ABSTRACT. We prove that small blow-ups or reverse flips (in the sense of the minimal model program) of rational symplectic manifolds with point centers create Floer-non-trivial Lagrangian tori. As applications, we demonstrate the existence of Hamiltonian non-displaceable Lagrangian tori in a plethora of symplectic manifolds, for example, representation varieties of punctured spheres. These results are part of a conjectural description of generators for the Fukaya category of a compact symplectic manifold with an orbifold running of the minimal model program.

Contents

1. Introduction 2
2. Fukaya algebras 8
2.1. $A_\infty$ algebras 10
2.2. Associahedra 11
2.3. Treed holomorphic disks 21
2.4. Transversality 29
2.5. Compactness 38
2.6. Composition maps 41
2.7. Spectral sequence 45
2.8. Divisor equation 46
2.9. Maurer-Cartan moduli space 49
3. Homotopy invariance 55
3.1. $A_\infty$ morphisms 55
3.2. Multiplihedra 60
3.3. Quilted holomorphic disks 64
3.4. Morphisms of Fukaya algebras 68
3.5. Homotopies 72
3.6. Stabilization 81
3.7. Canonical variations 82
4. Fukaya bimodules 86
4.1. $A_\infty$ bimodules 86
4.2. Treed strips 88
4.3. Hamiltonian perturbations 90
4.4. Morphisms 95
4.5. Homotopies 99
1. Introduction

Lagrangian Floer cohomology was introduced in [37] as a version of Morse theory for the space of paths from a Lagrangian submanifold to itself. Despite the fact that the theory was introduced almost thirty years ago, it has been far from clear which Lagrangian submanifolds have well-defined or non-trivial Floer cohomology, or whether a symplectic manifold contains any Lagrangians with non-trivial Floer cohomology at all. The purpose of this paper is to describe a method for producing Floer non-trivial Lagrangians via the surgeries that appear in the minimal model program, namely flips and blow-ups. In particular, if one has knowledge of a sufficiently nice minimal model program for the symplectic manifold (in a suitably modified sense) then one can read off a list of Floer non-trivial Lagrangians which conjecturally generate the Fukaya category.

Existence of Floer non-trivial Lagrangians in compact symplectic manifolds is known only in a special cases, such as symplectic manifolds with anti-symplectic involutions by Fukaya-Oh-Ohta-Ono [44], toric varieties by work of Cho-Oh [29] and Fukaya-Oh-Ohta-Ono [42] and in semi-Fano hypersurfaces by Seidel [97] and Sheridan [101]. The examples given in [29] are Lagrangian tori, which generalize the Clifford torus in complex projective space, while the examples in [97], [101] are Lagrangian spheres which can be seen as singular fibers in a Lagrangian fibration. A connection between the generating Lagrangians and the minimal model program was noted in work with González [47] on quantum cohomology of toric varieties; this gives a more conceptual picture for the generators. Recall that a running of the
mmp for a smooth birationally-Fano projective variety $X$ is a sequence of birational varieties $X = X_0, X_1, \ldots, X_k$ such that each $X_{i+1}$ is obtained from $X_i$ by an mmp transition. The simplest mmp transition is a blow-down of a divisor, or more generally a divisorial contraction. The other operations are flips which are birational isomorphisms on the complement of codimension four exceptional loci. The minimal model program ends (in the birationally-Fano case) with a Mori fiber space. Each blow-up or flip has a center $Z_i$ which is a subquotient of both $X_i$ and $X_{i+1}$. A flip, in the cases considered here, replaces a weighted-projective bundle $\mathbb{P}(N_i^+) \to Z_i$ over the center with another weighted-projective bundle $\mathbb{P}(N_{i+1}^-) \to Z_i$:

$$X_i \xleftarrow{\mathbb{P}(N_i^+)} Z_i \xrightarrow{\mathbb{P}(N_{i+1}^-)} X_{i+1}.$$ 

Each weighted-projective bundle above is obtained by removing the zero section from a vector bundle and taking the quotient by a linear $\mathbb{C}^\times$-action with only the zero section as fixed point set. In the case of a divisorial contraction, the map $\mathbb{P}(N_i^-) \to Z_i$ would be an isomorphism, while in the case of a Mori fibration, one may think of the center as the result of the transition: $\mathbb{P}(N_i^+) \cong X_i, Z_i \cong X_{i+1}$. Sometimes singularities in the spaces $X_i$ are unavoidable; however, in good cases (such as toric varieties) one may assume that the $X_i$’s are smooth orbifolds. In this case we say that $X$ has a smooth running of the mmp. Existence of mmp runnings is known for varieties of low dimension and in many explicit examples. For toric varieties, mmp runnings exist by work of Reid [89].

In this paper we prove the existence of Floer-non-trivial Lagrangian tori located near the exceptional loci for reverse mmp transitions. The version of Floer theory that we use requires the notion of bounding cochain of Fukaya-Oh-Ohta-Ono [41]. We begin with a brief discussion of what we mean by Fukaya algebras. There are several foundational systems which at the moment which are not known to be equivalent (or in some cases, completely written down.) Our foundational system uses stabilizing divisors as in Cieliebak-Mohnke [26]. While this foundational scheme is somewhat less general than the other approaches, it requires no discussion of virtual fundamental classes and so makes the necessary foundational arguments substantially shorter. In particular for any compact oriented spin Lagrangian submanifold equipped with a rank one local system there is a strictly unital Fukaya algebra, defined by counting weighted treed holomorphic disks, independent of all choices up to homotopy equivalence. The resulting moduli space of solutions to the weak Maurer-Cartan equation is independent of all choices and for each solution there exists a well-defined Floer cohomology group. We say that the Lagrangian is weakly-unobstructed if the space of solutions is non-empty and Floer non-trivial if the Floer cohomology group is non-zero for some solution to the weak Maurer-Cartan equation. The first result of this paper is the creation of Floer non-trivial Lagrangians at mmp transitions:

**Theorem 1.1.** Suppose that $X_+$ is a compact rational symplectic manifold obtained from a compact rational symplectic orbifold $X_-$ by a small reverse simple flip or
blow-up with trivial center with multiplicity

\[ m = \dim(QH(X_+)) - \dim(QH(X_-)). \]

In a contractible neighborhood of the exceptional locus there exists a Lagrangian torus \( L \subset X_+ \) with \( m \) distinct local systems with weakly unobstructed and non-trivial Floer cohomology, \( HF(L) \cong H(L) \neq 0 \).

In particular, the small blowup or flip of any such symplectic manifold with trivial center contains a Floer-non-trivial (hence Hamiltonian non-displaceable) Lagrangian torus; related discussions of blow-ups can be found in Smith [103]. In the case of toric manifolds, the Lagrangians are those that appear in the forthcoming work of Abouzaid-Fukaya-Oh-Ohta-Ono, see [2], [42]. We also prove that Floer-non-trivial Lagrangians survive further mmp surgeries in the following sense:

**Theorem 1.2.** Suppose that \( X_0, X_1, \ldots, X_k \) is a symplectic mmp running such that \( X_i \) is obtained from \( X_{i+1} \) and \( X_{i+1} \) is obtained from \( X_{i+2} \) by small reverse flips or blow-ups with non-singular, disjoint exceptional loci. Let \( L_{i+1} \subset X_{i+1} \) denote the Floer-non-trivial Lagrangian created by Theorem 1.1. Then for each of the brane structures above the image of \( L_{i+1} \) in \( X_i \) is weakly unobstructed and Floer-non-trivial.

As applications of Theorems 1.1 and 1.2 we show that various symplectic manifolds contain Hamiltonian non-displaceable Lagrangian tori.

**Example 1.3.** (Toric manifolds) In the case of a toric manifolds Theorems 1.1, 1.2 reproduce some results of Fukaya et al [42]. Recall that a toric variety is a normal variety with an action of a complex torus with an open orbit, see for example Cox-Little-Schenck [31]. An equivariant projective embedding of a smooth toric variety induces a symplectic structure and a Hamiltonian action of the unitary part of the complex torus. Hamiltonian torus actions with Lagrangian orbits are classified by a theorem of Delzant [32]. Let \( X \) be a smooth projective toric variety, \( T \) the unitary torus acting on \( X \), and \( \Psi : X \to t^\vee \) a moment map induced by a compatible symplectic structure. The **moment polytope** of \( X \) is the image

\[ P = \Psi(X) \subset t^\vee. \]

We write

\[ P = \{ \lambda \in t^\vee \mid \langle \lambda, \nu_j \rangle \geq c_j, \quad j = 1, \ldots, k \} \]

where \( \nu_j \in t\mathbb{Z} \) are the minimal lattice vectors that are inward pointing normal vectors to the facets of \( P \), and \( c_j \) are constants determining their position. The condition that \( X \) is smooth implies that the normal vectors \( \nu_j \) meeting each vertex form a lattice basis. Delzant’s classification theorem states that [32] the moment polytope and generic stabilizer classify compact connected Hamiltonian torus actions with Lagrangian torus orbits completely. Floer non-triviality of toric moment fibers was studied in Fukaya et al. [42], [43]. As a corollary of the main result Theorem 1.1 we have the following: Suppose that a compact toric manifold \( X \) is obtained by a small
reverse flip or blow-up and $t$ is the parameter representing the size of the exceptional locus $E \subset X$. Let $\lambda \in P$ denote the unique point satisfying

$$\langle \lambda, \nu_j \rangle - c_j = t, \quad \forall j, \quad \Psi^{-1}(\{\langle \lambda, \nu_j \rangle = c_j\}) \cap E \neq \emptyset$$

where $\nu_j$ ranges over facets whose inverse image intersects the exceptional locus $E$. Then the fiber $L = \Psi^{-1}(\lambda)$ is a Lagrangian torus with weakly unobstructed and non-trivial Floer cohomology.

In Figure 1 we show the case of a twice blow-up of the product of projective lines. Fibers corresponding to the blowups are those over the darkly shaded points shown in the Figure. These results are substantially weaker than the results in [42], [43], which give a whole collection of Floer non-trivial tori. For example, in the example in Figure 1, the results of [29], [42] show that the lightly shaded point in the middle also gives a Floer non-trivial torus. This ends the example.

**Figure 1.** Floer non-trivial tori for the twice blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$

**Example 1.4.** (Moduli of flat bundles on a compact oriented surface) The moduli space of flat bundles on a surface is a symplectic manifold that appears in several places in mathematical physics. It has a natural family of Lagrangian tori described by result of Goldman [45]. We show that some of Goldman’s tori have non-trivial Floer cohomology.

To set up notation let $\Sigma$ be a compact, oriented genus zero Riemann surface with $n$ boundary components. For any collection $\mu_1, \ldots, \mu_n \in [0, 1/2]$ of holonomy parameters let

$$\mathcal{R}(\mu_1, \ldots, \mu_n) \subset \text{Hom}(\pi_1(\Sigma), SU(2))/SU(2)$$

denote the moduli space of $SU(2)$-representations (isomorphism classes of flat $SU(2)$-bundles) on $\Sigma$ with holonomy around the $j$-th boundary component conjugate to $\text{diag}(\exp(\pm 2\pi i \mu_j))$ for $j = 1, \ldots, n$. For generic parameters $\mu_1, \ldots, \mu_n$, the moduli space $\mathcal{R}(\mu_1, \ldots, \mu_n)$ is a smooth symplectic manifold. Given a pants decomposition

$$\Sigma = \bigcup_{P \in \mathcal{P}} P$$

define a label map

$$(2) \quad \Psi_P : \mathcal{R}(\mu_1, \ldots, \mu_n) \to [0, 1/2]^{n-3}$$

by taking the logarithms of eigenvalues of the holonomies around the interior circles: If $Z_j$ is the $j$-th circle in the pants decomposition then (similar to the boundary condition in [93])

$$(\Psi_P)_j([\varphi]) = \mu_j, \quad [\text{diag}(\exp(\pm 2\pi i \mu_j))] = [\varphi([Z_j])].$$
Goldman [45] shows that the generic fibers of \( \Psi \) are Lagrangian tori. We say that a labelling \( \lambda \in \mathbb{R}^{n-3} \) is regular if for pair of pants \( P \in \mathcal{P} \) with boundary circles \( C_i, C_j, C_k \) with corresponding components \( \lambda_i, \lambda_j, \lambda_k \) of \( \lambda \), the quantity (which we call the looseness of the pair of pants)

\[
l(P) := \min(\lambda_i + \lambda_j - \lambda_k, \lambda_i + \lambda_k - \lambda_j, \lambda_j + \lambda_k - \lambda_i, 1 - \lambda_i - \lambda_j - \lambda_k)
\]

is independent of the choice of \( P \).

Theorem 1.1 implies the existence of a Floer non-trivial torus described as follows. Suppose that \( \mu_1, \ldots, \mu_n \) are generic labels and \( \mathcal{R}(\mu_1, \ldots, \mu_n) \) is the corresponding moduli space of flat \( SU(2) \)-bundles on the \( n \)-holed two-sphere. Let \( \mathcal{P} = \{P\} \) be a pants decomposition with ordered boundary circles \( C_1, \ldots, C_{n-3} \) and suppose that \( \lambda = (\lambda_1, \ldots, \lambda_{n-3}) \) is a regular labelling with looseness given by the first transition time in the mmp. Then the Goldman Lagrangian \( \Psi^{-1}(\lambda) \) has non-trivial Floer cohomology:

\[
HF(\Psi^{-1}(\lambda)) \cong H(\Psi^{-1}(\lambda)) \neq 0.
\]

Figure 2 gives an example of a labelled pants decomposition giving rise to a Floer-non-trivial Lagrangian, corresponding to an mmp transition at time \( t = .06 \). This ends the example.

The results in Theorems 1.1 1.2 are two steps towards the following conjecture. Denote by \( \text{Fuk}(X) \) the Fukaya category of \( X \). For any element \( w \) in the universal Novikov ring denoted by \( \text{Fuk}(X, w) \) the category whose objects are pairs \((L, b)\) where \( L \) is an object of \( X \) and \( b \) is a weakly bounding cochain satisfying the weak Maurer-Cartan equation with value \( w \) and whose morphism spaces are explained in [41]. Let \( D^\pi \text{Fuk}(X, w) \) be the idempotent completion of the derived Fukaya category and define

\[
D^\pi \text{Fuk}(X) = \bigsqcup_w D^\pi \text{Fuk}(X, w).
\]

**Conjecture 1.5.** Suppose \( X = X_0, X_1, \ldots, X_k = X' \) is a sequence of compact symplectic orbifolds such that each \( X_{i+1} \) is obtained from \( X_i \) by an mmp transition. Then
the idempotent-closure of the derived Fukaya category $D^\pi \text{Fuk}(X_0)$ is isomorphic to the disjoint union of categories of centers of the mmp transitions:

\[(3) \quad D^\pi \text{Fuk}(X) \cong D^\pi \text{Fuk}(X') \sqcup \bigsqcup_{i=1}^{k} D^\pi \text{Fuk}(Z_i)^{m_i}\]

where

\[m_i = \dim(QH(X_i)) - \dim(QH(X_{i+1}))\]

is the multiplicity of the $i$-th mmp transition given as the difference of the quantum cohomology rings $QH(X_{i+1}), QH(X_i)$.

By the disjoint union of categories we mean the category whose objects are the disjoint union, and whose morphism groups between elements of different sets in the disjoint union are trivial. Since our evidence is mostly in the birationally-Fano case, it is possible that the conjecture 1.5 needs some similar restriction. The decomposition above should be related to the decomposition by quantum multiplication of the first Chern class in the following sense: Recall that quantum multiplication by the first Chern class $c_1(X) \in H^2(X)$ induces an endomorphism $c_1(X) \star : QH(X) \rightarrow QH(X)$, $\alpha \mapsto c_1(X) \star \alpha$.

Here we work with the version of quantum cohomology that is a module over the \textit{universal Novikov field} $\Lambda$ of formal series in a single formal variable $q$, so that the structure coefficients are weighted with $q$-exponent given by the symplectic areas of pseudoholomorphic spheres. The eigenvalues $\lambda_1, \ldots, \lambda_k \in \Lambda$ which are non-zero have a well-defined $q$-valuation $\text{val}_q(\lambda_i) \in \mathbb{R}$, given by the exponent of the leading order term. We expect that the $q$-valuations of the eigenvalues of quantum multiplication by the first Chern class are the transition times in the minimal model program: the decomposition [3] should be preserved by $c_1(X) \star$ and the non-zero eigenvalues $\lambda$ of $c_1(X) \star$ on $\text{Fuk}(Z_i)^{m_i}$ have the property that $\text{val}_q(\lambda)$ is the transition time. The decomposition [3] is intended to be a geometric version of this eigenvalue decomposition, and the mmp is conjectured to (in good cases) produce generators for the summands. The conjecture [1.5] provides a method of attack on another conjecture of Kontsevich, that the Fukaya category $\text{Fuk}(X)$ of a symplectic manifold $X$ is expected to be a categorification of the quantum cohomology $QH(X)$, at least in many cases, in the following sense [60]: there should be an isomorphism

\[(4) \quad H(\text{Fuk}(X)) := \bigoplus_w H(\text{Fuk}(X, w)) \cong QH(X)\]

from the Hochschild cohomology $H(\text{Fuk}(X))$ of the Fukaya category to the quantum cohomology $QH(X)$; here $\text{Fuk}(X)_w$ is the summand of the Fukaya category with curvature $w$. For any pair of $A_\infty$ categories, the space of $A_\infty$ functors between them is itself an $A_\infty$ category; in particular, the space of endomorphisms of the identity functor is itself an $A_\infty$ algebra. The Hochschild cohomology is then the cohomology of the algebra of endomorphisms. We expect that if the centers of the mmp transitions have the property [4] then the manifold has the same property and hence also the quantum cohomology decomposes into summands corresponding to transitions.
The idea that the quantum cohomology of a symplectic manifold should behave well under minimal model transitions is not new, in for example Ruan [91], Lee-Lin-Wang [64], Bayer [11], Acosta-Shoemaker [4]; we also heard related results in talks of H. Iritani. The present paper is the natural generalization to Fukaya categories, which in many respects is easier than quantum cohomology since one can often explicitly identify the objects created by the transitions. See Li [67] for related result in the case of open symplectic manifolds.

The minimal model program has also proved useful for understanding the derived category of matrix factorizations: A method for producing generators for the derived category of matrix factorizations was studied by Bondal-Orlov [17], Kawamata [55] and others. The minimal model program on the symplectic side should correspond under mirror symmetry to a deformation of the mirror potential by a change of variables in the potential $W \mapsto \phi_t^*W$, $\phi_t(y) := yq^{tc_1(X)}$. This is a consequence of the fact that the areas of the holomorphic disks of index two change at the same rate. Because the mirror should be understood as a formal completion at $q = 0$, such a deformation changes the mirror by eliminating some critical loci which are killed by the formal completion.

The proof of Theorem 1.1 is a symplectic-field-theory argument. Locally near the exceptional locus the reverse flip is toric, and the computations in the toric case imply the existence of a Floer-non-trivial torus in the corresponding toric variety. This is the unique Lagrangian torus which collapses at the singularity of a running of the minimal model program; in this sense the Lagrangian is a “vanishing cycle”. The exceptional locus of the flip is separated from the rest of the symplectic manifold by a coisotropic submanifold fibered over a toric variety. Stretching the neck, as in symplectic field theory, produces a homotopy-equivalent broken Fukaya algebra associated to the Lagrangian, which counts maps to the pieces combined with Morse trajectories on the toric variety. (Actually, the homotopy equivalence is much stronger than the result that we need.) Similar arguments are common in the literature, for example, in the work of Iwao-Lee-Lin-Wang [64], [63]. One computes explicitly, using a Morse function arising as component of a moment map, that the resulting broken Fukaya algebra is weakly unobstructed and that the broken Floer cohomology of the Lagrangian is non-vanishing. Moduli spaces of holomorphic disks in toric varieties with invariant constraints are never isolated, and this implies the unobstructedness of Floer cohomology. The classification of disks of small area implies the existence of a critical point of the potential. The proof of Theorem 1.2 is a similar sft-style argument, except that now the Lagrangian is contained in the non-toric piece.

We thank Kai Cieliebak, Octav Cornea, Sheel Ganatra, François Lalonde, Dusa McDuff, Klaus Mohnke for helpful discussions and Sushmita Venugopalan and Guangbo Xu for pointing out several mistakes in earlier versions.

2. Fukaya algebras

The Fukaya algebra of a Lagrangian submanifold of a symplectic manifold was introduced by Fukaya in [40]. The Fukaya algebra is a homotopy-associative algebra
whose higher composition maps are counts of configurations involving perturbed pseudoholomorphic disks with boundary in the Lagrangian as in Figure 6. Stasheff’s homotopy-associativity equation follows from studying the boundary strata in the moduli space of treed disks as in Figure 3.

Figure 3. Moduli space of stable treed disks

Because the moduli spaces of disks involved in the construction are usually singular, there are technical issues involved in its construction similar to those involved in the construction of virtual fundamental classes for moduli spaces of pseudoholomorphic curves. Fukaya and collaborators [41] introduced a method of solving these issues using Kuranishi structures in which one first constructs local thickenings of the moduli spaces and then introduces perturbations constructed locally. The details involved are formidable and lengthy, which makes generalizations to other theories involving pseudoholomorphic curves challenging.

In this section we construct Fukaya algebras of Lagrangians in a compact rational symplectic manifold using a perturbation scheme that we find particularly convenient for various computations: the stabilizing divisors scheme introduced by Cieliebak-Mohnke [26]. We also incorporate Morse gradient trees introduced by Fukaya [40] and Cornea-Lalonde [30]. This construction allows us to take our Floer cochain spaces $\hat{CF}(L)$ to be finite-dimensional spaces of Morse cochains over a Novikov field $\Lambda$. Similar constructions appear in Seidel [96] and Charest [23] in the exact and monotone cases respectively. An approach using polyfolds is under development by J. Li and K. Wehrheim. The structure constants for the Fukaya algebras in the stabilizing divisors approach count holomorphic disks with Lagrangian boundary conditions and Morse gradient trajectories on the Lagrangians with domain-dependent almost complex structures and Morse functions depending on the position of additional markings mapping to a stabilizing divisor. Because the additional marked points must be ordered in order to obtain a domain without automorphisms, this scheme gives a multi-valued perturbation. The resulting structure maps

$$\mu^n : \hat{CF}(L)^{\otimes n} \rightarrow \hat{CF}(L), \quad n \geq 0$$

for the Fukaya algebra are defined only using rational coefficients.

