrangian $L' \cong (S^3)^n$, given by $b_1 = \ldots = b_n = 1$ transversally in a single point, namely the unique (up to conjugation) pair $a_{n+1}, b_{n+1}$ with $[a_{n+1}, b_{n+1}] = 1$. Hence $HF(L, L; \mathbb{Z}_2)$ is non-zero, which implies that $HF(L, L)$ is non-zero by the universal coefficient theorem. Putting everything together implies the claim.

Remark 4.4.3. (a) (Four-manifold invariants?) Naturally one expects these invariants to extend to invariants of four-manifolds, fitting into a bundle field theory in $2 + 1 + 1$ dimensions: a 2-functor from the 2-category of (2-manifolds, 3-bordisms, 4-bordisms) with bundles to the 2-category of (small categories, functors, natural transformations) via rank $r$ gauge theory, which associates to a bundle bordism $R$ between bordisms $Y_\pm$ a natural transformation $\Pi(R) : \Phi(Y_-) \to \Phi(Y_+)$. 

(b) (Surgery exact triangles) The surgery exact triangles for the theory (in the rank two case) are a consequence of a fibered generalization of a triangle for Dehn twists by Seidel, proved in [46].

References