For later applications, it is important that our Fukaya algebras have strict units so that disk potentials are defined. To achieve this we incorporate a slight enhancement, similar to that of homotopy units, in which perturbation systems compatible with breakings are homotoped to perturbation systems that admit forgetful maps. Then the maximum of the Morse function, if it is unique, defines a strict unit $e_L \in \hat{CF}(L)$. We also prove that the Fukaya algebras so constructed satisfy a natural convergence
property so that the Maurer-Cartan map

\[ \mu : \widehat{CF}(L) \to \widehat{CF}(L), \quad b \mapsto \sum_{n \geq 0} \mu^n \left( \underbrace{b, \ldots, b}_{n} \right) \]

is well-defined. Denote the space of solutions to the weak Maurer-Cartan equation.

\[ \widehat{MC}(L) := \mu^{-1}(\Lambda e_L) \subset \widehat{CF}(L) \]

where \( e_L \in \widehat{CF}(L) \) is the strict unit. The Floer cohomology of a Lagrangian brane is the complex of vector bundles \((\widehat{CF}(L)|_{\widehat{MC}(L)}, \partial)\). That is,

\[ HF(L) = \bigcup_{b \in \widehat{MC}(L)} HF(L)_b, \quad HF(L)_b := \ker(\partial_b)/\im(\partial_b), \quad b \in \widehat{MC}(L) \]

is the collection of cohomologies of the fiber. The Floer cohomology \( HF(L) \) is said to be non-vanishing if the fiber \( HF(L)_b \) is non-vanishing for some \( b \in \widehat{MC}(L) \). The main result of this chapter is the following:

**Theorem 2.1.** Let \((X, \omega)\) be a compact symplectic manifold with rational symplectic class \( [\omega] \in H^2(X, \mathbb{Q}) \) and \( L \subset X \) a compact embedded Lagrangian submanifold admitting a relative spin structure. For a comeager subset of perturbation data, counting weighted treed holomorphic disks defines a convergent \( A_\infty \) structure with strict unit

\[ \mu^n : \widehat{CF}(L)^{\otimes n} \to \widehat{CF}(L), \quad n \geq 0 \quad e_L \in \widehat{CF}(L) \]

independent of all choices up to convergent strictly-unital \( A_\infty \) homotopy. Furthermore, \( HF(L) \) is independent of all choices up to gauge equivalence (to be explained below).

This is a combination of Theorem 2.29 and Corollary 3.12 below.

### 2.1. \( A_\infty \) algebras.

The theory of homotopy-associative algebras was introduced by Stasheff [105] in order to capture algebraic structures on the space of cochains on loop spaces. Let \( g > 0 \) be an even integer. A \( \mathbb{Z}_g \)-graded \( A_\infty \) algebra consists of a \( \mathbb{Z}_g \)-graded vector space \( A \) together with for each \( d \geq 0 \) a multilinear composition map

\[ \mu^d : A^{\otimes d} \to A[2-d] \]

satisfying the \( A_\infty \) -associativity equations

(5) \[ 0 = \sum_{\substack{n, m \geq 0 \\ \ n + m \leq d \\ \ n + m \geq 0}} (-1)^{n+\sum_{i=1}^{n} |a_i|} \mu^{d-m+1}(a_1, \ldots, a_n, \mu^m(a_{n+1}, \ldots, a_{n+m}), a_{n+m+1}, \ldots, a_d) \]

for any tuple of homogeneous elements \( a_1, \ldots, a_d \) with degrees \( |a_1|, \ldots, |a_d| \in \mathbb{Z}_g \). The signs are the shifted Koszul signs, that is, the Koszul signs for the shifted grading in which the structure maps have degree one as in Kontsevich-Soibelman [61]. The notation \( [2-d] \) denotes a degree shift by \( 2-d \), so that \( \mu^1 \) has degree 1, \( \mu^2 \) has degree 0 etc. The element \( \mu^0(1) \in A \) (where 1 \( \in \Lambda \) is the unit) is called the curvature of
the algebra. The \( A_\infty \) algebra \( A \) is flat if the curvature vanishes. A strict unit for \( A \) is an element \( e_A \in A \) such that

\[
\mu^2(e_A, a) = a = (-1)^{|a|} \mu^2(a, e_A), \quad \mu^n(\ldots, e_A, \ldots) = 0, \forall n \neq 2.
\]

A strictly unital \( A_\infty \) algebra is an \( A_\infty \) algebra equipped with a strict unit. The cohomology of a flat \( A_\infty \) algebra \( A \) is defined by

\[
H(\mu^1) = \ker(\mu^1) / \text{im}(\mu^1).
\]

The algebra structure on \( H(\mu^1) \) is given by

\[
[a_1 a_2] = (-1)^{|a_1|} [\mu^2(a_1, a_2)].
\]

An element \( e_A \in A \) is a cohomological unit if \([e_A]\) is a unit for \( H(\mu^1) \). For certain curved \( A_\infty \) algebras we give a construction of a cohomology complex over the space of solutions to the weak Maurer-Cartan in Section 2.9 below. A result of Seidel [99, Corollary 2.14] implies that any flat \( A_\infty \) algebra with a cohomological unit is equivalent to an \( A_\infty \) algebra with strict unit. However, in our applications we will construct strict units by a homotopy unit construction. More precisely, let \( A \) be an \( A_\infty \) algebra over a field \( \Lambda \) and let \( e_A^* \) be formal symbols. By a homotopy unit for \( A \) we mean an \( A_\infty \) structure on the direct sum

\[
\hat{A} = A \oplus \Lambda e_A^* \oplus \Lambda e_A^*
\]

extending the given structure on \( A \) so that \( e_A^* \) is a strict unit, and \( e_A^* \in \mu^1(e_A^*) + A \). That is, up to coboundaries \( e_A^* \) lies in the subalgebra \( A \). Homotopy units for our Fukaya algebras will be constructed geometrically.

2.2. Associahedra. The combinatorics of the definition above is closely related to a sequence of spaces introduced by Stasheff under the name associahedra. We will use several related constructions of these spaces. The first of these is the moduli spaces of metric trees. A tree is a connected, cycle-free graph

\[
T = (\text{Edge}(T), \text{Vert}(T))
\]

where \( \text{Vert}(T) \) is the set of vertices and \( \text{Edge}(T) \) is the set of edges. A planar tree is a tree equipped with a planar structure: a cyclic ordering of the edges \( e \in \text{Edge}(T), e \ni v \) incident to each vertex \( v \in \text{Vert}(T) \). We introduce the following notation for the edges:

(a) if \( \text{Vert}(T) \) is non-empty, then the set of edges \( \text{Edge}(T) \) consists of

(i) combinatorially finite edges \( \text{Edge}_-(T) \) connecting two vertices and

(ii) semi-infinite edges \( \text{Edge}_+(T) \) with a single endpoint, or

(b) if \( \text{Vert}(T) \) is empty, then \( T \) has one infinite edge. We denote by \( \text{Edge}_-(T) \) its two ends.

From \( \text{Edge}_-(T) \), an open endpoint is distinguished as the root of the tree, the others being referred to as its leaves.
A moduli space of trees is obtained by allowing the finite edges to acquire lengths. A metric tree is a pair \( T = (\mathcal{T}, \ell) \) consisting of a tree \( \mathcal{T} \) equipped with a metric \( \ell \). By definition a metric is labelling
\[
\ell : \text{Edge}_-(\mathcal{T}) \to [0, \infty)
\]
of its combinatorially finite edges by elements of \([0, \infty)\) called lengths. An equivalence relation on metric trees is defined by collapsing edges of length zero: Given a tree with an edge of length zero, removing the edge and identifying its head and tail gives an equivalent metric tree.

The metric trees with edges of infinite length are often thought of as broken metric trees. More precisely, a broken metric tree is obtained from a finite collection of metric trees by gluing roots to leaves. Given two metric trees \( T_1, T_2 \) and semi-infinite root edge \( e_2 \in \text{Edge}_-(T_2) \) and leaf edge \( e_1 \in \text{Edge}_+(T_2) \), let \( \overline{T}_1 \) resp. \( \overline{T}_2 \) denote the space obtained by adding a point \( \infty_2 \) resp. \( \infty_1 \) at the open end of \( e_2 \) resp. \( e_1 \). The space
\[
T := \overline{T}_1 \cup_{\infty_1 \sim \infty_2} \overline{T}_2
\]
is a broken metric tree, the point \( \infty_1 \sim \infty_2 \) being called a breaking. See Figure 4.

![Figure 4. Creating a broken tree](image)

In general, broken metric trees are obtained by repeating this process in such a way that the resulting space is connected and has no non-contractible cycles. An equivalence relation on broken metric trees is defined by adding a breaking on edges of infinite length and the latter description will then be preferred. If a combinatorially finite edge has infinite length then one attaches an additional positive integer to that edge indicating its number of breakings, see [23].

In order to obtain a Hausdorff moduli space of broken trees a stability condition is imposed. A broken metric tree is stable if it has no automorphisms, or equivalently if each combinatorially semi-infinite edge is unbroken and each combinatorially finite edge is broken at most once. The moduli space of stable metric trees is a finite cell complex studied in, for example, Boardman-Vogt [14]; this is the first realization of Stasheff’s associahedron as a moduli space of geometric objects. However, the natural cell structure on this moduli space is a refinement of the canonical cell structure on the associahedra.

A realization of the associahedron that reproduces the canonical cell structure involves nodal disks with boundary markings. A nodal disk with a single boundary node is a topological space \( S \) obtained from a disjoint union of holomorphic disks \( S_1, S_2 \) by identifying pairs of boundary points \( w_{12} \in S_1, w_{21} \in S_2 \) on the boundary of each component so that
\[
S = S_1 \cup_{w_{12} \sim w_{21}} S_2.
\]
See Figure 5. The image of \( w_{12}, w_{21} \) in the space \( S \) is the nodal point. A nodal disk with multiple nodes is obtained by repeating this process with \( S_1 \) a nodal disk with fewer nodes. For an integer \( n \geq 0 \) a nodal disk with \( n + 1 \) boundary markings is a nodal disk \( S \) equipped with a finite ordered collection of points \( \bar{x} = (x_0, \ldots, x_n) \) on the boundary, disjoint from the nodes, in counterclockwise cyclic order around the boundary. An \((n + 1)\)-marked nodal disk is stable if each component has at least three special (nodal or marked) points. The moduli space of \((n + 1)\)-marked stable disks forms a compact cell complex, which is yet another geometric realization of the associahedron. More generally one can allow interior markings \( z_1, \ldots, z_n \in \text{int}(S) \) in the definition of marked nodal disks, and so that converging interior markings bubble off onto sphere components. A nodal disk with a single interior node is defined similarly to that of a boundary node, except in this case \( S_2 \) is a holomorphic sphere and \( w_{12}, w_{21} \) are points in the boundary. A nodal disk with interior markings arises from a nodal genus zero curve equipped with an anti-holomorphic involution with non-empty fixed point set by taking the quotient by the involution (see [23]); the components meeting the fixed point set give rise to disk components while the components disjoint from the fixed point set give rise to sphere components in the quotient. Similarly, markings in the fixed point set give rise to boundary markings while markings disjoint from the fixed point set give rise to interior markings.

A combination of the above constructions involves both trees and disks, as in Oh [87], Cornea-Lalonde [40], Biran-Cornea [13], and Seidel [96]. In order to incorporate spherical components we suppose that the set of vertices \( \text{Vert}(T) \) is equipped with a partition \( \text{Vert}_\circ(T) \) and \( \text{Vert}_\bullet(T) \) into vertices corresponding to disks and vertices correspond to spheres and similarly a partition \( \text{Edge}(T) \) into edges \( \text{Edge}_\circ(E) \) resp. \( \text{Edge}_\bullet(E) \) representing boundary nodes resp. interior nodes; for each \( v \in \text{Vert}_\circ(T) \), the edges \( e \in \text{Edge}_\circ(E) \) incident to \( v \) are equipped with a cyclic ordering; we call the resulting tree \( T \) equipped with a length function \( \ell : \text{Edge}_\circ(T) \to [0, \infty] \) also a metric tree.

**Definition 2.2.** A treed disk \( C \) is a triple \((T, S, o)\) consisting of

(a) a broken metric tree \( T = (T, \ell) \);

(b) a collection

\[
S = (S_v, \bar{x}_v, \bar{z}_v)_{v \in \text{Vert}(T)}
\]

of (boundary and interior) nodal disks resp. spheres for each vertex \( v \in \text{Vert}_\circ(T) \) resp. \( v \in \text{Vert}_\bullet(T) \) of \( T \), with number of boundary markings equal to the valence of \( v \), and
(c) a labelling $o : \text{Edge}_{\rightarrow}(T) \to \{1, \ldots, n\}$ of the set of interior leaves. We denote by $z_1, \ldots, z_n \in C$ the attaching points of the interior leaves and call them \textit{interior markings}.

The \textit{topological realization} of a treed disk $C$ is obtained by removing the vertices from $T$ and gluing in the nodal disks by attaching the boundary and interior markings $x_v, z_v$ to the edges of the tree meeting $v$. A treed disk $C = (T, S, o)$ is \textit{stable} iff

(a) the tree $T$ is stable; that is, each valence is at least three;
(b) each nodal disk $S_v, v \in \text{Vert}_o(T)$ is stable; that is, each disk contains at least three special boundary points or one special point in the interior and at least one special boundary point.
(c) each nodal sphere $S_v, v \in \text{Vert}_o(T)$ is stable, that is, has at least three special points.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{A treed disk with three disk components and one sphere component}
\end{figure}

See Figure 6. An \textit{isomorphism} of treed disks is an equivalence of broken metric trees together with an isomorphism of collections of nodal disks preserving the ordering of the interior markings. In particular, a treed disk with a boundary node will be equivalent to the same treed disk where an edge of length zero is substituted to the latter. We remark that for the purposes of constructing Fukaya algebras, one may in fact assume that the set of spherical vertices is empty (so any sphere component is attached to a nodal disk). However, in the construction of morphisms between Fukaya algebras and the proof of homotopy invariance, it is in fact necessary to consider sphere components.

In order to obtain Fukaya algebras with strict units, we attach additional parameters to certain of the semi-infinite edges called \textit{weightings}. A \textit{weighting} of a (broken) treed disk $C = (T, S, o)$ consists of

(a) (Weighted, forgettable, and unforgettable edges) A partition of the boundary semi-infinite edges

\[ \text{Edge}^w(T) \sqcup \text{Edge}^f(T) \sqcup \text{Edge}^u(T) = \text{Edge}_{\rightarrow}(T) \]

into \textit{weighted} resp. \textit{forgettable} resp. \textit{unforgettable} edges, and

(b) (Weighting) a map

\[ \rho : \text{Edge}_{\rightarrow}(T) \to [0, \infty] \]
satisfying the property: each of the semi-infinite \( e \) edges is assigned a weight \( \rho(e) \) such that
\[
\rho(e) = \begin{cases} 
0 & e \in \text{Edge}^\ast(T) \\
[0,\infty) & e \in \text{Edge}^\ast(T) \\
\{\infty\} & e \in \text{Edge}^\ast(T) 
\end{cases}
\]

If the outgoing edge is unweighted (forgettable or unforgettable) then an isomorphism of weighted treed disks is an isomorphism of treed disks that preserves the types of semi-infinite edges and weightings:
\[
\rho(e) = \rho'(e') \text{ for all corresponding edges } e \in \text{Edge}_{\infty\rightarrow}(T), e' \in \text{Edge}_{\infty\rightarrow}(T').
\]

The case that the outgoing edge is weighted is very rare in our examples and should be considered an exceptional case. If the outgoing edge is weighted then an isomorphism of weighted treed disks is an isomorphism of treed disks preserving the types of semi-infinite edges \( e \in \text{Edge}_{\infty\rightarrow}(T) \) and the weights \( \rho(e), e \in \text{Edge}_{\infty\rightarrow}(T) \) up to scalar multiples:
\[
\exists \lambda \in (0,\infty), \forall e \in \text{Edge}_{\infty\rightarrow}(T), e' \in \text{Edge}_{\infty\rightarrow}(T'), \rho(e) = \rho'(e').
\]

In particular, any weighted tree with no vertices \( \text{Vert}(T) = \emptyset \) and a single edge \( e \in \text{Edge}_{\infty\rightarrow}(T) \) that is weighted is isomorphic to any other such configuration with a different weight.

A well-behaved moduli space of weighted treed disks is obtained after imposing a stability condition. A weighted treed disk is stable if either
\begin{enumerate}
\item there is at least one disk component \( S_v, v \in \text{Vert}_{\infty}(T) \), each disk component \( S_v, v \in \text{Vert}_{\infty}(T) \) at least three edges attached \( e \in \text{Edge}(T), e \ni v \), and if the outgoing edge is weighted \( e_0 \in \text{Edge}^\ast(T) \) then at least one incoming edge \( e_i \in \text{Edge}_{\infty\rightarrow}(T), i > 0 \) is also weighted \( e_i \in \text{Edge}^\ast(T) \); or
\item if there are no disks, so that \( \text{Vert}(T) = \emptyset \), there is a single weighted leaf \( e_1 \in \text{Edge}^\ast(T) \) and an unweighted (forgettable or unforgettable) root \( e_0 \in \text{Edge}^\ast(T) \cup \text{Edge}^\ast(T) \).
\end{enumerate}

These conditions guarantee that the moduli space is expected dimension, see Remark 2.3 below. Because a configuration with no disks is allowed, the stability condition is not equivalent to the absence of infinitesimal automorphisms.

The combinatorial type of any weighted treed disk is the corresponding tree with additional data recording which lengths resp. weights are zero or infinite. Namely if \( C = (T,S,o) \) is a weighted treed disk then its combinatorial type is the graph \( \Gamma = \Gamma(C) \) obtained by gluing together the combinatorial graphs \( \Gamma(S_v) \) of the disks \( S_v \) along the edges corresponding to the edges of \( T \); and equipped with the additional data of
\begin{enumerate}
\item the subsets
\[
\text{Edge}^\ast(T) \text{ resp. } \text{Edge}^-\ast(T) \text{ resp. } \text{Edge}^\ast(T) \subset \text{Edge}_{\infty\rightarrow}(T)
\]
of weighted, resp. forgettable, resp. unforgettable semi-infinite edges;
\item the subsets
\[
\text{Edge}^{\infty}(T) \text{ resp. } \text{Edge}^{0}(T) \text{ resp. } \text{Edge}^{(0,\infty)}(T) \subset \text{Edge}_{\infty}(T)
\]
\end{enumerate}
of combinatorially finite edges of infinite resp. zero length resp. non-zero finite length;

(c) the subset

\[ \text{Edge}^{\infty}(T) \text{ resp. } \text{Edge}^{0}(T) \subset \text{Edge}^{\ast}(T) \]

of weighted edges with infinite resp. zero weighting.

The set of vertices admits a partition into interior and boundary vertices

\[ \text{Vert}(\Gamma) = \text{Vert}_{\bullet}(\Gamma) \cup \text{Vert}_{\circ}(\Gamma) \]

corresponding to spheres or disks. The length function on edges of \( T \) extends to the edges of \( \Gamma \) corresponding to disk and spherical nodes, by setting the lengths of those edges to be zero. We will from now on refer to the edges of \( T \) as being edges of \( \Gamma \).

The moduli spaces of stable weighted treed disks are naturally cell complexes with multiple cells of top dimension. For integers \( n, m \geq 0 \) denote by \( \mathcal{M}_{n,m} \) the moduli space of isomorphism classes of stable weighted treed disks with \( n \) leaves and \( m \) interior markings. For each combinatorial type \( \Gamma \) denote by \( \mathcal{M}_{\Gamma} \subset \mathcal{M}_{n,m} \) the set of isomorphism classes of weighted stable treed disks of type \( \Gamma \) so that the moduli spaces decomposes into strata of fixed type

\[ \mathcal{M}_{n,m} = \bigcup_{\Gamma} \mathcal{M}_{\Gamma}. \]

The dimension of \( \mathcal{M}_{\Gamma} \) is equal to the \( n + 2m - 2 \) plus the number of weighted leaves. In particular, if \( \Gamma \) has no vertices then the dimension is zero. In Figure 7 a subset of the moduli space with one interior marking is shown, where the interior marking is constrained to lie on the line half-way between the special points on the boundary.

Remark 2.3. The moduli spaces of weighted treed disks are related to unweighted moduli spaces by taking products with intervals: If \( \Gamma \) has at least one vertex and \( \Gamma' \) denotes the combinatorial type of \( \Gamma \) obtained by setting the weights to zero and the outgoing edge of \( \Gamma \) is unweighted then

\[ \mathcal{M}_{\Gamma} \cong \mathcal{M}_{\Gamma'} \times (0, \infty)^{1|\text{Edge}^{\ast}(0, \infty)(\Gamma)|}. \]

If the outgoing edge is weighted and at least one incoming edge is weighted then

\[ \mathcal{M}_{\Gamma} \cong \mathcal{M}_{\Gamma'} \times (0, \infty)^{1|\text{Edge}^{\ast}(0, \infty)(\Gamma)|-2} \]

since only the ratios of the weightings of leaves must be preserved by the isomorphisms. In particular, if \( \Gamma \) is a type with a single weighted leaf and no vertices, the outgoing edge may be unforgettable or forgettable, assigning a weighting on the leaf will be irrelevant and \( \mathcal{M}_{\Gamma} \) will be a point.

The moduli spaces admit universal curves, which admit partitions into one and two-dimensional parts. For any combinatorial type \( \Gamma \) let \( \overline{U}_{\Gamma} \) denote universal treed disk consisting of isomorphism classes of pairs \( (C, z) \) where \( C \) is a treed disk of type \( \Gamma \) and \( z \) is a point in \( C \), possibly on a disk component, sphere component, or one of the edges of the tree. The map

\[ \overline{U}_{\Gamma} \to \mathcal{M}_{\Gamma}, \quad [C, z] \to [C] \]
is the universal projection. Because of the stability condition, there is a natural bijection
\[ \mathcal{U}_\Gamma = \bigcup_{|C| \in \mathcal{M}_\Gamma} C. \]
We denote by
\[ \mathcal{S}_\Gamma = \{ |C = S \cup T, z| \in \mathcal{U}_\Gamma | z \in S \} \]
the locus where \( z \) lies on a disk or sphere of \( C \). Denote by
\[ \mathcal{T}_\Gamma = \{ |C = S \cup T, z| \in \mathcal{U}_\Gamma | z \in T \} \]
the locus where \( z \) lies on an edge of \( C \). Hence
\[ \mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma \]
and \( \mathcal{S}_\Gamma \cap \mathcal{T}_\Gamma \) is the set of points on the boundary of the disks meeting the edges of the tree. In case \( \Gamma \) has no vertices we define \( \mathcal{U}_\Gamma \) to be the real line, considered as a fiber bundle over the point \( \mathcal{M}_\Gamma \). The tree part splits into interior and boundary tree parts depending on whether the edge is attached to an interior point or a boundary point of a disk or sphere:
\[ \mathcal{T}_\Gamma = \mathcal{T}_{\text{int}, \Gamma} \cup \mathcal{T}_{\text{bnd}, \Gamma}. \]

![Figure 7. Treed disks with interior leaves](image)

Later we will need local trivializations of the universal treed disk and the associated families of complex structures and metrics on the domains. For a stable combinatorial type \( \Gamma \) let
\[ \mathcal{U}_\Gamma^i \to \mathcal{M}_\Gamma^i \times C, , i = 1, \ldots, l \]
be a collection of local trivializations of the universal treed disk, identifying each nearby fiber with \( (C, z, w) \) such that each point in the universal treed disk is contained in one of these local trivializations. The complex structures on the fibers induce a family
\[ \mathcal{M}_\Gamma^i \to \mathcal{J}(S), \ m \mapsto j(m) \]
of complex structures on the two-dimensional locus \( S \subset C \).

The following operations on treed disks will be referred to in the coherence conditions on perturbation data.
Definition 2.4. (Morphisms of graphs) A *morphism* of graphs \( \Upsilon : \Gamma' \to \Gamma \) is a surjective morphism of the set of vertices \( \text{Vert}(\Gamma') \to \text{Vert}(\Gamma) \) obtained by combining the following *elementary morphisms*:

\[
\begin{align*}
\text{(a) (Cutting edges)} \quad \Upsilon & \text{ cuts an edge with infinite length if there exists } e \in \text{Edge}(\Gamma'), \quad \ell(e) = \infty \\
& \text{so that the map } \text{Vert}(\Gamma) \to \text{Vert}(\Gamma') \text{ on vertices is a bijection, but } \\
& \text{Edge}(\Gamma) \cong \text{Edge}(\Gamma') - \{e\} + \{e_+, e_-\}
\end{align*}
\]

where \( e_\pm \in \text{Edge}(\Gamma) \) are attached to the vertices contained in \( e \). Since our graphs are trees, \( \Gamma \) is disconnected with pieces \( \Gamma_-, \Gamma_+ \). The edge corresponds to a broken segment and \( \Gamma_-, \Gamma_+ \) are types of stable treed disks. The ordering on \( \text{Edge}(\Gamma) \) induces one on \( \text{Edge}(\Gamma_\pm) \) by viewing the latter as a subset of the former.

The weighting and type of the cut edges are defined as follows. Suppose that \( \Gamma_- \) is the component of \( \Gamma \) not containing the root edge. If \( \Gamma_- \) has any interior leaves, set \( \rho(e_\pm) = 0 \) and \( e_\pm \in \text{Edge}(\Gamma) \). Otherwise (and these are relatively rare exceptional cases in our examples and used only for the construction of strict units) if there are no interior leaves let \( e_1, \ldots, e_k \) denote the incoming edges to \( \Gamma_- \).

(i) If any of \( e_1, \ldots, e_k \) are unforgettable then \( e_\pm \in \text{Edge}(\Gamma) \) are also unforgettable.

(ii) If none of \( e_1, \ldots, e_k \) are unforgettable and at least one of \( e_1, \ldots, e_k \) is weighted then \( e_\pm \in \text{Edge}(\Gamma) \) are also weighted.

(iii) If \( e_1, \ldots, e_k \) are forgettable then \( e_\pm \in \text{Edge}(\Gamma) \) are also forgettable. See Figure 9 for an example. Define the weighting on the cut edges

\[
\rho(e_\pm) = \min(\rho(e_1), \ldots, \rho(e_k)).
\]

In particular if \( \Gamma_- \) has all zero weights \( \rho(e_l) = 0, l = 1, \ldots, k \) then \( \rho(e_\pm) = 0 \).

(b) (Collapsing edges) \( \Upsilon \) *collapses an edge* if the map on vertices is a bijection except for a single vertex \( v' \in \text{Vert}(\Gamma) \)

\[
\text{Vert}(\Upsilon) : \text{Vert}(\Gamma') \to \text{Vert}(\Gamma), \quad \text{Vert}(\Upsilon)^{-1}(v') = \{v_-, v_+\}
\]

that has two pre-images \( v_\pm \in \text{Vert}(\Gamma') \). The vertices \( v_-, v_+ \) are connected by an edge \( e \in \text{Edge}(\Gamma') \) so that

\[
\text{Edge}(\Gamma) \cong \text{Edge}(\Gamma') - \{e\}.
\]

See Figure 8.
(c) (Making an edge length finite or non-zero) $\Upsilon$ makes an edge finite resp. non-zero if $\Gamma'$ is the same graph as $\Gamma$ and the lengths of the edges are the same except for a single edge $e$:

$$\ell|_{\text{Edge}_-(\Gamma) - \{e\}} = \ell'|_{\text{Edge}_-(\Gamma') - \{e\}}.$$  

For the edge $e$ we require

$$\ell'(e) = \infty \text{ resp. 0, } \ell(e) \in (0, \infty).$$

(d) (Forgetting tails) $\Upsilon : \Gamma' \to \Gamma$ forgets a tail (semi-infinite edge) and collapses edges to make the resulting combinatorial type stable. The ordering on $\text{Edge}_{\to}(\Gamma')$ then naturally defines one on $\text{Edge}_{\to}(\Gamma)$ viewing the latter as a subset. See Figure 10.

(e) (Making an edge weight finite or non-zero) makes a weight finite or non-zero if $\Gamma'$ is the same graph as $\Gamma$ and the weights of the edges $\rho(e), e \in \text{Edge}^*(\Gamma)$ are the same except for a single edge $e$,

$$\rho|_{\text{Edge}_-(\Gamma) - \{e\}} = \rho'|_{\text{Edge}_-(\Gamma') - \{e\}}.$$  

For the edge $e$ we have

$$\rho'(e) = \infty \text{ resp. 0, } \rho(e) \in (0, \infty).$$

It will be important for our construction of perturbations later that the operations of cutting edges commute. For example, if $\Gamma$ is obtained from $\Gamma'$ by cutting two edges, then the induced weighting on $\Gamma$ is independent of the order of the cutting. This follows from the identity $\min(\rho(e_1), \ldots, \rho(e_j), \min(\rho(e_{j+1}), \ldots, \rho(e_{j+k})), \ldots, \rho(e_i)) = \min(\rho(e_1), \ldots, \rho(e_i))$.

Each of the above operations on graphs corresponds to a map of moduli spaces of stable marked treed disks.

**Definition 2.5.** (Morphisms of moduli spaces)
(a) (Cutting edges) Suppose that $\Gamma$ is obtained from $\Gamma'$ by cutting an edge $e$. There are diffeomorphisms

$$\mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma'}, \ [C] \to [C']$$

obtained as follows. Given a treed disk $C$ of type $\Gamma'$, let $z_+, z_-$ denote the endpoints at infinity of the edge corresponding to $e$. Form a treed disk $C'$ by identifying $z_+ \sim z_-$ and choosing the labelling of the interior leaves to be that of $\Gamma'$.

(b) (Collapsing edges) Suppose that $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge. There is an embedding $\iota_{\Gamma'} : \mathcal{M}_{\Gamma'} \to \mathcal{M}_\Gamma$. In the case of an edge of $\text{Edge}_0^-(\Gamma')$, the image of $\iota_{\Gamma'}(\mathcal{M}_{\Gamma'})$ is a 1-codimensional corner of $\mathcal{M}_\Gamma$. In the case of an edge of $\text{Edge}_\sim(\Gamma')$ the image $\iota_{\Gamma'}(\mathcal{M}_{\Gamma'})$ is a 2-codimensional submanifold of $\mathcal{M}_\Gamma$.

(c) (Making an edge or weight finite resp. non-zero) If $\Gamma$ is obtained from $\Gamma'$ by making an edge finite resp. non-zero then $\mathcal{M}_{\Gamma'}$ also embeds in $\mathcal{M}_\Gamma$ as the 1-codimensional corner. The image is the set of configurations where the edge $e$ reaches infinite resp. zero length $\ell(e)$ or weight $\rho(e)$.

(d) (Forgetting tails) Suppose that $\Gamma$ is obtained from $\Gamma'$ by forgetting the $i$-th tail (either in $\text{Edge}_\sim(\Gamma')$ or $\text{Edge}_\circ(\Gamma')$). Forgetting the $i$-th marking and collapsing the unstable components and their distance to the stable components (if any) defines a map $\mathcal{M}_{\Gamma'} \to \mathcal{M}_\Gamma$. Each weighted semi-infinite edge for $\Gamma'$ defines a weighted semi-infinite edge for $\Gamma$ with the same weight.

Each of the maps involved in the operations (Collapsing edges/Making edges/weights finite or non-zero), (Forgetting tails), (Cutting edges) extends to a smooth map of universal treed disks. In the case that the type is disconnected we have

$$\Gamma = \Gamma_1 \sqcup \Gamma_2 \implies \mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2}.$$ 

In this case the universal disk $\overline{U}_\Gamma$ is the disjoint union of the pullbacks of the universal disks $\overline{U}_{\Gamma_1}$ and $\overline{U}_{\Gamma_2}$. If $\pi_1, \pi_2$ are the projections on the factors above then

$$\overline{U}_\Gamma = \pi_1^* \overline{U}_{\Gamma_1} \sqcup \pi_2^* \overline{U}_{\Gamma_2}.$$ 

Orientations on the main strata (i.e. of maximal dimension) of the moduli space of (non-weighted) treed disks may be constructed as follows.

**Definition 2.6.** (Orientations on moduli of treed disks)

(a) (A single disk) On the strata made of treed disks having a single disk, choosing an orientation amounts to choosing orientations on the spaces of stable marked disks. One can identify a smooth disk $D$ with $n + 1$ boundary markings $x_0, \ldots, x_n$ and $m \geq 1$ attaching points of interior leaves $z_1, \ldots, z_m$ with the positive half-space $\mathbb{H} \subset \mathbb{C}$ by a map

$$\phi : D \setminus \{x_0\} \to \mathbb{H}, \ z_i \mapsto i$$

so that the boundary markings $x_i, i \geq 1$ map to an ordered tuple in $\mathbb{R} \subset \mathbb{C}$. For $m = 0$ interior markings, then there are $n + 1 \geq 3$ boundary markings.
We identify
\[ \phi : D - \{x_0\} \to \mathbb{H}, \quad x_1 \mapsto 0, \quad x_2 \mapsto 1. \]
The remaining boundary markings \( x_i, i \geq 3 \) map to an ordered tuple of \([1, \infty[ \subset \mathbb{R} \subset \mathbb{C}\). The moduli space \( \mathcal{M}_\Gamma \) of disks of this type then inherits an orientation from the canonical orientation on \( \mathbb{R}^{n-2} \times \mathbb{C}^m \).

(b) (Multiple disks) One can extend these orientations to the top dimensional strata of treed disks having more than a single disk as follows. The closures of the main strata are attached to strata with fewer edges with finite non-zero lengths via isomorphisms of treed disks identifying a boundary node with an edge of length zero. The addition of an edge of finite non-zero length corresponds to identifying the closures of two main strata on a 1-codimensional corner strata. For \( n, m \geq 0 \) we choose orientations \( \mathcal{O}_{n,m} \) on the main strata \( \mathcal{M}_\Gamma \) of \( \mathcal{M}_{n,m} \) so that they fit together to an oriented topological manifold over the latter 1-codimensional strata. The closures of the top-dimensional strata \( \mathcal{M}_\Gamma \) then fit together to an oriented manifold with corners (see [23]).

2.3. Treed holomorphic disks. The composition maps in the Fukaya algebra will be obtained by counting treed holomorphic disks, which we now define.

**Definition 2.7.**

(a) (Gradient flow lines) Let \( L \) be a compact connected smooth manifold. We denote by \( \mathcal{G}(L) \) the space of smooth Riemannian metrics on \( L \). Fix a metric \( G \in \mathcal{G}(L) \) and a Morse function \( F : L \to \mathbb{R} \) having a unique maximum \( x_M \in L \). Let \( I \subset \mathbb{R} \) be a connected subset containing at least two elements, that is, an open or closed interval. The gradient vector field of \( F \) is defined by
\[ \text{grad}_F : L \to TL, \quad G(\text{grad}_F, \cdot) = dF \in \Omega^1(L). \]
A gradient flow line for \( F \) is a map
\[ u : I \to L, \quad \frac{d}{ds} u = -\text{grad}_F(u) \]
where \( s \) is a unit velocity coordinate on \( I \). Given a time \( s \in \mathbb{R} \) let
\[ \phi_s : L \to L, \quad \frac{d}{ds} \phi_s(x) = \text{grad}_F(\phi_s(x)), \quad \forall x \in L \]
denote the time \( s \) gradient flow of \( F \).

(b) (Stable and unstable manifolds) Denote by
\[ \mathcal{I}(L) := \mathcal{I}(L, F) := \text{crit}(F) \subset L \]
the space of critical points of \( F \). Taking the limit of the gradient flow determines a discontinuous map
\[ L \to \text{crit}(F), \quad y \mapsto \lim_{s \to \pm \infty} \phi_s(y). \]
By the stable manifold theorem each $x \in \mathcal{I}(L)$ determines stable and unstable manifolds

$$W^\pm_x := \left\{ y \in L \mid \lim_{s \to \pm \infty} \phi_s(y) = x \right\} \subset L$$

consisting of points whose downward resp. upwards gradient flow converges to $x$. We denote by

$$i : \mathcal{I}(L) \to \mathbb{Z}_{\geq 0}, \quad x \mapsto \dim(W^-_x)$$

the index map. The pair $(F, G)$ is Morse-Smale if the intersections

$$W^+_x \cap W^-_x \subset L$$

are transverse for $x_+, x_- \in \mathcal{I}(L)$, and so a smooth manifold of dimension $i(x_+) - i(x_-)$.

(c) (Almost complex structures) Let $(X, \omega)$ be a symplectic manifold. An almost complex structures on $(X, \omega)$ given by $J : TX \to TX$, $J^2 = -I$ is tamed iff $\omega(\cdot, J\cdot)$ is a positive definite and compatible if it is in addition symmetric, hence a Riemannian metric on $X$. We denote by $\mathcal{J}_\tau(X)$ the space of tamed almost complex structures. The space $\mathcal{J}_\tau(X)$ has a natural manifold structure locally isomorphic to the space of sections

$$\delta J : X \to \text{End}(TX), \quad J(\delta J) = -(\delta J)J.$$

In order to obtain the necessary transversality our Morse functions and almost complex structures must be allowed to depend on a point in the domain. Fix a compact subset $\mathcal{T}\Gamma \subset \mathcal{T}_\Gamma$ containing, in its interior, at least one point on each edge. Also fix a compact subset $\mathcal{S}_\Gamma \subset \mathcal{S}_\Gamma - \{w_e \in \mathcal{S}_\Gamma, \ e \in \text{Edge}_-(\Gamma)\}$ disjoint from the boundary and spherical nodes, containing in its interior at least one point on each sphere and disk component. Thus the complement $\mathcal{T}_\Gamma - \mathcal{S}_\Gamma \subset \mathcal{T}_\Gamma$ is a neighborhood of infinity on each edge. Furthermore, the complement $\mathcal{S}_\Gamma - \mathcal{S}_\Gamma^{\text{op}} \subset \mathcal{S}_\Gamma$ is a neighborhood of the boundary and nodes.

**Definition 2.8.**

(a) (Domain-dependent Morse functions) Suppose that $\Gamma$ is a type of stable treed disk, and $\mathcal{T}_\Gamma \subset \mathcal{U}_\Gamma$ is the tree part of the universal treed disk, and $\mathcal{T}_{0,\Gamma}$ its boundary part as in (8). Let $(F, G)$ be a Morse-Smale pair. For an integer $l \geq 0$ a domain-dependent perturbation of $F$ of class $C^l$ is a $C^l$ map

$$(10) \quad F_{\Gamma} : \mathcal{T}_{0,\Gamma} \times L \to \mathbb{R}$$

equal {\text{to the given function}} {F} \text{ away from the compact part:}$

$$F_{\Gamma}|(\mathcal{T}_{0,\Gamma} - \mathcal{T}_{0,\Gamma}^{\text{op}}) = \pi_2^* F$$

where $\pi_2$ is the projection on the second factor in (8).

(b) (Domain-dependent almost complex structure) Let $J \in \mathcal{J}_\tau(X)$ be a tamed almost complex structure. Let $l \geq 0$ be an integer. A domain-dependent almost complex structure of class $C^l$ for treed disks of type $\Gamma$ and base $J$ is a
map from the two-dimensional part $\mathfrak{S}_\Gamma$ of the universal curve $U_\Gamma$ to $J_\tau(X)$ given by a $C^1$ map

$$J_\Gamma : \mathfrak{S}_\Gamma \times X \to \text{End}(TX)$$

equal to the given $J$ away from the compact part:

$$J_\Gamma | (\mathfrak{S}_\Gamma - \mathfrak{S}^{\text{cp}}_\Gamma) = \pi_2^* J$$

where $\pi_2$ is the projection on the second factor in $\mathfrak{S}_\Gamma$.

Let $(X,\omega)$ be a compact symplectic manifold and $L \subset X$ a Lagrangian submanifold. Let $J \in J_\tau(X)$ be a base almost complex structure and $G \in G(L)$ a base metric so that $(F,G)$ is Morse-Smale.

**Definition 2.9.** (Perturbation data) A perturbation datum is pair $P_\Gamma = (F_\Gamma, J_\Gamma)$ consisting of a domain-dependent Morse function $F_\Gamma$ and a domain-dependent almost complex structure $J_\Gamma$.

The following are three operations on perturbation data:

**Definition 2.10.**

(a) (Cutting edges) Suppose that $\Gamma$ is a combinatorial type obtained by cutting an edge of $\Gamma'$. A perturbation datum for $\Gamma$ gives rise to a perturbation datum for $\Gamma'$ by pushing forward $P_\Gamma$ under the map $\pi_\Gamma'^{-1} : U_\Gamma \to U_{\Gamma'}$. That is, define

$$J_{\Gamma'}(z', x) = J_{\Gamma}(z, x), \quad \forall z \in (\pi_\Gamma'^{-1})^{-1}(z).$$

This is well-defined by the (Constant near the nodes and markings) axiom. The definition for $F_{\Gamma'}$ is similar.

(b) (Collapsing edges/making an edge/weight finite/non-zero) Suppose that $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making an edge/weight finite/non-zero. Any perturbation datum $P_\Gamma$ for $\Gamma$ induces a datum for $\Gamma'$ by pullback of $P_\Gamma$ under $\iota_\Gamma'^{-1} : U_{\Gamma'} \to U_\Gamma$.

(c) (Forgetting tails) Suppose that $\Gamma$ is a combinatorial type of stable treed disk is obtained from $\Gamma'$ by forgetting a semi-infinite edge. Consider the map of universal disks $f_{\Gamma'} : U_{\Gamma'} \to U_{\Gamma}$ given by forgetting the edge and stabilizing. Any perturbation datum $P_\Gamma$ induces a datum $P_{\Gamma'}$ by pullback of $P_\Gamma$.

We are now ready to define coherent collections of perturbation data. These are data that behave well with each type of operation in Definition 2.10.

**Definition 2.11.** (Coherent families of perturbation data) A collection of perturbation data $P = (P_\Gamma)$ is coherent if it is compatible with the morphisms of moduli spaces of different types in the sense that

(a) (Cutting edges) if $\Gamma$ is obtained from $\Gamma'$ by cutting an edge of infinite length, then $P_{\Gamma'}$ is the pushforward of $P_\Gamma$;

(b) (Collapsing edges/making an edge/weight finite/non-zero) if $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making an edge/weight finite/non-zero, then $P_{\Gamma'}$ is the pushback of $P_\Gamma$.
(c) (Products) if \( \Gamma \) is the union of types \( \Gamma_1, \Gamma_2 \) obtained by cutting an edge of \( \Gamma' \), then \( P_1 \) is obtained from \( P_{\Gamma_1} \) and \( P_{\Gamma_2} \) as follows: Let

\[
\pi_k : \mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2} \to \mathcal{M}_{\Gamma_k}
\]

denote the projection on the \( k \)th factor. Then \( \mathcal{U}_\Gamma \) is the union of \( \pi_1^* \mathcal{U}_{\Gamma_1} \) and \( \pi_2^* \mathcal{U}_{\Gamma_2} \). Then we require that \( P_1 \) is equal to the pullback of \( P_{\Gamma_k} \) on \( \pi_k^* \mathcal{U}_{\Gamma_k} \):

\[
(11) \quad P_1|_{\mathcal{U}_{\Gamma_k}} = \pi_k^* P_{\Gamma_k}.
\]

There is one important clarification corresponding to the case that the configuration is constant on one of the pieces: suppose that \( \Gamma_1 \) corresponds to a configuration with a single unmarked disk and two incoming leaves, one of which is weighted resp. forgettable as in the bottom row in Figure 13. Then by our conventions for (Cutting Edges) the corresponding incoming leaf of \( \Gamma_2 \) is weighted resp. forgettable, with the same weight as the leaf of \( \Gamma_1 \), and require \( (11) \).

(d) (Infinite weights) If a weight parameter \( \rho(e_i) \) is equal to infinity, then \( P_{\Gamma_1} \) is pulled back under the forgetful map forgetting the \( e_i \) semi-infinite edge and stabilizing from the perturbation datum \( P_1 \) given by (Forgetting tails). The last sentence of the previous item guarantees that this condition is compatible with the product axiom, in the case that forgetting an edge with infinite weight leads to a collapse of a disk component (so that the stabilization of \( \Gamma \) is \( \Gamma_2 \), in the notation of the previous paragraph.)

Let \( C \) be a possibly unstable treed disk of type \( \Gamma \). The stabilization of \( C \) is the stable treed disk of some type \( s(\Gamma) \) obtained by collapsing unstable surface and tree components. Thus the stabilization of any treed disk is the fiber of a universal treed disk \( \mathcal{U}_s(\Gamma) \). Given perturbation datum for the stabilization of the type \( \Gamma \), we obtain a domain-dependent almost complex structure and Morse function for \( C \), still denoted \( J_\Gamma, F_\Gamma \), by pull-back under the map \( C \to \mathcal{U}_\Gamma \).

**Definition 2.12.** (Perturbed holomorphic treed disks) Given perturbation datum \( P_\Gamma \), a holomorphic treed disk in \( X \) with boundary in \( L \) consists of a treed disk \( C = S \cup T \) and a continuous map \( u = (u_S, u_T) : C \to X \) such that the following holds: Let \( T = T_c \cup T_s \) be the splitting into boundary and interior parts as in \( (8) \).

(a) (Boundary condition) The Lagrangian boundary condition holds \( u(\partial S \cup T_c) \subset L \).

(b) (Surface equation) On the surface part \( S \) of \( C \) the map \( u \) is \( J_\Gamma \)-holomorphic for the given domain-dependent almost complex structure: if \( j \) denotes the complex structure on \( S \) then

\[
J_{\Gamma, u(z), \bar{z}} \, du_S = du_S \, j.
\]

(c) (Boundary tree equation) On the boundary tree part \( T_c \subset C \) the map \( u \) is a collection of gradient trajectories:

\[
\frac{d}{ds} u_T = -\text{grad}_{F_{\Gamma, (s, u(z))}} u_T.
\]
where $s$ is a local coordinate with unit speed. Thus for each edge $e \in \text{Edge}_-(\Gamma)$ the length of the trajectory is given by the length $u|_{e \subset T}$ is equal to $\ell(e)$.

(d) (Interior tree part) On the interior tree part $T_\circ \subset T$ the map $u$ is constant.

The last condition means that the interior parts of the tree are essentially irrelevant from our point of view. However, from a conceptual viewpoint if one is going to replace boundary markings with edges then one should also replace interior markings with edges; this conceptual point will become important in the proof of homotopy invariance. The stability condition for weighted treed disks is the following.

**Definition 2.13.** A weighted treed disk $u : C \to X$ with interior nodes $(z_1, \ldots, z_k)$ and boundary nodes $w_1, \ldots, w_m$ is stable if

(a) each disk component on which $u$ is constant has at least three special boundary points or one boundary node and one interior special point

$$du(C_i) = 0, \quad C_i \text{ disk} \quad \implies \quad 2\#\{z_k, w_k \in \text{int}(C_i)\} + \#\{w_k \in \partial C_i\} \geq 3;$$

(b) each sphere $C_i \subset C$ component on which $u$ is constant has at least three special points:

$$du(C_i) = 0, \quad C_i \text{ sphere} \quad \implies \quad \#\{z_k, w_k \in C_i\} \geq 3$$

and

(c) each infinite line on which $u$ is constant has a weighted leaf and an unforgettable or forgettable root:

$$du(C_i) = 0, \quad C_i \text{ line} \quad \implies \quad e_0 \in \text{Edge}^*(\Gamma) \text{ and } e_1 \in \text{Edge}^*(\Gamma) \cup \text{Edge}^\circ(\Gamma).$$

The stability condition in Definition 2.13 is not quite the same as having no automorphisms because of the third item. The latter is an exceptional case in which the trajectory does admit automorphisms, corresponding to translations of the line. However, the moduli space is still of expected dimension because of the condition on the weighting.

Equivalence of weighted treed disks is defined as follows. Given a non-constant holomorphic treed disk $u : C \to X$ with leaf $e_i$ for which the weighting $\rho(e_i) = \infty$ resp. 0, we declare $u$ to be equivalent to the holomorphic treed disk $u' : C \to X$ obtained by adding a constant trajectory with weighted incoming and forgettable resp. unforgettable outgoing edge. See Figure 11.

Removing a constant segment and relabelling gives equivalent holomorphic treed disks. Also, any two configurations with an outgoing weighted edge with the same underlying metric tree are considered equivalent. See Figure 12.

We introduce notation for various moduli spaces of equivalence classes of weighted treed disks. For integers $n, m$ denote by $\overline{\mathcal{M}}_{n,m}(L)$ the moduli space of equivalence classes of connected treed holomorphic disks with $n$ leaves and $m$ interior markings. For any connected combinatorial type $\Gamma$ of treed holomorphic disk, denote by $\mathcal{M}_{\Gamma}(L)$ the subset of type $\Gamma$. Define

$$\hat{\mathcal{I}}(L) = \mathcal{I}(L) - \{x_M\} \cup \{x_M^*, x_M^\circ, x_M^\check{\circ}\}.$$
Figure 11. Equivalent weighted treed disks

Figure 12. Equivalent weighted treed disks, ctd.

Thus $\mathcal{I}(L)$ is the union of the critical points of $F$, with the maximum $x_M$ replaced by three copies $x_M^*, x_M^!, x_M^\dagger$. We extend the index map on $\mathcal{I}(L)$ to $\hat{\mathcal{I}}(L)$ by

$$i(x_M^*) = i(x_M^!) = 0, \quad i(x_M^\dagger) = -1.$$ 

We define

$$\hat{\mathcal{I}}_d(L) = \{ x \in \hat{\mathcal{I}}(L) \mid i(x) = d \}.$$ 

The moduli spaces above break into components depending on the limits along the semi-infinite edges. An *admissible labelling* of a (non-broken) weighted treed disk $C$ with leaves $e_1, \ldots, e_n$ and outgoing edge $e_0$ is a sequence $x = (x_0, \ldots, x_n) \in \hat{\mathcal{I}}(L)$ satisfying:

(a) (Label axiom) If $x_i = x_M^*$ resp. $x_M^!$ resp. $x_M^\dagger$ then the corresponding semi-infinite edge is required to be weighted resp. forgettable resp. unforgettable, that is,

$$x_i = x_M^* \text{ resp. } x_M^! \text{ resp. } x_M^\dagger \implies e_i \in \text{Edge}^*(\Gamma) \text{ resp. } e_i \in \text{Edge}^!(\Gamma) \text{ resp. } e_i \in \text{Edge}^\dagger(\Gamma).$$
Furthermore, in this case the limit along the $i$-th semi-infinite edge is required to be $x_M$:

$$\lim_{s \to \infty} u(\varphi_e(s)) = x_M.$$ 

In every other case the semi-infinite edge is required to be unforgettable:

$$x_i \notin \{x_M^i, x_M^\ast, x_M^*\} \implies e_i \in \text{Edge}^e(\Gamma).$$

(b) (Outgoing edge axiom)

(i) The outgoing edge $e_0$ is weighted, $e_0 \in \text{Edge}^e(\Gamma)$ only if there are two incoming leaves $e_1, e_2 \in \text{Edge}_{\partial C}(\Gamma)$, exactly one of which, say $e_1$ is weighted with the same weight $\rho(e_1) = \rho(e_0)$, and the other, say $e_2$ is forgettable with weight $\rho(e_2) = \infty$, and there is a single disk with no markings.

(ii) The outgoing edge $e_0$ can only be forgettable, that is, $e_0 \in \text{Edge}^e(\Gamma)$ if either

- there are two forgettable incoming leaves $e_1, e_2 \in \text{Edge}^e(\Gamma)$, or
- there is a single leaf $e_1 \in \text{Edge}^e(\Gamma)$ that is weighted and the configuration has no interior leaves, that is, $\text{Edge}_{\partial C}(\Gamma) = \emptyset$.

See Figure 13.

The (Outgoing edge axiom) treats the case of constant holomorphic treed disks. As is typical in Floer theory, constant configurations must be treated with great care. Denote by $\mathcal{M}_\Gamma(L, x) \subset \mathcal{M}_\Gamma(L)$ the moduli space of isomorphism classes of stable holomorphic treed disks with boundary in $L$ and admissible labelling $x = (x_0, \ldots, x_n)$.

Remark 2.14. (Constant trajectories) If $x_1 = x_M^\ast$ and $x_0 = x_M^*$, resp. $x_0 = x_M^\ast$, then the moduli space $\mathcal{M}(L, x_0, x_1)$ contains a configuration with no disks and single edge on which $u$ is constant, corresponding to a weighted leaf and an outgoing end that is unforgettable resp. forgettable. These trajectories are pictured in Figure 13.

The moduli spaces are cut out locally by Fredholm maps in suitable Sobolev completions. Let $p \geq 2$ and $k \geq 0$ be integers. Denote by $\text{Map}^{k,p}(C, X, L)$ the space of continuous maps $u$ from $C$ to $X$ of Sobolev class $W^{k,p}$ on each disk, sphere and edge component such that the boundaries $\partial C := \partial S \cup T$ of the disks and edges mapping to $L$. In each local chart for each component of $C$ and $X$ the map $u$ is given by a collection of continuous functions with $k$ partial derivatives of class $L^p$. The space $\text{Map}^{k,p}(C, X, L)$ has the structure of a Banach manifold, with a local chart at $u \in \text{Map}^{k,p}(C, X, L)$ given by the geodesic exponential map

$$W^{k,p}(C, u^*TX, (u|_{\partial C})^*TL) \to \text{Map}^{k,p}(C, X, L), \quad \xi \mapsto \exp_u(\xi)$$

where we assume that the metric on $X$ is chosen so that $L$ is totally geodesic, that is, preserved by geodesic flow. Denote by

$$\text{Map}^{k,p}_\Gamma(C, X, L, D) \subset \text{Map}^{k,p}(C, X, L)$$

the subset of maps such that $u$ has the prescribed homology class on each component. This is a $C^q$ Banach submanifold where $q < k - n/p$. For each local trivialization of
the universal tree disk as in \textbf{(118)} we consider the ambient moduli space defined as follows. Let $\text{Map}_{k,p}(C, X, L)$ denote the space of maps of Sobolev class $k \geq 1, p > 2$ mapping the boundary of $C$ into $L$ and constant on each disk with no interior marking. Let $l \gg k$ be an integer and

\[ B_{k,p,\Gamma} := \mathcal{M}_{\Gamma} \times \text{Map}_{k,p}(C, X, L). \]

Consider the map given by the local trivialization

\[ \mathcal{M}_{\Gamma}^{\text{univ},i} \rightarrow \mathcal{J}(S), \ m \mapsto j(m). \]

Consider the fiber bundle $\mathcal{E}^i = \mathcal{E}_{k,p,\Gamma}^i$ over $B_{k,p,\Gamma}$ given by

\[ (\mathcal{E}_{k,p,\Gamma}^i)_{m,u,J} \subset \Omega^{0,1}_{j,\Gamma}(S, u^*_S TX)_{k-1,p} \oplus \Omega^1(T, u^*_T TL)_{k-1,p} \]

the space of 0,1-forms with respect to $j(m), J$ that vanish to order $m(e) - 1$ at the node or marking corresponding to each contact edge $e$. Local trivializations of $\mathcal{E}_{k,p,\Gamma}^i$ are defined by geodesic exponentiation from $u$ and parallel transport using the Hermitian connection defined by the almost complex structure

\[ \Phi_{\xi} : \Omega^{0,1}(S, u^*_S TX)_{k-1,p} \rightarrow \Omega^{0,1}(S, \exp_{u_S}(\xi)^* TX)_{k-1,p} \]

see for example \textbf{[77], p. 48}. The Cauchy-Riemann and shifted gradient operators applied to the restrictions $u_S$ resp. $u_T$ of $u$ to the two resp. one dimensional parts of $C = S \cup T$ define a $C^0$ section

\[ \tilde{\partial}_{\Gamma} : B_{k,p,\Gamma} \rightarrow \mathcal{E}_{k,p,\Gamma}^i, \quad (m, u, J, F) \mapsto \left( \tilde{\partial}_{j(m),J} u_S, \left( \frac{d}{ds} - \text{grad}_F \right) u_T \right) \]

where

\[ \tilde{\partial}_{j(m),J} u := \frac{1}{2} (J du_S - du_S j(m)), \]

\[ \partial_{j(m),J} u := \frac{1}{2} (J du_S - du_S j(m)), \]
and $s$ is a local coordinate with unit speed. The local moduli space is
\[ \mathcal{M}^\text{univ, i}_\Gamma(L) = \overline{\partial}^{-1} B_{k,p, \Gamma}^i \]
where $B_{k,p, \Gamma}^i$ is embedded as the zero section. Define a non-linear map on Banach spaces using the local trivializations
\[ \mathcal{F}_u : \Omega^0(C^\times, u^*(TX, TL))_{k,p} \times \mathcal{M}_\Gamma \rightarrow \Omega^{0,1}(C^\times, u_S^*TX)_{k-1,p} \oplus \Omega^1(C^\times, u_T^*TL)_{k-1,p} \]
\[ , \quad (\xi, C) \mapsto \Phi_t^{-1} \overline{\partial}_J \exp_u(\xi). \]
Let
\[ D_u = D_0 \mathcal{F}_u = \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}_u(t\xi) \]
denote the linearization of $\mathcal{F}_u$ (see Floer-Hofer-Salamon [38, Section 5]). Standard arguments show that the operator $D_u$ is Fredholm.

2.4. Transversality. In this section we use domain-dependent almost complex structures and metrics to regularize the moduli space of holomorphic disks with boundary in a Lagrangian submanifold. In order to use domain-dependent almost complex structures, the surface part of the domains must be stable. We begin by recalling the existence of stabilizing divisors for Lagrangian manifolds in Charest-Woodward [25], see also Cieliebak-Mohnke [27]. For a submanifold $L \subset X$, let $h_2 : \pi_2(X, L) \rightarrow H_2(X, L)$ be the degree two relative Hurewicz morphism. Denote by $[\omega]^\vee : H_2(X, L) \rightarrow \mathbb{R}$ the map induced by pairing with $[\omega] \in H^2(X, L)$.

**Definition 2.15.** (Rational Lagrangians)

(a) A symplectic manifold $X$ with two-form $\omega \in \Omega^2(X)$ is rational if the class $[\omega] \in H^2(X, \mathbb{R})$ is rational, that is, in the image of $H^2(X, \mathbb{Q}) \rightarrow H^2(X, \mathbb{R})$. Equivalently, $(X, \omega)$ is rational if there exists a linearization of $X$: a line bundle $\tilde{X} \rightarrow X$ with a connection whose curvature is $(2\pi k/i)\omega$ for some integer $k > 0$.

(b) A Lagrangian $L \subset X$ of a rational symplectic manifold $X$ with linearization $\tilde{X}$ is strongly rational if $\tilde{X}|L$ has a covariant constant section $L \rightarrow \tilde{X}|L$.

(c) An immersed Lagrangian submanifold $L \subset X$ is rational if
\[ [\omega]^\vee \circ h_2(\pi_2(L)) = \mathbb{Z} \cdot e \subset \mathbb{R} \]
for some $e > 0$.

The rationality assumptions guarantee that we can find stabilizing divisors in the following sense.

**Definition 2.16.** (Stabilizing divisors)

(a) A divisor in $X$ is a closed codimension two symplectic submanifold $D \subset X$. An almost complex structure $J : TX \rightarrow TX$ is adapted to a divisor $D$ if $D$ is an almost complex submanifold of $(X, J)$.

(b) A divisor $D \subset X$ is stabilizing for a Lagrangian submanifold $L \subset X$ iff

(i) it is disjoint from $L$, that is, $D \subset X - L$, and
(ii) any disk \( u : (C, \partial C) \to (X, L) \) with non-zero area \( \omega([u]) > 0 \) intersects \( D \) in at least one point.

(c) A divisor \( D \subset X \) is weakly stabilizing for a Lagrangian submanifold \( L \subset X \) iff

(a) it is disjoint from \( L \), that is, \( D \subset X - L \), and

(b) there exists an almost-complex structure \( J_D \in J(X, \omega) \) adapted to \( D \) such that any non-trivial \( J_D \)-holomorphic disk \( u : (C, \partial C) \to (X, L) \) intersects \( D \) in at least one point.

Recall from Charest-Woodward \[25\] the existence result for stabilizing divisors:

**Theorem 2.17.** \[25\] Section 4] There exists a divisor \( D \subset X - L \) that is weakly stabilizing for \( L \) representing \( k[\omega] \) for some sufficiently large integer \( k \). Moreover, if \( L \) is rational resp. strongly rational then there exists a divisor \( D \subset X - L \) that is stabilizing for \( L \) resp. stabilizing for \( L \) representing \( k[\omega] \) for some large \( k \) and such that \( L \) is exact in \( (X - D, \omega|_{X - D}) \).

The existence of the divisors in the theory is an application of the theory of Donaldson-Auroux-Gayet-Mohsen stabilizing divisors \[33, 10\]. In the case that \( X \) is a smooth projective algebraic variety, stabilizing divisors may be obtained using a result of Borthwick-Paul-Uribe \[15\]. Roughly speaking one chooses an approximately holomorphic section concentrated on the Lagrangian; then a generic perturbation defines the desired divisor. For example, in the case \( L \) is a circle in the symplectic two-sphere \( X \) then a stabilizing divisor \( D \) is given by choosing a point in each component of the complement of \( L \). There is also a time-dependent version of this result which will be used later to prove independence of the homotopy type of the Fukaya algebra from the choice of base almost complex structure: If \( J^t, t \in [0, 1] \) is a smooth path of compatible almost complex structures on \( X \) then (see \[25\] Lemma 4.20) there exists a path of \( J^t \)-stabilizing divisors \( D^t, t \in [0, 1] \) connecting \( D^0, D^1 \).

**Definition 2.18.** (Adapted stable treed disks) Let \( L \) be a compact Lagrangian and \( D \) a codimension two submanifold disjoint from \( L \). A stable treed disk \( u : C \to X \) with boundary in \( L \) is adapted to \( D \) iff

(a) (Stable domain) The surface part of \( C \) is stable (that is, each disk component \( C_v \subset C, v \in \text{Vert}_v(\Gamma) \) has either one interior special point and one special point on the boundary \( w_j \in \partial C_i \) or three special points on the boundary \( w_i, w_j, w_k \in \partial C_i \); and each sphere component \( C_v, v \in \text{Vert}_v(\Gamma) \) has at least three special points;

(b) (Non-constant spheres) Each component \( C_i \) of \( C \) that maps to \( D \) is constant:

\[
u(C_i) \subset D \implies du|C_i = 0.
\]

That is, there are no non-constant sphere components of \( C \) mapping entirely to \( D \).

(c) (Markings) Each marking \( z_e \) corresponding to an interior semi-infinite leaf \( e \in \text{Edge}_s, \Gamma \) maps to \( D \), that is, \( u(z_e) \in D \), and each connected component of \( u^{-1}(D) \) contains an interior leaf:

\[
u(e_i) \subset D, \quad C_i \subset u^{-1}(D) \text{ connected component} \implies \text{Edge}_{s, i}(\Gamma) \cap C_i \neq \emptyset.
\]
We denote by $z_i$ the point where $e_i$ attaches to the surface part and call it an interior marking.

Thus $u : C \to X$ consists of components on which $u$ is non-constant which meet $D$ in finitely many points and components on which $u$ is constant. Note that any ghost component (component on which $u$ is constant) containing an interior marking is required to map to $D$.

The moduli space of treed holomorphic disks is stratified by combinatorial type as follows. We denote by $\Pi(X)$ resp. $\Pi(X, L)$ the space of homotopy classes of maps from the two sphere resp. disk with boundary in $L$. For point $z_e \in C$ mapping to $D$ near which $u$ is non-constant, we denote by $m(e) \in \mathbb{Z}_{\geq 0}$ the multiplicity of the intersection with the divisor $D$ at $z_e$. This is the winding number of small loop around $z_e$ in the complement of the zero section in a tubular neighborhood of $D$ in $X$. The combinatorial type of a holomorphic treed disk $u : C \to X$ adapted to $D$ consists of

(a) the combinatorial type $\Gamma = (\text{Vert}(\Gamma), \text{Edge}(\Gamma))$ of its domain $C$ together with

(b) the labelling

$$d : \text{Vert}(\Gamma) \to \Pi(X) \cup \Pi(X, L)$$

of each vertex $v$ of $\Gamma$ corresponding to a disk or sphere component with the corresponding homotopy class and

(c) the labelling

$$m : \text{Edge}_*(\Gamma) \to \mathbb{Z}_{\geq 0}$$

recording the order of tangency of the map $u$ to the divisor $D$ at each of the interior markings $z_e, e \in \text{Edge}_*(\Gamma)$ and spherical nodes $w_e, e \in \text{Edge}_{*,-}(\Gamma)$ mapping to a divisor; if the node does not map to the divisor the order is by definition zero.

Note that at any spherical node mapping to the divisor, at least one component on one side of the node is constant mapping to the divisor by the (Markings) axiom. We take the order of tangency at this node to be the order of tangency on this constant component, or zero if the components on both side map to the divisor. Let $\mathcal{M}(L, D)$ the moduli space of stable treed marked disks in $X$ with boundary in $L$ adapted to $D$ and $\mathcal{M}_{\Gamma}(L, D)$ the locus of combinatorial type $\Gamma$. For $x \in \mathcal{T}(L)^n$ let

$$\mathcal{M}_{\Gamma}(L, D, x) \subset \mathcal{M}_{\Gamma}(L, D)$$

denote the adapted subset made of holomorphic treed disks of type $\Gamma$ adapted to $D$ with limits $x = (x_0, \ldots, x_n) \in \mathcal{T}(L)$ along the root and leaves.

The expected dimension of the moduli space is given by a formula involving the Maslov indices of the disks and the indices and number of semi-infinite edges. If $u_i = u|_{C_i}, i = 1, \ldots, k$ of $u$, denote by the Maslov index $I(u_i)$. The expected
The dimension of $\mathcal{M}_\Gamma(L, x)$ at $[u : C \to X]$ is given by

$$i(\Gamma, x) := i(x_0) - \sum_{i=1}^{n} i(x_i) + \sum_{i=1}^{k} I(u_i) + n - 2 - |\text{Edge}_0^\Gamma| - |\text{Edge}_\infty^\Gamma|$$

$$- 2|\text{Edge}_{\infty}^\Gamma| - \sum_{e \in \text{Edge}_{\infty}^\Gamma(\Gamma)} 2m(e) - \sum_{e \in \text{Edge}_{\infty}^\Gamma(\Gamma)} 2m(e).$$

In order to obtain transversality we begin by fixing an open subset of the universal curve on which the perturbations will vanish. Let

$$\mathcal{U}_\Gamma^{\text{cp}} \subset \mathcal{U}_\Gamma$$

be a compact subset disjoint from the nodes and attaching points of the edges such that the interior of $\mathcal{U}_\Gamma^{\text{cp}}$ in each two and one-dimensional component is open. Suppose that perturbation data $P_\Gamma$ for all boundary types $\mathcal{U}_\Gamma^{\text{cp}}$ have been chosen. Let

$$\mathcal{P}_\Gamma^l(X, D) = \{ P_\Gamma = (F_\Gamma, J_\Gamma) \}$$

denote the space of perturbation data $P_\Gamma = (F_\Gamma, J_\Gamma)$ of class $C^l$ that are

- equal to the given pair $(F, J)$ on $\mathcal{U}_\Gamma \setminus \mathcal{U}_\Gamma^{\text{cp}}$, and such that
- the restriction of $P_\Gamma$ to $\mathcal{U}_\Gamma^{\text{cp}}$ is equal to $P_\Gamma$, for each boundary type $\Gamma'$, that is, type of lower-dimensional stratum $\mathcal{M}_{\Gamma'} \subset \mathcal{M}_\Gamma$.

The second condition will guarantee that the resulting collection satisfies the (Collapsing edges/Making edges/weights finite or non-zero) axiom of the coherence condition Definition 2.11. Let $\mathcal{P}_\Gamma(X, D)$ denote the intersection of the spaces $\mathcal{P}_\Gamma^l(X, D)$ for $l \geq 0$.

One cannot expect, using stabilizing divisors, to obtain transversality for all combinatorial types. The reason is a rather trivial analog of the multiple cover problem: once one has a ghost bubble mapping to the divisor and containing a marking then one has configurations with arbitrary number of markings on that component, whose expected dimension goes to minus infinity but which are all non-empty. A type $\Gamma$ will be called uncrowded if each maximal ghost component contains at most one marking. Write $\Gamma' \leq \Gamma$ iff $\Gamma$ is obtained from $\Gamma'$ by (Collapsing edges/making edge lengths or weights finite/non-zero) or $\Gamma'$ is obtained from $\Gamma$ by (Forgetting a forgettable tail).

**Theorem 2.19.** (Transversality) Suppose that $\Gamma$ is an uncrowded type of stable treed marked disk of expected dimension $i(\Gamma, x) \leq 1$, see (6.17). Suppose regular coherent perturbation data for types of stable treed marked disk $\Gamma'$ with $\Gamma' \leq \Gamma$ are given. Then there exists a comeager subset

$$\mathcal{P}_\Gamma^{\text{reg}}(X, D) \subset \mathcal{P}_\Gamma(X, D)$$

of regular perturbation data for type $\Gamma$ coherent with the previously chosen perturbation data such that if $P_\Gamma \in \mathcal{P}_\Gamma^{\text{reg}}(X, D)$ then

(a) (Smoothness of each stratum) the stratum $\mathcal{M}_\Gamma(L, D)$ is a smooth manifold of expected dimension;
FLOER COHOMOLOGY AND FLIPS
33

(b) (Tubular neighborhoods) if \( \Gamma \) is obtained from \( \Gamma' \) by collapsing an edge of \( \text{Edge}_c(\Gamma') \) or making an edge or weight finite/non-zero or by or by gluing \( \Gamma' \) at a breaking then the stratum \( \mathcal{M}_{\Gamma'}(L, D) \) has a tubular neighborhood in \( \mathcal{M}_{\Gamma}(L, D) \); and

(c) (Orientations) there exist orientations on \( \mathcal{M}_{\Gamma}(L, D) \) compatible with the morphisms (Cutting an edge) and (Collapsing an edge/Making an edge/weight finite/non-zero) in the following sense:

(i) If \( \Gamma \) is obtained from \( \Gamma' \) by (Cutting an edge) then the isomorphism \( \mathcal{M}_{\Gamma'}(L, D) \to \mathcal{M}_{\Gamma}(L, D) \) is orientation preserving.

(ii) If \( \Gamma \) is obtained from \( \Gamma' \) by (Collapsing an edge) or (Making an edge/weight finite/non-zero) then the inclusion \( \mathcal{M}_{\Gamma'}(L, D) \to \mathcal{M}_{\Gamma}(L, D) \) has orientation (using the decomposition

\[ T\mathcal{M}_{\Gamma}(L, D)|\mathcal{M}_{\Gamma'}(L, D) \cong \mathbb{R} \oplus T\mathcal{M}_{\Gamma'}(L, D) \]

and the outward normal orientation on the first factor) given by a universal sign depending only on \( \Gamma, \Gamma' \).

**Proof.** The transversality statement is an application of the Sard-Smale theorem. Let \( \Gamma \) be a combinatorial type represented by a treed disk \( C \). In the first part of the proof we assume that \( \Gamma \) has no forgettable leaves and construct a perturbation datum \( P_{\Gamma} \) by extending the given perturbation data on the boundary of the universal moduli space. Let \( p \geq 2 \) and \( k \geq 0 \) be integers. Denote by \( \text{Map}^{k,p}(C, X, L) \) the space of continuous maps \( u \) from \( C \) to \( X \) of Sobolev class \( W^{k,p} \) on each disk, sphere and edge component such that the boundaries \( \partial C := \partial S \cup T \) of the disks and edges mapping to \( L \). In each local chart for each component of \( C \) and \( X \) the map \( u \) is given by a collection of continuous functions with \( k \) partial derivatives of class \( L^p \).

The space \( \text{Map}^{k,p}(C, X, L) \) has the structure of a Banach manifold, with a local chart at \( u \in \text{Map}^{k,p}(C, X, L) \) given by the geodesic exponential map

\[ W^{k,p}(C, T_X, (u|_{\partial C})^*TL) \to \text{Map}^{k,p}(C, X, L), \quad \xi \mapsto \exp_u(\xi) \]

where we assume that the metric on \( X \) is chosen so that \( L \) is totally geodesic, that is, preserved by geodesic flow. For any combinatorial type \( \Gamma \) with vertices labelled by homology classes denote by \( \text{Map}^{k,p}_{\Gamma}(C, X, L) \subset \text{Map}^{k,p}(C, X, L) \) the subset of maps such that \( u \) has the prescribed homology class on each component.

For each local trivialization of the universal tree disk as in \([18]\) we consider the ambient moduli space defined as follows. Let \( \text{Map}^{k,p}_{\Gamma}(C, X, L, D) \) denote the space of maps of Sobolev class \( k \geq 1, p > 2 \) mapping the boundary of \( C \) into \( L \), the interior markings into \( D \), and constant on each disk with no interior marking, and with the prescribed order of vanishing at the intersection points with \( D \). Let \( l \gg k \) be an integer and

\[ B^i_{k,p,l,\Gamma} := \mathcal{M}_{\Gamma}^i \times \text{Map}^{k,p}_{\Gamma}(C, X, L, D) \times \mathcal{P}^l_{\Gamma}(X, D). \]

Consider the map given by the local trivialization

\[ \mathcal{M}_{\Gamma}^{\text{univ},i} \to J(S), \quad m \mapsto j(m). \]
Consider the fiber bundle $\mathcal{E}^i = \mathcal{E}^i_{k,p,l,\Gamma}$ over $\mathcal{B}^i_{k,p,l,\Gamma}$ given by

$$(\mathcal{E}^i_{k,p,l,\Gamma})_{m,u,J} \subset \Omega^0_{j,\Gamma}(S,(u_S)^*TX)_{k-1,p} \oplus \Omega^1(T,(u_T)^*TL)_{k-1,p}$$

the space of 0, 1-forms with respect to $j(m)$, $J$ that vanish to order $m(e) - 1$ at the node or marking corresponding to each contact edge $e$. The Cauchy-Riemann and shifted gradient operators applied to the restrictions $u_S$ resp. $u_T$ of $u$ to the two resp. one dimensional parts of $C = S \cup T$ define a $C^q$ section

$$(18) \quad \bar{\partial}_T : \mathcal{B}^i_{k,p,l,\Gamma} \to E^i_{k,p,l,\Gamma}, \quad (m,u,J,F) \mapsto \left( \bar{\partial}_{j(m),J}u_S,(\frac{d}{ds} - \text{grad}_F)u_T \right)$$

where

$$(19) \quad \bar{\partial}_{j(m),J}u := \frac{1}{2}(Jdu_S - du_Sj(m)),$$

and $s$ is a local coordinate with unit speed. The local universal moduli space is

$$\mathcal{M}^\text{univ,i}(L,D) = \bar{\partial}^{-1}\mathcal{B}^i_{k,p,l,\Gamma}$$

where $\mathcal{B}^i_{k,p,l,\Gamma}$ is embedded as the zero section.

We claim that the local universal moduli space is cut out transversally. To see this, suppose that

$$\eta = (\eta_2,\eta_1) \in \Omega^0_{j,\Gamma}(S,(u_S)^*TX)_{k-1,p} \oplus \Omega^1(T,(u_T)^*TL)_{k-1,p}$$

is in the cokernel of derivative $D_{u,P}$ of $\mathcal{E}^i_{k,p,l,\Gamma}$ with 2-dimensional part $\eta_2$ and one-dimensional part $\eta_1$. Variation of $\mathcal{E}^i_{k,p,l,\Gamma}$ with respect to the section $\xi_1$ on the one-dimensional part $T$ gives

$$0 = \int_T(D_{u_1}\xi_1,\eta_1)ds = \int_T(\xi_1,D_{u_1}^*\eta_1)ds, \forall \xi \in \Omega^0_c(u_T^*TL)$$

where $D_{u_1}$ is the operator of $(104)$, hence

$$(20) \quad \nabla_\xi \ast \eta_1 = 0.$$ 

On the other hand, the linearization of $\mathcal{E}^i_{k,p,l,\Gamma}$ with respect to the Morse function $F \mapsto -\text{grad}_F u_T$ is pointwise surjective:

$$\{ -\text{grad}_F u_T(s) \mid F_T \in C^\ell_c(\mathcal{T}_c,\Gamma \times L) \} = T_{u_T(s)}L.$$ 

This implies that

$$(21) \quad \eta_1(u_T(s)) = 0, \quad \forall s \in \mathcal{U}_\Gamma - \mathcal{U}_\Gamma^\text{thin}.$$ 

Combining $(20)$ and $(21)$ implies that $\eta_1 = 0$. By similar arguments the two-dimensional part $\eta_2$ satisfies

$$(22) \quad (D_u\xi_2,\eta_2) = 0, \quad \int_S(Y \circ du \circ j) \wedge \eta_2 = 0$$

for every $\xi_2 \in \Omega^0(S,u^*TX)$ with given orders of vanishing at the intersection points with $D$ and variation of almost complex structure $Y \in \Omega^0(S,u^*\text{End}(TX))$ as in [77, Chapter 3]. Hence in particular $D^*_u\eta_2 = 0$ away from the intersection points with $D$. So if $du(z) \neq 0$ for some $z \in S$ not equal to an intersection point then $\eta_2$ vanishes in
an open neighborhood of $z$. By unique continuation $\eta_2$ vanishes everywhere on the component of $S$ containing $z$ except possibly at the intersection points with $D$. At these points $\eta_2$ could in theory be a sum of derivatives of delta functions; That $\eta_2$ also vanishes at these intersection points follows from [26, Lemma 6.5, Proposition 6.10].

It remains to consider components of the two-dimensional part on which the map is constant. Let $S' \subset S$ be a union of disk and sphere components of $S \subset C$. If $u : C \to X$ is a map that is constant on $S' \subset C$ the linearized operator is constant on each disk component of $S'$ and surjective by a doubling trick. However, we also must check that the matching conditions at the nodes are cut out transversally. Let $S''$ denote the normalization of $S'$, obtained by replacing each nodal point in $S'$ with a pair of points in $S''$. Since the combinatorial type of the component is a subgraph of a tree, the combinatorial type must itself be a tree. We denote by $T_uL$ the tangent space at the constant value of $u$ on $S'$. Taking the differences of the maps at the nodes defines a map

$$\delta : \ker(D_u|S'') \cong T_uL^k \to T_u^{m} , \quad \xi \mapsto (\xi(w^i_+) - \xi(w^-_i))_{i=1}^m.$$  

An explicit inverse to $\delta$ is given by defining recursively as follows. Consider the orientation on the combinatorial type $\Gamma'' \subset \Gamma$ induced by the choice of outgoing semi-infinite edge of $\Gamma$. For $\eta \in T_u^m$ define an element $\xi \in T_uL^k$ by

$$\xi(t(e)) = \xi(h(e)) + \eta(e)$$

whenever $t(e), h(e)$ are the head and tail of an edge $e$ corresponding to a node. This element may be defined recursively starting with the edge $e''_0$ of $\Gamma''$ closest to the outgoing edge of $\Gamma$ and taking $\xi(v) = 0$ for $v$ the vertex corresponding to the disk component closest to outgoing edge. The matching conditions at the nodes connecting $S'$ with the complement $S - S'$ are also cut out transversally. Indeed on the adjacent components the linearized operator restricted to sections vanishing at the node is already surjective as in [26, Lemma 6.5]. This implies that the conditions at the boundary nodes cut out the moduli space transversally. A similar discussion for collections of sphere components on which the map is constant implies that the matching conditions at the spherical nodes are also cut out transversally. This completes the proof that the parametrized linearized operator is surjective.

Using the surjectivity of the parametrized linearized operator in the previous paragraphs we apply the implicit function theorem to conclude that the local universal moduli space is a Banach manifold. More precisely, $\mathcal{M}_{\Gamma}^{\text{univ,}i}(L, D)$ is a Banach manifold of class $C^q$, and the forgetful morphism $\varphi_i : \mathcal{M}_{\Gamma}^{\text{univ,}i}(L, D) \to \mathcal{P}_L(L, D)_i$ is a $C^q$ Fredholm map. Let $\mathcal{M}_{\Gamma}^{\text{univ,}i}(L, D)_d \subset \mathcal{M}_{\Gamma}^{\text{univ,}i}(L, D)$ denote the component on which $\varphi_i$ has Fredholm index $d$. By the Sard-Smale theorem, for $k, l$ sufficiently large the set of regular values $\mathcal{P}_L^{l, \text{reg}}(X)_l$ of $\varphi_i$ on $\mathcal{M}_{\Gamma}^{\text{univ,}i}(L, D)_d$ in $\mathcal{P}_L(X)_l$ is comeager. Let

$$\mathcal{P}_L^{l, \text{reg}}(X)_l = \bigcap_i \mathcal{P}_L^{l, \text{reg}}(X)_l.$$  

A standard argument shows that the set of smooth domain-dependent $\mathcal{P}_L^{\text{reg}}(L, D)$ is also comeager. Fix $(J_\Gamma, F_\Gamma) \in \mathcal{P}_L^{\text{reg}}(L, D)$. By elliptic regularity, every element
of $\mathcal{M}_\Gamma(L,D)$ is smooth. The transition maps for the local trivializations of the universal bundle define smooth maps

$$\mathcal{M}_\Gamma(L,D)|_{\mathcal{M}_\Gamma \cap \mathcal{M}'_\Gamma} \to \mathcal{M}'_\Gamma(L,D)|_{\mathcal{M}_\Gamma \cap \mathcal{M}'_\Gamma}.$$ 

This construction equips the space $\mathcal{M}_\Gamma(L,D) = \cup_i \mathcal{M}_i(L,D)$ with a smooth atlas. Since $\mathcal{M}_\Gamma$ is Hausdorff and second-countable, so is $\mathcal{M}_\Gamma(L,D)$ and it follows that $\mathcal{M}_\Gamma(L,D)$ has the structure of a smooth manifold.

Suppose that the combinatorial type $\Gamma$ has a forgettable leaf. In this case the choice of perturbation datum is constrained by the requirement that the perturbation datum is pulled back under the forgetful map if there is at least one interior marking on $\Gamma$, or at least three leaves. In this case, there exists a type $\Gamma'$ obtained by forgetting a forgettable leaf of $\Gamma$. By assumption a regular perturbation datum $P_{\Gamma'}$ has been chosen, and define $P_{\Gamma}$ by pull-back under the forgetful morphism as in Definition 2.11 (d). The linearized operator is automatically surjective, and the moduli space cut out transversally. On the other hand, suppose that $\Gamma$ has no interior markings and only two incoming leaves. In this case, let $P_{\Gamma}$ be the trivial perturbation datum. That is, let $P_{\Gamma}$ consist of the constant almost complex structure $J_{\Gamma} = J$ and Morse function $F_{\Gamma} = F$. The moduli space $\mathcal{M}_\Gamma$ is in this case simply the intersection $W_{x_0}^{-} \cap W_{x_1}^{-}$ of the unstable manifolds of the labels on the incoming leaves. Since the incoming leaves are labelled $x_0 = x_1 = x_M$, the maximum of the Morse function, the intersection is transverse and the perturbation data is regular.

The construction of tubular neighborhoods and orientations is similar to the case treated in [25], with the added ingredient of estimates for gluing Morse trajectories in the case of a broken Morse trajectory from, e.g., Schwarz [94].

Remark 2.20. The strata $\mathcal{M}_\Gamma(L,D)$ of expected dimension one have boundary points that are of two possible types.

(a) The first possibility is that $\Gamma$ has a broken edge, as in Figure 6.10

![Figure 14. Treed disk with a broken edge](image)

In the case that no component with no vertex is involved, the normal bundle to the stratum has fiber canonically isomorphic to $\mathbb{R}_{\geq 0}$, corresponding
to deformations that make the length finite. Note that the broken edge may be a leaf and that that leaf may have label $x_M^r$ and weight 0 or $\infty$ as in Remark 2.14. In this case any configuration $u$ of type $\Gamma$ is constant on the first segment in the leaf and the normal bundle is again isomorphic to $\mathbb{R}_{\geq 0}$, corresponding to deformations that replace that segment with an unbroken segment labelled $x_M^r$ and deform the weight away from 0 resp. $\infty$.

(b) The second possibility is that $\Gamma$ corresponds to a stratum with a boundary node: either $T$ has an edge of length zero or equivalently $S$ has a disk with a boundary node. See Figure 15.

We call $\mathcal{M}_\Gamma(L, D)$ in the first resp. second case a true resp. fake boundary stratum. A fake boundary stratum is not a part of the boundary of the moduli space of adapted treed disks in the sense that the space is not locally homeomorphic to a manifold with boundary near such a stratum since any such treed disk may be deformed either by deforming the disk node, or deforming the length of the edge corresponding to the node to a positive real number.

Orientations on moduli spaces of treed holomorphic disks are defined in the presence of relative spin structures on the Lagrangian. A relative spin structure for an oriented Lagrangian $L \subset X$ is a lift of the Čech class of its tangent bundle to relative non-abelian cohomology with values in Spin relative to the map $i : L \to X$; equivalently, a lift of the transition maps $\psi_{\alpha\beta}$ for $TL$ with respect to some cover $\{U_\alpha\}$ of $L$ to Spin satisfying the cocycle condition $\psi_{\alpha\beta}\psi_{\alpha\gamma}^{-1}\psi_{\beta\gamma} = i^*\epsilon_{\alpha\beta\gamma}$ where $\epsilon_{\alpha\beta\gamma}$ is a 2-cycle on $X$. Given a relative spin structure, the determinant line on the treed holomorphic disk is oriented as follows: Fix coherent orientations on the stable and unstable manifolds of the Morse function on $L$. Let $[u : C \to X] \in \mathcal{M}_\Gamma(L, D)$ be an equivalence class of stable adapted holomorphic treed disks of combinatorial type $\Gamma$ of expected dimension 0. Assuming that regularity has been achieved, the linearization $D\bar{\partial}_T|_{\bar{\partial}T}$ of the section $\bar{\partial}_T$ of (106) restricted to the perturbation $P_T$ is an isomorphism. Choose orientations on the stable and unstable manifolds of the
Morse function $W^\pm(x_i)$ for $x_i \in \mathcal{I}(L)$ so that the map
\[ T_{x_i}W^-(x_i) \oplus T_{x_i}W^+(x_i) \to T_{x_i}L \]
induces an isomorphism of determinant lines
\[ \det(T_{x_i}W^-(x_i) \oplus T_{x_i}W^+(x_i)) \to \det(T_{x_i}L). \]
In case $x_i = x_M$ we define $W^\pm(x_M) = W^\pm(x_M) \times \mathbb{R}$ and choose orientations similarly. One naturally obtains an orientation of the determinant line of the linearized operator from the isomorphism
\[
\det(D\bar{\partial}_t|_{P_t}) \to \det(TM_t) \otimes \det(TW^+(x_0)) \otimes \bigotimes_{i=1}^n \det(TW^-(x_i)),
\]
the orientations on the stable and unstable manifolds and the orientation on the underlying moduli space of treed disks. See [23], [109] for similar discussions. The case of the type $\Gamma$ of a trivial trajectory with no disks is treated separately: In the case of a trajectory connecting $x_M$ with $x_0$ resp. $x_M$ the moduli space is a point and we define the orientation to agree resp. disagree with the standard orientation.

2.5. Compactness. In this section we show that the subset of the moduli space satisfying an energy bound is compact for suitable perturbation data. For arbitrary combinatorial types, compactness of the spaces of adapted treed disks (so that they have stable surface part domains) can fail since bubbles mapping entirely to the stabilizing divisor can develop.

**Definition 2.21.** For $E > 0$, an almost complex structure $J_D \in \mathcal{J}_r(X)$ is $E$-stabilized by a divisor $D$ iff

(a) (Non-constant spheres) $D$ contains no non-constant $J_D$-holomorphic spheres of energy less than $E$; and

(b) (Sufficient intersections) each non-constant $J_D$-holomorphic sphere $u : C \to X$ resp. $J_D$-holomorphic disk $u : (C, \partial C) \to (X, L)$ with energy less than $E$ has at least three resp. one intersection points resp. point with the divisor $D$:
\[ E(u) < E \implies \#u^{-1}(D) \geq 1 + 2(\chi(C) - 1) \]
where $\chi(C)$ is the Euler characteristic.

Denote by $H_2(X, \mathbb{Z})_\bullet \subset H_2(X, \mathbb{Z})$ the set of classes representing non-constant $J_D$-holomorphic spheres, and $H_2(X, \mathbb{Z})_c \subset H_2(X, \mathbb{Z})$ the set of classes representing non-constant $J_D$-holomorphic disks with boundary in $L$. A divisor $D$ with Poincaré dual $[D]^\vee = k[\omega]$ for some $k \in \mathbb{N}$ has sufficiently large degree for an almost complex structure $J_D$ iff
\[
([D]^\vee, \alpha) \geq 2(c_1(X), \alpha) + \dim(X) + 1 \quad \forall \alpha \in H_2(X, \mathbb{Z})_\bullet,
([D]^\vee, \beta) \geq 1 \quad \forall \beta \in H_2(X, L, \mathbb{Z})_c.
\]

We introduce the following notation for spaces of almost complex structures sufficiently close to the given one. Given $J \in \mathcal{J}_r(X, \omega)$ denote by
\[ \mathcal{J}_r(X, D, J, \theta) = \{ J_D \in \mathcal{J}_r(X, \omega) \mid ||J_D - J|| < \theta, \quad J_D(TD) = TD \} \]
the space of tamed almost complex structures close to \( J \) in the sense of [26, p. 335] and preserving \( TD \).

**Lemma 2.22.** For \( \theta \) sufficiently small, suppose that \( D \) has sufficiently large degree for an almost complex structure \( \theta \)-close to \( J \) in the space of tamed almost complex structures close to \( J \) in the sense of [26, p. 335] and preserving \( TD \).

For \( \theta \) sufficiently small, suppose that \( D \) has sufficiently large degree for an almost complex structure \( \theta \)-close to \( J \). For each energy \( E > 0 \), there exists an open and dense subset \( \mathcal{J}^*(X, D, J, \theta, E) \) in \( \mathcal{J}_\tau(X, D, J, \theta) \) such that if \( J_D \in \mathcal{J}^*(X, D, J, \theta, E) \), then \( J_D \) is \( E \)-stabilized by \( D \). Similarly, if \( D = (D^t) \) is a family of divisors for \( J^t \), then for each energy \( E > 0 \), there exists a dense and open subset \( \mathcal{J}^*(X, D^t, J^t, \theta, E) \) in the space of time-dependent tamed almost complex structures \( \mathcal{J}_\tau(X, D^t, J^t, \theta) \) such that if \( J_D^t \in \mathcal{J}^*(X, D^t, J^t, \theta, E) \), then \( J_D^t \) is \( E \)-stabilized for all \( t \).

We restrict to perturbation data taking values in \( \mathcal{J}^*(X, D, J, \theta, E) \) for a (weakly or strictly) stabilizing divisor \( D \) having sufficiently large degree for an almost-complex structure \( \theta \)-close to \( J \). Let \( J_D \in \mathcal{J}_\tau(X, D, J, \theta) \) be an almost complex structure that is stabilized for all energies, for example, in the intersection of \( \mathcal{J}^*(X, D, J, \theta, E) \) for all \( E \). For each energy \( E \), there is a contractible open neighborhood \( \mathcal{J}^{**}(X, D, J_D, \theta, E) \) of \( J_D \) in \( \mathcal{J}^*(X, D, J, \theta, E) \) that is \( E \)-stabilized.

The construction of perturbation data satisfying compactness depends crucially on the following relationship between the energy and the number of interior markings. Let \( \Gamma \) be a type of stable treed disk. Disconnecting the components that are connected by boundary nodes with positive length one obtains types \( \Gamma_1, \ldots, \Gamma_l \), and a decomposition of the universal curve \( U_\Gamma \) into components \( U_{\Gamma_1}, \ldots, U_{\Gamma_l} \). Let \( n(\Gamma_i) \) denote the number of markings on \( U_{\Gamma_i} \). We assume that \( D \) has Poincaré dual given by \( k[\omega] \). The following relation between energy and intersection number follows from the exactness of \( L \) in the complement, see Proposition 2.23.

**Proposition 2.23.** Any stable treed disk \( u : C \to X \) with domain of type \( \Gamma \) and only transverse intersections with the divisor has energy equal to

\[
E(u|C_i) \leq n(\Gamma_i, k) := \frac{n(\Gamma_i)}{k}
\]

on the component \( C_i \subset C \) contained in \( U_{\Gamma_i} \).

**Definition 2.24.** A perturbation datum \( P_\Gamma = (F_\Gamma, J_\Gamma) \) for a type of stable treed disk \( \Gamma \) is **stabilized** by \( D \) if \( J_\Gamma \) takes values in \( \mathcal{J}^*(X, D, J, \theta, n(\Gamma_i, k)) \) on \( U_{\Gamma_i} \) (in particular, if \( J_\Gamma \) takes values in \( \mathcal{J}^{**}(X, D, J_D, \theta, n(\Gamma_i, k)) \)).

In other words, perturbations sufficiently close to the almost complex structure \( J_D \) have the property that any component of an adapted disk or sphere in the domain is stabilized by its intersections with the stabilizing divisor.

In order to rule out configurations involving markings on ghost bubbles appearing in the compactification we introduce a second further assumption on the perturbations. A perturbation data \( J_\Gamma \) for a type of stable strip \( \Gamma \) is **ghost-marking-independent** if the following holds: Let \( \Gamma' \) denote the combinatorial type obtained by forgetting all markings on ghost components. Then \( J_\Gamma \) is obtained from \( J_{\Gamma'} \) under the map forgetting all markings on ghost components, that is, components with trivial homology classes.
Theorem 2.25. (Compactness for fixed type) For any collection \( P = (P_i) \) of coherent, regular, stabilizing, ghost-marking-independent perturbation data and any uncrowded type \( \Gamma \) of expected dimension at most one, the moduli space \( \overline{\mathcal{M}}_{\Gamma}(L, D) \) of adapted stable treed marked disks of type \( \Gamma \) is compact and the closure of \( \mathcal{M}_{\Gamma}(L, D) \) contains only configurations with disk bubbling.

Proof. (c.f. [25, Proof of Proposition 4.10].) Because of the existence of local distance functions, similar to [77, Section 5.6], it suffices to check sequential compactness. Let \( u_\nu : C_\nu \to X \) be a sequence of stable treed disks of type \( \Gamma \), necessarily of fixed energy \( E(\Gamma) \). The sequence of stable disks \( [C_\nu] \) converges to a limiting stable disk \( [C] \) in \( \overline{\mathcal{M}}_{\Gamma} \). The sequence of maps \( u_\nu : C_\nu \to X \) has a stable Gromov-Floer limit \( u : \hat{C} \to X \), where \( \hat{C} \) is a possibly unstable sphere or disk with stabilization \( C \) of type \( \Gamma_\infty \). We show that \( u \) is adapted. From the (Compatible with the divisor) axiom, we have

\[
(J_\Gamma|\overline{\mathcal{M}}_\Gamma)|D = J_D|D.
\]

Now \( J_D \in J^*(X, D, J, \theta, n(\Gamma_i, k)) \) was chosen so that that \( D \) contains no \( J_D \)-holomorphic spheres. Hence the limit \( u \) satisfies the (Non-constant spheres) property.

To show the (Markings) property, note that each connected component \( C_i \) of \( u_\nu^{-1}(D) \) has surface part \( S \cap C_i \) either a point \( z_i \) or a union of sphere and disk components. In the first case, the intersection multiplicity \( m(z_i) \) with the divisor at the point \( z_i \) is positive while in the second, the intersection multiplicity at each node connecting the component \( C_i \) with the rest of the domain \( C \) is positive. In the first case, by topological invariance of the intersection multiplicity there exists a sequence \( z_\nu \) of isolated points in \( u_\nu^{-1}(D) \) with positive intersection multiplicity converging to \( z_i \). By the (Markings) property of \( u_\nu \), the points \( z_\nu \) must be markings. Hence \( z_i \) is a marking as well. In the second case, the positivity of the intersection multiplicity implies that there exists a sequence of points \( z_{\nu,i} \in u_\nu^{-1}(D) \subset C_\nu \) converging to \( C_i \). Either \( z_{\nu,i} \) are isolated, or lie on some sequence of connected components \( C_{\nu,i} \) of \( u_\nu^{-1}(D) \subset C_\nu \) on which \( u_\nu \) is constant in which case any limit point of \( C_{\nu,i} \) lies in \( C_i \). Since each \( C_{\nu,i} \) contains a marking by the (Markings) property, so does \( C_i \).

Note that if \( u_\nu(z_{i,\nu}) \in D \) then \( u(z_i) \in D \), by convergence on compact subsets of complements of the nodes. This shows the (Markings) property.

To show the (Stable domains) property note that since \( D \) is stabilizing for \( L \), any disk component \( \tilde{C}_i \) of \( \hat{C} \) must have at least one interior intersection point, call it \( z \), with \( D \). Since \( z \) lies in the interior of \( C_i \), and the boundary \( \partial C_i \) has at least one special point the component \( C_i \) is stable. Next consider a spherical component \( \tilde{C}_i \) of \( \hat{C} \) attached to a component of \( C \). First suppose that \( u \) is non-constant on \( \tilde{C}_i \). Suppose that \( \tilde{C}_i \) is attached at a point of \( C \) contained in \( \overline{\Gamma} \), for some type \( \Gamma_i \), so that the energy of \( u|_{\tilde{C}_i} \) is at most \( n(\Gamma_i, k) \). Since \( J_{\Gamma_i} \) is constant and equal to an element of \( J^*(X, D, J, \theta, n(\Gamma_i, k)) \) on \( \tilde{C}_i \), the restriction of \( u \) to \( \tilde{C}_i \) has at least three intersection points resp. one intersection point with \( D \). By definition these points must be markings, which contradicts the instability of \( \tilde{C}_i \). Hence the stable map \( u \)
must be constant on $\hat{C}_i$, and thus $\hat{C}_i$ must be stable. This shows that $\hat{C}$ is equal to $C$ and the (Stable domain) property follows.

It remains to rule out sphere bubbling. Suppose $C$ has a spherical component $C_i$ on which $u$ is non-constant. After forgetting all but one marking on maximal ghost components we obtain a configuration in an uncrowded stratum $M_{1\nu}(L,D)$. Since transversality as been achieved for such strata, the moduli space $M_{1\nu}(L,D)$ is of expected dimension. But since the configuration $u$ has a spherical node, this expected dimension is at least two less than the expected dimension of $M_{1\nu}(L,D)$, which is at most one. Hence the dimension is negative, a contradiction.

By the previous paragraph, all spherical components of $C$ are ghost components. Suppose that $\hat{C}_v$ is a maximal ghost component of $C$. Necessarily, $\hat{C}_v$ contains at least two markings, say $z_i, z_j$ that are limits of markings $z_{i,\nu}, z_{j,\nu}$ on $C_\nu$. By invariance of local intersection degree, the node $w$ at which $\hat{C}_v$ attaches to the remainder of $\hat{C}$ has intersection degree at least two with the divisor $D$. Such a configuration lies in a stratum with negative expected dimension, a contradiction. So $C$ contains only disk components, which completes the proof. □

2.6. Composition maps. In this section we use holomorphic treed disks to define the structure coefficients of the Fukaya algebra. Let $q$ be a formal variable and $\Lambda$ the universal Novikov field of formal sums with rational coefficients

$$\Lambda = \left\{ \sum_i c_i q^{\rho_i} \mid c_i \in \mathbb{C}, \rho_i \in \mathbb{R}, \rho_i \to \infty \right\}$$

We denote by $\Lambda_{\geq 0}$ resp. $\Lambda_{> 0}$ the subalgebra with only non-negative resp. positive powers. We denote by

$$\Lambda^\times = \left\{ c_0 + \sum_{i>0} c_i q^{\rho_i} \subset \Lambda_{\geq 0} \mid c_0 \neq 0 \right\}$$

the subgroup of formal power series with invertible leading coefficient.

We define the space of Floer cochains for Lagrangians with additional data called brane structures. Let $X$ be a compact symplectic manifold and let $\text{Lag}(X)$ denote the fiber bundle over $X$ whose fiber $\text{Lag}(X)_x$ at $x$ consists of Lagrangian subspaces of $T_x X$. Let $g$ be an even integer. A Maslov cover is an $g$-fold cover $\text{Lag}^g(X) \to \text{Lag}(X)$ such that the induced two-fold cover

$$\text{Lag}^2(X) := \text{Lag}^g(X)/\mathbb{Z}_{g/2} \to \text{Lag}(X)$$

is the oriented double cover. A Lagrangian submanifold $L$ is admissible if $L$ is compact and oriented; we also assume for simplicity that $L$ is connected. A grading on $L$ is a lift of the canonical map

$$L \to \text{Lag}(X), \quad l \mapsto T_l L$$

to $\text{Lag}^g(X)$, see Seidel [95]. A brane structure on a compact oriented Lagrangian is a relative spin structure together with a grading and a flat $\Lambda_x$ line bundle. (The grading will be irrelevant until we discuss Fukaya bimodules.) An admissible Lagrangian
brane is an admissible Lagrangian submanifold equipped with a brane structure. For $L$ an admissible Lagrangian brane define the space of Floer cochains

$$\hat{CF}(L) = \bigoplus_{d \in \mathbb{Z}} \hat{CF}_d(L), \quad \hat{CF}_d(L) = \bigoplus_{x \in \hat{I}_d(L)} \Lambda^x$$

where $\hat{I}_d(L)$ are as in [12]. Let $CF(L) \subset \hat{CF}(L)$ the subspace generated by $x \in \hat{I}(L)$.

The composition maps are defined in the following. Given a Lagrangian brane $L$, we denote by $\text{Hol}_L(u) \in \Lambda^* \times$ the evaluation of the local system on the homotopy class of loops defined by going around the boundary of the treed disk once. We denote by $\sigma([u])$ the number of interior markings of $[u] \in \mathcal{M}_{\Gamma}(L,D,x)$. 

**Definition 2.26.** (Higher composition maps) For regular stabilizing coherent perturbation data $(P_{\Gamma})$ define

$$\mu^n : \hat{CF}(L)^\otimes \to \hat{CF}(L)$$

on generators by

$$\mu^n(x_1, \ldots, x_n) = \sum_{x_0, [u] \in \overline{\mathcal{M}}_{\Gamma}(L,D,x_0,x_0)} (-1)^{\nabla} (\sigma([u])!)^{-1} \text{Hol}_L(u) q^{E([u])} \epsilon([u]) x_0$$

where $\nabla = \sum_{i=1}^n i |x_i|$.

**Remark 2.27.** (Zero-th composition map is a quantum correction) Any configuration with a only single outgoing edge must have at least one non-constant disk. Hence $\mu^0(1)$ has no term with coefficient $q^0$.

**Remark 2.28.** (Leading order term in the first composition map) The constant trajectories at the maximum $x_M$ with weighted leaf and outgoing unweighted root are elements of $\overline{\mathcal{M}}_{\Gamma}(L,D,x_M,x_0)_0$, as in Remark 2.3. The orientations on these trajectories are determined by the orientation on $\mathcal{M}_{\Gamma}$ which by the discussion after (24) is negative resp. positive for $x_0 = x_M$ resp. $x_0 = x_M$. Hence

$$\mu^1(x_M) = x_M x_M - x_M + \sum_{x_0, [u] \in \overline{\mathcal{M}}_{\Gamma}(L,D,x_M,x_0), E([u]) > 0} (-1)^{\nabla} (\sigma([u])!)^{-1} q^{E([u])} \epsilon([u]) \text{Hol}_L(u)x_0.$$ 

This formula is similar to that in Fukaya-Oh-Ohta-Ono [41] (3.3.5.2). Presumably the discussion here is a version of their treatment of homotopy units.

**Theorem 2.29.** ($A_\infty$ algebra for a Lagrangian) For any coherent regular stabilizing perturbation system $P = (P_{\Gamma})$ the maps $\mu^n_{\geq 0}$ satisfy the axioms of a (possibly curved) $A_\infty$ algebra $\hat{CF}(L)$ with strict unit. The subspace $CF(L)$ is a subalgebra without unit.

**Proof.** By Theorems 2.19 and 2.25 the one-dimensional component of the moduli space with bounded energy is a finite union of compact oriented one manifolds, the
boundary points of the moduli space come in pairs. This gives the identity

\[ 0 = \sum_{\Gamma \in \mathcal{T}_{n,m}} \sum_{[u] \in \partial \mathcal{M}_\Gamma(L, D, x)} (\sigma([u])!)^{-1} q^{E(u)} \epsilon([u]) \text{Hol}_L(u). \]

In case the moduli spaces do not involve weightings then each combinatorial type \( \Gamma \) with a single interior edge of infinite length is obtained by gluing together graphs \( \Gamma_1, \Gamma_2 \) with \( n - n_2 + 1 \) and \( n_2 \) leaves along a single leaf, say with \( m_1 \) resp. \( m_2 \) interior markings, and by the (Products) axiom we have an isomorphism

\[ \mathcal{M}_\Gamma(L, D, x) = \bigcup_{y} \mathcal{M}_{\Gamma_1}(L, D, x_0, x_1, \ldots, x_{i-1}, y, x_{i+n_2+1}, \ldots, x_{n}) \times \mathcal{M}_{\Gamma_2}(L, D, y, x_i, \ldots, x_{i+n_2-1}). \]

Say \( \sigma([u]) = m \). Since there are \( m \) choose \( m_1, m_2 \) ways of distributing the interior markings to the two components graphs,

\[ 0 = \sum_{\begin{array}{c} i, m_1 + m_2 = m \\ [u_1] \in \mathcal{M}_{\Gamma_1}(L, D, x_0, x_1, \ldots, x_{i-1}, y, x_{i+n_2+1}, \ldots, x_{n}) \\ [u_2] \in \mathcal{M}_{\Gamma_2}(L, D, y, x_i, \ldots, x_{i+n_2-1}) \end{array}} (m!)^{-1} \binom{m}{m_1} q^{E([u_1]) + E([u_2])} \epsilon([u_1])\epsilon([u_2]) \text{Hol}_L([u_1]) \text{Hol}_L([u_2]) \]

which is the \( A_\infty \) axiom up to signs. In the case of a weighted marking one has additional boundary components corresponding to when the weighting becomes zero or infinity, but those configurations correspond to splitting off a constant Morse trajectory with weighted leaf and outgoing forgettable or unforgettable root, that is, the terms \( x_M^\bullet \) and \( x_M^\cdot \) in \( \mu^1(x_M) \).

The signs in the \( A_\infty \) axiom are derived as follows, using the orientations constructed in [24]. Recall the moduli space \( \mathcal{D}_{n,m} \) of stable disks discussed in Section 2.1. Any stratum \( \mathcal{D}_\Gamma \subset \mathcal{D}_{n,m} \) corresponding to disks with \( \nu \) nodes has a neighborhood homeomorphic to a neighborhood of \( \mathcal{D}_\Gamma \) in \( \mathcal{D}_\Gamma \times [0, \infty)^\nu \). Suppose that we are in the codimension one case, that is, \( \nu = 1 \) so that \( \mathcal{D}_\Gamma \cong \mathcal{D}_{\Gamma_1,1} \times \mathcal{D}_{\Gamma_2,1} \cong \mathcal{M}_{n-i+1,m_1} \times \mathcal{M}_{i,m_2} \) for some \( i \geq 2 \) or \( m_2 \geq 1 \). The homeomorphism has inverse given by a gluing map of the form

\[ (0, \epsilon) \times \mathcal{D}_{n-i+1,m_1} \times \mathcal{D}_{i,m_2} \rightarrow \mathcal{D}_{n,m} \]

\[ (\delta, (x_1, \ldots, x_{n-i+1}, z_2, \ldots, z_{m_2}), (x_1', \ldots, x_i', z_2', \ldots, z_{m_1}')) \rightarrow (x_1, \ldots, x_j, x_{j+1} + \delta^{-1} x_1', \ldots, x_{j+1} + \delta^{-1} x_i', x_{j+2}, \ldots, x_{n-i+1}, x_{j+1} + \delta^{-1} i, x_{j+1} + \delta^{-1} z_2', \ldots, x_{j+1} + \delta^{-1} z_{m_1}', z_2, \ldots, z_{m_2}). \]

The inclusion \( \mathcal{D}_{n-i+1,m_1} \times \mathcal{D}_{i,m_2} \rightarrow \mathcal{D}_{n,m} \) then has orientation sign \((-1)^{(n-i-j)+j} \).

By construction, the inclusion of \( \mathcal{M}_{n-i+1,m_1} \times \mathcal{M}_{i,m_2} \) in \( \mathcal{M}_{n,m} \) has the same sign as that of the inclusion of \( \mathcal{D}_{n-i+1,m_1} \times \mathcal{D}_{i,m_2} \) in \( \mathcal{D}_{n,m} \).

The signs for gluing treed holomorphic disks are as follows. We omit interior markings to simplify notation, since the deformations associated to these are even
dimension and so do not affect the computation. If \( i \geq 2 \) or \( m_2 \geq 1 \) then the gluing construction for treed holomorphic disks gives rise to a gluing map

\[
(33) \quad (0, \epsilon) \times \mathcal{M}_{n-i+1}(x_0, x_1, \ldots, x_j, y, x_{j+i+1}, \ldots, x_n) \times \mathcal{M}_{i}(y, x_{j+1}, \ldots, x_{j+i}) \rightarrow \mathcal{M}_n(x_0, \ldots, x_n). 
\]

In the special case \( i = 1 \) and \( m_2 = 0 \) where the breaking is that of a trivial trajectory, the gluing map is given by replacing the \( i \)-th weight with the gluing parameter resp. its reciprocal if \( y = x_{\lambda}^d \) resp. \( x_{\lambda}^d \). The orientation on the determinant line of \( \mathcal{M}_i(y, x_{j+1}, \ldots, x_{j+i}) \) is induced from an isomorphism

\[
(34) \quad \det(TM_{i}(y, x_{j+1}, \ldots, x_{j+i})) \rightarrow \det(TM_{i} \oplus TL \oplus T_yW^{+} \oplus T_{x_{j+1}}W^{-} \oplus \cdots \oplus T_{x_{j+i}}W^{-}).
\]

Here \( T_{x_i}W^- \) is the negative part of the tangent space \( T_{x_i}L \) with respect to the Hessian of \( F \), with an additional factor of \( \mathbb{R} \) if \( x_i = x_{\lambda}^d \). Similarly the orientation on the determinant line of \( TM_{n-i+1}(x_0, x_1, \ldots, y, \ldots, x_n) \) is induced from an isomorphism

\[
(35) \quad \det(TM_{n-i+1}(x_0, x_1, \ldots, y, \ldots, x_n)) \rightarrow \det(TM_{n-i+1,m_1} \oplus TL \oplus TW_{x_0}^{+} \oplus TW_{x_1}^{-} \oplus \cdots \oplus TW_{x_n}^{-}).
\]

Transposing \( TM_{i} \) with \( TL \oplus T_{x_0}W^{+} \oplus T_{x_1}W^{-} \oplus \cdots \oplus TW_{y}^{-} \oplus \cdots \oplus T_{x_n}W^{-} \) yields a sign of \((-1)^i\) to the power \( i(n - i + 1) \). Transposing \( TL \oplus T_{y}W^{+} \oplus T_{x_{j+1}}W^{-} \oplus \cdots \oplus T_{x_{j+i}}W^{-} \) with \( TL \oplus TW_{x_0}^{+} \oplus TW_{x_{j+1}}^{-} \oplus \cdots \oplus TW_{x_n}^{-} \) yields a sign \((-1)^i\) to the power \( i(n - i - j + j) \). The gluing map \( \mathcal{M}_i \times \mathcal{M}_{n-i+1} \rightarrow \mathcal{M}_n \) has sign \((-1)^i\) to the power \( i(n - i - j + j) \), see (32). Comparing the contributions from \((-1)^i\) gives a sign of \((-1)^i\) to the power

\[
(36) \quad \sum_{k=1}^{n} k|x_k| + \sum_{k=1}^{i} k|x_{j+k}| + \sum_{k=1}^{j} k|x_k| + \sum_{k=1}^{n-i-j} (j + k + 1)|x_{j+i+k}| + (j + 1)|y| \\
\quad \equiv 2 j(|y| + i) + (i - 1)(\sum_{k=1}^{n-i-j} |x_{j+i+k}|) + (j + 1)|y| \\
\quad \equiv 2 |y| + j(i + 1)(\sum_{k=1}^{n-i-j} |x_{j+i+k}|).
\]
while the sign in the $A_\infty$ axiom contributes $\sum_{k=1}^j (|x_k| - 1)$. Combining the signs one obtains in total

\begin{equation}
\begin{aligned}
i \left( \sum_{k=i+j+1}^n |x_k| \right) + & i(n-i+1) + i(n-i-j) + j + ji + (i-1) \left( \sum_{k=i+j+1}^n |x_k| \right) + |y| + \sum_{k=1}^j (|x_k| - 1) \\
& = \sum_{k=1}^j (|x_k| - 1) + |y| + i + \sum_{k=i+j+1}^n |x_k| \equiv 2 \sum_{k=1}^n |x_k|
\end{aligned}
\end{equation}

which is independent of $i,j$. A similar computation holds in the case of splitting off a trivial trajectory: in that case, $i = 1$ and $m_2 = 0$ then transposing the gluing parameter with the determinant lines of $TM_n, TL, T_{x_0}^+ W$, and $T_{x_k}^- W, 1 \leq k \leq j$ yields a sign $(-1)^{n+\sum_{k \leq j} |x_k|}$, while the signs $\heartsuit$ for the various moduli spaces combine to $(-1)^j$. Combining with the sign in the $A_\infty$ axiom gives $(-1)^{n+j-|x_0|+j} = (-1)^{\sum_{k=1}^n |x_k|}$ as in the previous case. The $A_\infty$ axiom follows.

The existence of strict units follows from the (Infinite weights axiom). We claim that a strict unit is given by the element $e_L = x^\gamma_M \in \hat{CF}(L)$. The (Infinite weights) axiom implies that the perturbation data $P_T$ used to define $\overline{\mathcal{M}}_T(L, D, \ldots, x_{i-1}, x_i = x_M^M, x_{i+1}, \ldots)$ is pulled back from data $P_{T'}$ for $\overline{\mathcal{M}}_{T'}(L, D, \ldots, x_{i-1}, x_i = x_M^M, x_{i+1}, \ldots)$ for the type $\Gamma'$ obtained from $\Gamma$ by forgetting the corresponding incoming leaf, as long as the type $\Gamma$ has a stabilization. It follows that for dimension reasons, the compositions $\mu^n(x, \ldots, x_M^M, \ldots)$ vanish except for the case that the resulting type has no stabilization: That is, $n = 2$ and the underlying configuration has no non-constant disks and the map is constant on the domain. One obtains from the configuration with constant values $x$ (or $x_M$ if $x \in \{x_M^M, x_M^M, x_M^M \}$)

$$\mu^2(x, x_M) = (-1)^{\deg(x)} \mu^2(x_M^M, x) = x, \quad \forall x \in \hat{\mathcal{I}}(L)$$

as in Figure [13].

\begin{proof}
\end{proof}

2.7. Spectral sequence. In the case that the Floer cohomology is weakly unobstructed there is a Morse-to-Floer spectral sequence which relates the Floer cohomology with the Morse cohomology, discussed in the monotone case in Oh [86]. Let

$$\mu^{0,n} : CF(L) \otimes^n \to CF(L)$$

denote the $q^0$ terms in the higher composition map $\mu^n$. These are, by construction, the higher composition maps in the Morse $A_\infty$ algebra for $L$. In particular, the cohomology of $\mu^{1,0}$ is the Morse cohomology and so isomorphic to the singular cohomology of $L$:

$$H(\mu^{1,0}) \cong H(L).$$

Proposition 2.30. Suppose that a compact Lagrangian brane $L$ is equipped with a bounding cochain $b \in MC(L)$ so that $(\mu^b_1)^2 = 0$. The complex $(CF(L), \mu^b_1)$ admits a filtration whose associated graded is the Morse complex $(CF(L), \mu^{1,0})$, with spectral sequence converging to the Floer cohomology, that is, cohomology of $\mu^b_1$. 

Proof. As in Fukaya-Oh-Ohta-Ono [41], consider the $\mathbb{R}$-filtration of Floer cochains by $q$-degree on $CF(L, \Lambda \geq 0)$, induced by the filtration on $\Lambda \geq 0$:

$$CF(L, \Lambda \geq 0) = \bigcup_{\rho > 0} q^\rho CF(L, \Lambda \geq 0).$$

Define an associated $\mathbb{Z}$-filtration induced by $q^{kE}$, where $E < \hbar$

$$CF(L, \Lambda \geq 0) = \bigcup_{k \geq 0} q^{kE} CF(L, \Lambda \geq 0).$$

Configurations of zero energy and a single leaf contain no disks. Indeed let $u : C \to X$ be a treed disk with zero energy. Necessarily any disk is constant. By the dimension constraint, $C$ contains no interior markings. By the stability condition, any disk component $C_i \subset C$ has at least two special points. One of these special points $z$ in $C_i$ is connected neither to the root edge nor to the leaf without crossing $C_i$ itself; following a non-self-crossing path from $z$ away from $C_i$ one eventually locates a terminal disk or sphere component of $C$, that is, a disk or sphere component $C_k \subset C$ with at most one special point. But $u$ is also constant on $C_k$, which contradicts stability.

Using the description of zero-energy configurations from the previous paragraph we compute the differential on the associated graded complex:

$$Gr(CF(L, \Lambda \geq 0)) = \bigcup_{k \geq 0} q^{kE} CF(L, \Lambda \geq 0)/q^{(k+1)E} CF(L, \Lambda \geq 0)$$

is the associated graded of the Morse differential $\mu^{1,0}$:

$$Gr(\mu^{1}) = Gr(\mu^{1,0})$$

The filtration leads to a spectral sequence $E^r$ with first page equal to the Morse homology

$$E^1 = H(\mu^{1,0}, \Lambda \geq 0)$$

and such that the spectral sequences converges to the Floer cohomology:

$$E^r \implies HF(L, \Lambda \geq 0).$$

Convergence of the spectral sequence is discussed in in Fukaya-Oh-Ohta-Ono [41, Theorem 16.3.28]. The two assumptions needed are that $CF(L)$ is finite (which follows since $L$ is compact) and the differential is “gapped” in the language of [41], which follows from energy quantization for disks. \hfill \Box

2.8. Divisor equation. The moduli spaces do not admit forgetful morphisms in general, hence the divisor axiom is not satisfied. However, there is a weak version given as follows. For divisorial perturbations we have a forgetful map

$$f_1 : r : \mathcal{M}_r(X, L, D, l_1, l_0) \to \mathcal{M}_r(X, L, D, l_0)$$

obtained by forgetting the map on the incoming edge. The fiber over an element $[u : C \to X] \in \mathcal{M}_r(l_0)$ is the set of points in the boundary $\partial C$ mapping to $W^-(l_1)$,

$$(f_1^{-1}([u]) \cong (u|\partial C)^{-1}(W^-(l_1)).$$
To see that one can choose the perturbations $P_\Gamma$ for a single incoming edge to be divisorial, note that the pair $(F,G)$ is already assumed to be Morse-Smale. Thus transversality for configurations with no disk components are already regular without perturbation. Next consider the space of isomorphism classes of pairs

$\mathcal{M}_\Gamma(l_0; l_1) = \{ [u : C \to X, P_{f(\Gamma)}] \}$

where $C$ is a treed holomorphic disk of type $f(\Gamma)$, $P_{f(\Gamma)}$ is a perturbation datum for type $f(\Gamma)$, $z \in \partial S$ is a point on the boundary distinct from the tree part $T$, and $z$ maps to $W^+(l_1)$ under $u$. For a comeager set of perturbations $P_{f(\Gamma)}$, the moduli space $\mathcal{M}_\Gamma(l_0; l_1)$ is regular of expected dimension. Indeed perturbations of the map which vanish at the marking $z$ are enough to force an element $\eta$ in the cokernel of the linearized operator to satisfy $D^*_u \eta = 0$. Then the pullback $P_\Gamma$ of $P_{f(\Gamma)}$ by the forgetful map is divisorial. More complicated perturbations are necessary for more than one leaf because configurations with constant disks correspond to intersections of the unstable manifolds of the unperturbed Morse function, which are not transversally intersecting for obvious reasons (e.g. they may be equal.)

Divisorial perturbations give the following weak form of the divisor equation. Let $c \in \ker(\mu^{1,0})$ be a cocycle in the first page of the Morse-to-Floer spectral sequence and $[W^-(c)]$ the homology class of the union of unstable manifolds of the critical points appearing in $c$. Then

$$\mu^1(c) = \sum_{u \in \mathcal{M}_1(l_0)} q^A(u) \epsilon(u) \sigma(u)([\partial u], [W^-(c)])(\text{Hol}_L([\partial u]))$$

$$= \partial_{[c]} \mu^0(1).$$

The following argument of Fukaya et al [41] implies that critical points of the disk potential determine Floer-non-trivial brane structures: Let

$$\mathcal{R}(L) := \text{Hom}(\pi_1(L), \Lambda^\times)$$

denote the space of isomorphism classes of local systems on $L$, with tangent space at any point identified with $H^1(L, \Lambda_{\geq 0})$ via the exponential map. Suppose that $0 \in MC(L)$ for every choice of local system $y$. Set

$$W: \mathcal{R}(L) \to \Lambda_{\geq 0}, \quad W(y)e_L = \mu^0(1).$$

Thus by the divisor equation

$$\mu^1(c) = \partial_{[c]} \mu^0(1) = \partial_{[c]} W(y)e_L.$$

**Proposition 2.31.** Suppose that for some $\rho \in \mathcal{R}(L)$, for all $b \in H^1(L, \Lambda_{\geq 0})$, $\partial_b W(y) = 0$, and the classical cohomology is generated by $H^1(L)$. Then the Floer cohomology is the ordinary cohomology,

$$H(\mu^1_\rho, \Lambda_{\geq 0}) = H(L, \Lambda_{\geq 0}).$$

**Proof.** In the first page of the Morse-to-Floer spectral sequence we show by induction that $\mu^1(c) = 0$ for all classes $c \in H(L)$. As in Fukaya-Oh-Ohta-Ono [42, Lemma 13.1] we write

$$\mu^1(c) = \sum_\beta q^\beta \mu^{1,\beta}(c)$$
for some rational numbers $\mu^{1,\beta}$. Then comparing terms with coefficient $\beta$ in the $A_\infty$ associativity relation we obtain

\begin{equation}
(40) \quad \mu^{1,\beta}(\mu^{2,0}(c_1, c_2)) = \sum_{\beta_1 + \beta_2 = \beta} \pm \mu^{2,\beta_1}(\mu^{1,\beta_2}(c_1), c_2) + \sum_{\beta_1 + \beta_2 = \beta, \beta_2 > 0} \pm \mu^{1,\beta_1}(\mu^{2,\beta_2}(c_1), c_2).
\end{equation}

The first two terms on the right vanish by the inductive hypothesis, since $\max(\beta_1, \beta_2) < \beta$, while the last term vanishes by the inductive hypothesis as well since $\beta_1 < \beta$.

It follows that if for some $\rho \in \mathcal{R}(L)$

\[ \forall c \in H^1(L, \Lambda^x), \quad \partial_c \mathcal{W}(y) = 0 \]

and the classical cohomology is free and generated by $H^1(L)$, then the Floer cohomology

\[ H(\mu^{1,\beta}_b, \Lambda) = H(L, \Lambda) \]

is non-trivial. Indeed, by the above discussion $H(\mu^{1,\beta}_b, \Lambda_{\geq 0}) = H(L, \Lambda_{\geq 0})$. The result for the Novikov field $\Lambda$ follows from the universal coefficient theorem. □

Brane structures corresponding to non-degenerate critical points are particularly well-behaved. We say that a Lagrangian torus $L$ equipped with brane structure is non-degenerate if the leading order part $\mathcal{W}_0$ of the potential $\mathcal{W}$ (that is the sum of terms with lowest $q$-degree) has a non-degenerate critical point at some element $c \in \mathcal{R}(L)$. Existence of a non-degenerate critical point is invariant under perturbation by Fukaya-Oh-Ohhta-Ono [42, Theorem 10.4], and in particular there exists a bijection

\[ \text{Crit}(\mathcal{W}_0) \cong \text{Crit}(\mathcal{W}) \]

between the set $\text{Crit}(\mathcal{W}_0)$ of critical points of $\mathcal{W}_0$ and critical points $\text{Crit}(\mathcal{W})$ of $\mathcal{W}$. This is a version of the formal implicit function theorem adapted to the setting Novikov rings with real coefficients; essentially one may, given a critical point $y_0$ of $\mathcal{W}_0$ one may solve for a critical point $y$ of $\mathcal{W}$ order by order, using non-degeneracy of the Hessian of $\mathcal{W}$ at $y_0$; and conversely any critical point of $\mathcal{W}$ determines a critical point of $\mathcal{W}_0$. By the previous paragraph, existence of a non-degenerate critical point implies that $L$ has non-trivial Floer cohomology.

Example 2.32. (Potential for the projective line) Suppose that $X = S^2$ with area $A$ and $L \cong S^1$ separates $X$ into pieces of areas $A_1, A_2$. Thus the primitive disks have areas $A_1, A_2$ with opposite boundary homotopy classes and the potential is

\[ \mathcal{W}(y) = q^{A_1}y + q^{A_2}/y. \]

The Floer differential vanishes iff

\[ 0 = y\partial \mathcal{W}/\partial y = q^{A_1}y - q^{A_2}/y \]

for some $y$. The equation $y^2 = q^{A_2-A_1}$ has a solution in $\mathcal{R}(L) \cong \Lambda^x$ iff $A_1 = A_2$. Thus only in this case (the case that the Lagrangian is Hamiltonian isotopic to the equator) one has non-trivial Floer cohomology.
2.9. Maurer-Cartan moduli space. In this section we define a cohomology complex associated to an \( A_\infty \) algebra satisfying suitable convergence conditions which is a complex of vector bundles over a space of solutions to a weak Maurer-Cartan equation. Let \( A \) be an \( A_\infty \) algebra defined over \( \Lambda_{\geq 0} \) with strict unit \( e_A \). Suppose that the vector space underlying \( A \) is a finite rank and free \( \Lambda_{\geq 0} \)-module and furthermore admits a \( \mathbb{Z} \)-grading; we write

\[
A = \bigoplus_{d \in [d_-,d_+]} A^d, \quad A^{\leq 0} = \bigoplus_{d \geq 0} A^d, \quad A^{<0} = \bigoplus_{d < 0} A^d
\]

for some integers \( d_- \leq d_+ \); define \( A^{\geq 0} \) and \( A^{>0} \) similarly. Consider the decomposition

\[
\text{Hom}(A^\otimes n, A) = \bigoplus_m \text{Hom}(A^\otimes n, A)_m
\]

where elements of \( \text{Hom}(A^\otimes n, A)_m \) have degree \( m \). Write the map \( \mu^n \) in terms of its components,

\[
\mu^n = \sum_m \mu^{n,m}, \quad \mu^{n,m} \in \text{Hom}(A^\otimes n, A)_{2-n+m}
\]

so that

\[
\mu^{n,m}(A^{d_1} \otimes \ldots \otimes A^{d_n}) \subset A^{d_1+\ldots+d_n+2-n+m}.
\]

(\( n > 0 \).

For a Fukaya algebra, \( \mu^{n,m} \) is the contribution to \( \mu^n \) of disks of Maslov index \( m \).) We say that an \( A_\infty \) algebra \( A \) as above is convergent iff there exists a sequence \( E_m \to \infty \) such that

\[
\mu^{n,m}(A) \subset q^{E_m} A, \quad \forall n \geq 0.
\]

(\( n > 0 \).

For a Fukaya algebra, the contributions with a given energy bound have only finitely indices contributing.)

We introduce a suitable domain for the weak Maurer-Cartan equation as follows. Given a convergent \( A_\infty \) algebra, define

\[
A^+ := A^{\leq 0} + \Lambda_{>0} A^{>0}, \quad A^{++} := A^{<0} + \Lambda_{>0} A^{\geq 0}.
\]

In other words, in positive degree (resp. non-negative degree) the elements of \( A^+ \) resp. \( A^{++} \) must have positive exponents of the formal variable \( q \).

**Lemma 2.33.** For \( b \in A^+ \) the sum

\[
\mu^0_b(1) := \mu^0(b) + \mu^1(b,b) + \ldots
\]

is well-defined.

**Proof.** For \( b \in A^+ \) write

\[
b = b_{\leq 0} + q^E b_{>0}, \quad b_{\leq 0} \in A^{\leq 0}, \quad b_{>0} \in A^{>0}
\]

for some \( E > 0 \). We claim that for any \( n_0 \), the contributions to \((43)\) of \( q \)-exponent less than \( En_0 \) are finite. Consider the set

\[
S_{n_0} = \{(a_1, \ldots, a_n) \in \{b_{\leq 0}, b_{>0}\}^n, \#\{a_i = b_{>0}\} \leq n_0\}
\]
of tuples \((a_1, \ldots, a_n)\) where at most \(n_0\) of the elements \(a_i\) are equal to \(b_{>0}\) and the remaining elements are equal to \(b_{\leq0}\). Since the tuples \((a_1, \ldots, a_n)\) not in \(S_{n_0}\) have \(\mu^n(a_1, \ldots, a_n) \in q^{En_0}A\), it suffices to show that the sum
\[
\sum_{n,(a_1, \ldots, a_n) \in S_{n_0}} \mu^n(a_1, \ldots, a_n)
\]
converges in \(A\). By definition of \(d_\pm\) the elements \(b_{>0}\) are of degree at most \(d_+\) and the output must be of degree at least \(d_-\). Hence
\[
\mu^n(a_1, \ldots, a_n) = \sum_{m \geq n - 2 - d_+ n_0 + d_-} \mu^{n,m}(a_1, \ldots, a_n).
\]
By the convergent condition (76), we have
\[
\sum_{n \geq n_1, (a_1, \ldots, a_n) \in S_{n_0}} \mu^n(a_1, \ldots, a_n) \in q^{E_{n-2+d_+n_0-d_-}}A
\]
so the sum (44) converges. \(\square\)

More generally the same argument implies convergence of the deformed composition map
\[
\mu^n_b(a_1, \ldots, a_n) = \sum_{i_1, \ldots, i_{n+1}} \mu^{n+i_1+\ldots+i_{n+1}}(b_{i_1}, b_{i_2}, \ldots, b_{i_{n+1}}, a_1, a_2, \ldots, a_n)
\]
over all possible combinations of insertions of the element \(b \in A^+\) between (and before and after) the elements \(a_1, \ldots, a_n\), is convergent for similar reasons. The maps \(\mu^n_b\) define an \(A_\infty\) structure on \(A\). In particular
\[
(\mu^1_b)^2(a_1) = \mu^2_b(\mu^0_b(a_1), a_1) - \mu^2_b(a_1, \mu^0_b(a_1)).
\]
The weak Maurer-Cartan equation for \(b \in A^+\) is
\[
\mu^0_b(1) = \mu^0(1) + \mu^1(b) + \mu^2(b, b) + \ldots \in \Lambda e_A.
\]
Denote by \(\widetilde{MC}(A)\) the space of solutions to the weak Maurer-Cartan equation (45). Any solution to the weak Maurer-Cartan equation defines an \(A_\infty\) algebra such that \((\mu^1_b)^2 = 0\) and so has a well-defined cohomology
\[
H(\mu^1_b) = \frac{\ker(\mu^1_b)}{\im(\mu^1_b)}.
\]
An \(A_\infty\) algebra is weakly unobstructed if there exists a solution to the weak Maurer-Cartan equation.

We introduce a notion of gauge equivalence for solutions to the weak Maurer-Cartan equation, so that cohomology is invariant under gauge equivalence. For \(b_1, \ldots, b_n \in A^+\) define
\[
\mu^n_{b_0, b_1, \ldots, b_n}(a_1, \ldots, a_n) = \sum_{i_1, \ldots, i_{n+1}} \mu^{n+i_1+\ldots+i_{n+1}}(b_{i_1}, \ldots, b_{i_{n+1}}, a_1, a_2, \ldots, a_n, b_{i_{n+1}}).
\]
Two cochains $b_0, b_1$ are gauge equivalent iff

$$\exists h \in A^{++}, \ b_1 - b_0 = \mu_{b_0, b_1}(h),$$

see (42). We then write $b_0 \sim_h b_1$.

Gauge equivalence is an equivalence relation, by a discussion parallel to that in Seidel [99, Section 1.1]. To show transitivity, if $b_0 \sim_{h_0} b_1$ and $b_1 \sim_{h_1} b_2$ then $b_0 \sim_{h_0 h_1} b_2$ where

$$h_{02} = h_{01} + h_{12} + \mu_{b_0, b_1, b_2}^2(h_{01}, h_{12}).$$

To prove symmetry define suppose that $b_0 \sim_{h_0} b_1$. Define

$$\phi(h_{10}) = h_{10} - (-1)^{|h_{10}|} \mu_{b_1, b_0, b_1}(h_{10}, h_{01})$$
$$\psi(h_{11}) = h_{11} + (-1)^{|h_{11}|} \mu_{b_1, b_1, b_0}(h_{01}, h_{11})$$

From the identities

$$\mu_{b_1, b_0}^1 = \mu_{b_1, b_1} - \mu_{b_0, b_1, b_1}^2(\cdot, b_1 - b_0), \quad \mu_{b_1, b_1}^1 = \mu_{b_0, b_1}^1 + \mu_{b_0, b_1, b_1}^2(b_1 - b_0, \cdot)$$

one sees that $\phi, \psi$ are chain maps:

$$\phi \mu_{b_1, b_0}^1 = \mu_{b_1, b_1}^1 \phi, \quad \psi \mu_{b_1, b_1}^1 = \mu_{b_0, b_1}^1 \psi.$$

Indeed,

$$\phi \mu_{b_1, b_0}^1(h_{10}) = \mu_{b_1, b_0}^1(h_{10}) - (-1)^{|h_{10}|} \mu_{b_1, b_0, b_1}^2(\mu_{b_1, b_0}^1(h_{10}), h_{01})$$
$$= \mu_{b_1, b_1}^1(h_{10}) + \mu_{b_0, b_1, b_1}^2(h_{10}, \mu_{b_0, b_1}^1(h_{01})) - (-1)^{|h_{10}|} \mu_{b_1, b_0, b_1}^2(\mu_{b_1, b_0}^1(h_{10}), h_{01})$$
$$= \mu_{b_1, b_1}^1(h_{10}) - (-1)^{|h_{10}|} \mu_{b_1, b_1}^1(\mu_{b_0, b_1, b_1}^2(h_{10}, h_{01}))$$
$$= \mu_{b_1, b_1}^1 \phi(h_{10})$$

and similarly for $\psi$. Since the $q = 0$ part of $\mu_{b_1, b_0, b_1}^2(\cdot, h_{01})$ resp. $\mu_{b_1, b_1, b_0}^2(h_{01}, \cdot)$ has negative degree, the maps $\phi, \psi$ are invertible. Furthermore,

$$\phi(b_0 - b_1) = \mu_{b_1, b_1}^1(h_{01})$$
$$\psi(\mu_{b_1, b_1}^1(h_{01})) = (-1)^{|h_{01}|} \mu_{b_0, b_1}^1 \mu_{b_0, b_1, b_1}^2(h_{01}, h_{01}) + b_0 - b_1.$$

Hence if we define

$$h_{10} := (\psi \circ \phi)^{-1}(-h_{01} - (-1)^{|h_{01}|} \mu_{b_0, b_1, b_1}^2(h_{01}, h_{01}))$$

then $b_1 \sim_{h_{10}} b_0$:

$$\mu_{b_1, b_0}^1(h_{10}) = \mu_{b_1, b_0}^1 \phi^{-1} \psi^{-1}(-h_{01} - (-1)^{|h_{01}|} \mu_{b_0, b_1, b_1}^2(h_{01}, h_{01}))$$
$$= \phi^{-1} \psi^{-1} \mu_{b_0, b_1}^1(-h_{01} - (-1)^{|h_{01}|} \mu_{b_0, b_1, b_1}^2(h_{01}, h_{01}))$$
$$= \phi^{-1} \psi^{-1}((-1)^{|h_{01}|} \mu_{b_1, b_1}^1 \mu_{b_0, b_1, b_1}^2(h_{01}, h_{01}) + b_0 - b_1)$$
$$= b_0 - b_1.$$

Also $b \sim_0 b$ for any $b$, hence $\sim$ is reflexive.
We define the potential of the algebra as a function on the moduli space of solutions to the weak Maurer-Cartan equation. Denote by $\tilde{MC}(A)$ the quotient of $MC(A)$ by the gauge equivalence relation,

$$MC(A) = \tilde{MC}(A)/\sim$$

which we call the moduli space of solutions to the weak Maurer-Cartan equation. Define a potential

$$\tilde{W} : \tilde{MC}(A) \rightarrow \Lambda$$

on the space of solutions to the weak Maurer-Cartan equation by

$$\tilde{W}(b)_{1A} = \mu^0_0(1).$$

We remark that $\tilde{W}$ would be related to the potential $W$ defined in (39) via the divisor equation, but the divisor equation in the perturbation system given here only holds in weak form.

**Lemma 2.34.** The potential $\tilde{W}$ is gauge-invariant and so descends to a potential $W : MC(A) \rightarrow \Lambda$.

**Proof.** Suppose $b_0 \sim_h b_1$ so $b_1 - b_0 = \mu^1_{b_0,b_1}(h)$. We have

$$\mu^1_{b_1}(1) - \mu^1_{b_0}(1) = \sum_{i,j} \mu^{i+j+1}(b_0,\ldots,b_0,b_1 - b_0,b_1,\ldots,b_1)$$

$$\quad = \sum_{i,j} \mu^{i+j+1}(b_0,\ldots,b_0,\mu^1_{b_0,b_1}(h),b_1,\ldots,b_1)$$

$$\quad = \sum_{i,j,k} \mu^{i+j+k+2}(b_0,\ldots,b_0,\mu^0_0(1),b_0,\ldots,b_0,h,b_1,\ldots,b_1)$$

$$\quad + (-1)^{|h|+|b_1|+j+1} \sum_{i,j,k} \mu^{i+j+k+1}(b_0,\ldots,b_0,h,b_1,\ldots,b_1,\mu^0_{b_1}(1),b_1,\ldots,b_1)$$

$$\quad = \mu^2(\tilde{W}(b_0)e_A,h) - (-1)^{|h|} \mu^2(h,\tilde{W}(b_1)e_A)$$

$$\quad = (\tilde{W}(b_0) - \tilde{W}(b_1))h.$$

It follows that

$$(e_A + h)\tilde{W}(b_1) = (e_A + h)\tilde{W}(b_0).$$

Since $h \in A^{++} = A_{<0} + \Lambda_{>0}A_{\geq 0}$ (that is, the components of non-negative degree have positive $q$-degree) the sum $e_A + h$ is non-zero. Hence $\tilde{W}(b_1) = \tilde{W}(b_0)$ as claimed.

**Corollary 2.35.** If $b_0 \sim_h b_1$, then $\mu^1_{b_0,b_1}$ is a differential.

**Proof.** Using the $A_{\infty}$ relations and strict unitality we have

$$(\mu^1_{b_0,b_1})^2(a) = \mu^2(\mu^0_{b_0}(1),a) - (-1)^{|a|} \mu^2(a,\mu^0_{b_1}(1))$$

$$= (\tilde{W}(b_1) - \tilde{W}(b_0))a = 0$$

as claimed.
The cohomology of an $A_{\infty}$ algebra is a complex of bundles over the space of solutions to the weak Maurer-Cartan equation. For any $b \in \widetilde{MC}(A)$ define

$$H(b) := H(\mu^1_b) = \frac{\ker(\mu^1_b)}{\text{im}(\mu^1_b)}.$$ 

The cohomology complex is the resulting complex of sheaves over $\widetilde{MC}(A)$:

$$A \times \widetilde{MC}(A) \to A \times \widetilde{MC}(A), \quad (a, b) \mapsto (\mu^1_b(a), b)$$

The complex (49) might be viewed as an object in the derived category of bounded complexes of coherent sheaves of $O_{\widetilde{MC}(A)}$-modules. Since we are working over Novikov fields the algebraic geometry here is non-standard and we do not attempt to discuss it here. The “stalks” of the cohomology complex fit together to the “cohomology sheaf”

$$H(A) := \bigcup_{b \in \widetilde{MC}(A)} H(b).$$

Lemma 2.36. The cohomology sheaf $H(A)$ is gauge-equivariant in the sense that if $b_0 \sim_{h_{10}} b_1$, then $H(b_0) \cong H(b_1)$.

Proof. One can verify that

$$\mu^2_{b_2, b_0, b_0}(h_{10}, a) = \sum_{n_1, n_2, n_3} \mu^{2+n_1+n_2+n_3}_{b_1, b_1, b_0, b_0}(h_{10}, b_1, b_1, b_0, a, b_0, \ldots, b_0)$$

satisfies

$$\mu^2_{b_1, b_0, b_0}(h_{10}, \mu^1_{b_0}(a)) - \mu^1_{b_1, b_0}(\mu^2_{b_1, b_0, b_0}(h_{10}, a)) = \mu^1_{b_0}(a) - \mu^1_{b_1, b_0}(a).$$

Hence the operator

$$\mu^2_{b_1, b_0, b_0}(h_{10}, \_ \_ \_ ) - \text{Id}(\_ \_ \_ ) : (A, \mu^1_{b_0}) \to (A, \mu^1_{b_1, b_0}),$$

is a chain morphism, see (47). For the same reasons,

$$\mu^2_{b_0, b_1, b_0}(h_{10}, \_ \_ \_ ) - \text{Id}(\_ \_ \_ ) : (A, \mu^1_{b_1, b_0}) \to (A, \mu^1_{b_0}),$$

is a chain morphism. Consider the map

$$A \to A, \quad a \mapsto H^1_{b_0, b_1, b_0, b_0}(h_{01}, h_{10}, a)$$

where

$$H^1_{b_0, b_1, b_0, b_0}(h_{01}, h_{10}, a) := \sum_{n_1, n_2, n_3, n_4} \mu^{3+n_1+n_2+n_3+n_4}_{b_0, b_1, b_0, b_0, b_0}(h_{01}, b_1, b_1, b_1, b_1, \ldots, b_0, a, b_0, \ldots, b_0).$$

This map is a chain homotopy between

$$(\mu^2_{b_0, b_1, b_0}(h_{10}, \_ \_ \_ ) - \text{Id}(\_ \_ \_ )) \circ (\mu^2_{b_1, b_0, b_0}(h_{10}, \_ \_ \_ ) - \text{Id}(\_ \_ \_ ))$$

and the chain map

$$\Phi(a) = a + \mu^2_{b_0}(h_{01} + h_{10} + \mu^2_{b_0, b_1, b_0}(h_{01}, h_{10}), a).$$
The latter can be seen to induce an isomorphism of \( H(b_0) \). In the same way, their reverse composition is homotopic to a map inducing an isomorphism of

\[
H(b_1, b_0) := H(A, \mu^1_{b_1, b_0}).
\]

Similarly, \( \mu^1_{b_1, b_0}(-, h_{10}) - \text{Id}(-) \) and \( \mu^2_{b_1, b_0, b_i}(-, h_{01}) - \text{Id}(-) \) define a chain morphism from \((A, \mu^1_{b_1})\) to \((A, \mu^1_{b_1, b_0})\) and \((A, \mu^1_{b_1, b_0})\) to \((A, \mu^1_{b_1})\), respectively. This shows that \( H(b_1, b_0) \cong H(b_1) \). Hence \( H(b_1, b_0) \cong H(b_1, b_0) \cong H(b_0) \) as claimed. \( \square \)

The proposition implies that the cohomology bundle descends to the Maurer-Cartan moduli space but we are unsure about whether this point of view is really helpful.

We show that the perturbation system may be chosen so that the Fukaya algebra associated to a Lagrangian is convergent:

**Definition 2.37.** A perturbation system \( P = (P_{\Gamma}) \) is **convergent** if for each energy bound \( E \), there exists a constant \( c(E) \) such that for any \( \Gamma \) and any treed \( J_\Gamma \)-holomorphic disk \( u : C \to X \) of type \( \Gamma \), the total Maslov index \( I(u) := \sum I(u_i) \) satisfies

\[
(E(u) < E) \implies (I(u) < c(E)).
\]

**Lemma 2.38.** Any convergent, coherent, regular, stabilizing perturbation system \( P = (P_{\Gamma}) \) defines a convergent Fukaya algebra \( \widehat{CF}(L, P) \).

**Proof.** We claim that the sequence \( E_m := \inf \{ E(u) | I(u) \geq m \} \) converges to infinity. To prove the claim, note that otherwise there would exist a sequence \( u_m \) with \( I(u_m) \geq m \) and \( E(u_m) \) bounded, which would violate (52). The convergence condition \((76)\) follows. \( \square \)

**Proposition 2.39.** There exist convergent, coherent, regular, stabilizing perturbations \( P = (P_{\Gamma}) \).

**Proof.** First, note that for a fixed almost complex structure \( J_D \) and energy bound \( E \), the perturbation system given by taking \( J_{\Gamma} \) constant equal to \( J_D \) is convergent. Indeed, the number of disk and sphere components is bounded by \( E/h \), while the number of homotopy classes of disks and spheres with energy at most \( E \) is finite by Gromov compactness.

Next, we may suppose that the surface part \( S_\Gamma \) of the universal bundle is equipped with a metric so that \( C^2 \) norms are well-defined. Recall that any fiber \( S_\Gamma \) of \( S_\Gamma \) breaks up into disjoint components \( S_{\Gamma_i} \) with number of interior markings \( n(\Gamma_i, k) \). There exists a function \( d(n) \) such that if

\[
\| J_{\Gamma} - J_D \|_{C^2(\Sigma_{\Gamma} \times X, \text{End}(TX))} \leq d(n(\Gamma_i, k))
\]

then the homotopy classes of spheres and disks appearing in \( P_{\Gamma} = (J_{\Gamma}, F_{\Gamma}) \)-holomorphic treed disks of type \( \Gamma \) are contained in those for \( P = (J_D, F) \). Indeed, otherwise there exists a sequence of types \( \Gamma_{\nu} \), a sequence \( J_{\Gamma_{\nu}}, F_{\Gamma_{\nu}} \) of perturbation data, an integer \( i \), and a sequence of components \( S_{\Gamma_{\nu}} \subset S_{\Gamma_{\nu}} \) with \( n(\Gamma_{\nu, i}, k) \) markings such that \( J_{\Gamma_{\nu}} \to J_D \) in \( C^2 \) on \( S_{\Gamma_{\nu}} \) such that the homotopy type of spheres and disks in
u_{\nu}|_{S_{\Gamma_{i}, \nu}} does not occur for $J_{D}$-holomorphic spheres and disks with $n(\Gamma_{i}, k)$ markings. By Gromov compactness, $S_{\Gamma_{i}, \nu}$ converges to a stable disk or sphere and $u_{\nu}$ converges to a limit $u: S \to X$ with the same homotopy type as $u_{\nu}$ for $\nu$ sufficiently large, a contradiction. Hence the inequality (52) holds for $(J_{\Gamma}, F_{\Gamma})$-holomorphic treed disks as well, as long as the perturbations are chosen $C^{2}$-small in the above sense. Since the $C^{2}$ bound depends only on the number of markings $n(\Gamma_{i}, k)$ on the connected components of the fibers $S \subset S_{\Gamma}$, this restriction is compatible with the coherence and regularity conditions. Furthermore, since the condition is open then the restriction does not affect the construction of regular almost complex structures either. The statement of the proposition follows.

Consider the space of solutions to the weak Maurer-Cartan equation:

$$\tilde{MC}(L) := \mu^{-1}(\Lambda e_{L}) \subset CF(L), \quad e_{L} = <x^{\tilde{M}}_{M}>$$

The following Lemma will be used elsewhere to show that Lagrangians are weakly unobstructed.

**Lemma 2.40.** Suppose that $\mu^{0}(1) \in \Lambda x^{\ast}_{M}$ and every non-constant disk has positive Maslov index. Then $\tilde{MC}(L)$ is non-empty.

**Proof.** Suppose $\mu^{0}(1) = W x^{\ast}_{M}$ and the condition in the Lemma holds. Equation (28) becomes $\mu^{1}(x^{\ast}_{M}) = x^{\ast}_{M} - x^{\ast}_{M}$. Hence

$$\mu(W x^{\ast}_{M}) = \mu^{0}(1) + W \mu^{1}(x^{\ast}_{M}) = W x^{\ast}_{M} + W(x^{\ast}_{M} - x^{\ast}_{M}) = W x^{\ast}_{M} \in \Lambda x^{\ast}_{M}$$

which implies $W x^{\ast}_{M} \in \tilde{MC}(L)$. □

### 3. Homotopy invariance

In this and following sections we show that the Fukaya algebra constructed above is independent, up to $A_{\infty}$ homotopy invariance, of the choice of perturbation system. The argument uses moduli spaces of quilted treed disks, introduced without trees in Ma’u-Woodward [75].

**3.1. $A_{\infty}$ morphisms.** We begin by recalling the definitions of $A_{\infty}$ morphisms and related notions.

**Definition 3.1.** (a) ($A_{\infty}$ morphisms) Let $A_{0}, A_{1}$ be $A_{\infty}$ algebras. An $A_{\infty}$ morphism $F$ from $A_{0}$ to $A_{1}$ consists of a sequence of linear maps

$$F^{d}: A_{0}^{\otimes d} \to A_{1}[1 - d], \quad d \geq 0$$

such that the following holds:

$$\sum_{i+j \leq d} (-1)^{i+\sum_{j=1}^{i} |a_{j}|} F^{d-j+1}(a_{1}, \ldots, a_{i}, \mu^{j}_{A_{0}}(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_{d}) = \sum_{i_{1}+\ldots+i_{m}=d} \mu_{A_{1}}^{m}(F^{i_{1}}(a_{1}, \ldots, a_{i_{1}}), \ldots, F^{i_{m}}(a_{i_{1}+\ldots+i_{m-1}+1}, \ldots, a_{d}))$$

(53)
where the first sum is over integers $i, j$ with $i + j \leq d$, the second is over partitions $d = i_1 + \ldots + i_m$. An $A_\infty$ morphism $F$ is unital if

$$F^1(e_{A_0}) = e_{A_1}, \quad F^k(a_1, \ldots, a_i, e_{A_0}, a_{i+2}, \ldots, a_k) = 0$$

for every $k \geq 2$ and every $0 \leq i \leq k - 1$.

(b) (Composition of $A_\infty$ morphisms) The composition of $A_\infty$ morphisms $F_0, F_1$ is defined by

$$\mathcal{F}_0 \circ \mathcal{F}_1)^d(a_1, \ldots, a_d) = \sum_{i_1 + \ldots + i_m = 0} F_0^{i_1}(a_1, \ldots, a_i),$$

$$\ldots, F_1^{i_m}(a_d, a_{d-1}, \ldots, a_{d-i_m+1}, \ldots, a_d)).$$

(c) (Homology morphism) Any $A_\infty$ morphism $F : A_0 \to A_1$ defines an ordinary morphism

$$H(F) : H(A_0) \to H(A_1), \quad [a] \mapsto (-1)^{|a|}[F(a)].$$

(d) ($A_\infty$ natural transformations) Let $F_0, F_1 : A_0 \to A_1$ be $A_\infty$ morphisms. A pre-natural transformation $\mathcal{T}$ from $F_0$ to $F_1$ consists of for each $d \geq 0$ a multilinear map

$$\mathcal{T}^d : A_0^d \to A_1.$$

Let $\text{Hom}(F_0, F_1)$ denote the space of pre-natural transformations from $F_0$ to $F_1$. Define a differential on $\text{Hom}(F_0, F_1)$ by

$$\mu_1^\text{Hom}(F_0, F_1) \mathcal{T}^d(a_1, \ldots, a_d) = \sum_{k, m} \sum_{i_1, \ldots, i_m} (-1)^{i_1} \mu^m_{A_2}(F_0^{i_1}(a_1, \ldots, a_i), F_0^{i_2}(a_{i+1}, \ldots, a_{i+i_2}), \ldots,$$

$$\mathcal{T}^k(a_{i_1+\ldots+i_{k-1}+1}, \ldots, a_{i_1+\ldots+i_k}), F_1^{i_{k+1}}(a_{i_1+\ldots+i_k+1}, \ldots, \ldots, F_1^{i_m}(a_d, a_{d-1}, \ldots, a_{d-i_m+1}, \ldots, a_d))$$

$$- \sum_{i, e} (-1)^{i+i_1+\ldots+i_k} a_1 |a_1| + |i_1| + \ldots + |a_i| + |i_k| + |a_{i+1}| + \ldots + |a_{i+i_2}| + \ldots,$$

$$\mathcal{T}^d(a_1, \ldots, a_i, a_{i+1}, \ldots, a_{i+i_2}, \ldots, a_d),$$

where

$$\hat{t} = (|\mathcal{T}| - 1)(|a_1| + \ldots + |a_i| + \ldots + |a_{i+i_2}| - i_1 - \ldots - i_k) .$$

A natural transformation is a closed pre-natural transformation. We say that

an $A_\infty$ (pre-)natural transformation $\mathcal{T}$ from a unital morphism $F_0$ to a unital morphism $F_1$ is unital if $\mathcal{T}^k(a_1, \ldots, a_i, e_{A_0}, a_{i+2}, \ldots, a_k) = 0$ for every $k \geq 1$ and every $0 \leq i \leq k - 1$.

(e) (Composition of natural transformations) Given two pre-natural transformations $\mathcal{T}_1 : F_0 \to F_1, \mathcal{T}_2 : F_1 \to F_2$, define $\mu^2(\mathcal{T}_1, \mathcal{T}_2)$ by

$$\mu^2(\mathcal{T}_1, \mathcal{T}_2)^d(a_1, \ldots, a_d) = \sum_{m, k, l, i_1, \ldots, i_m} (-1)^{i_1} \mu^m_{A_2}(F_0^{i_1}(a_1, \ldots, a_i), \ldots,$$

$$\mathcal{T}_1^{i_k}(a_{i_1+\ldots+i_{k-1}+1}, \ldots, a_{i_1+\ldots+i_k}), F_1^{i_{k+1}}(\ldots), \ldots, F_1^{i_{l-1}}(\ldots),$$

$$\mathcal{T}_2^{i_l}(a_{i_1+\ldots+i_{l-1}+1}, \ldots, a_{i_1+\ldots+i_l}), F_2^{i_{l+1}}(\ldots), \ldots, F_2^{i_m}(a_d, a_{d-1}, \ldots, a_d))$$
where
\[
\hat{t} = \sum_{i=1}^{i_1+\ldots+i_{k-1}} (|T_1| - 1)(|a_i| - 1) + \sum_{i=1}^{i_1+\ldots+i_{l-1}} (|T_2| - 1)(|a_i| - 1).
\]

Let \( \text{Hom}(A_0, A_1) \) denote the space of \( A_\infty \) morphisms from \( A_0 \) to \( A_1 \), with morphisms given by pre-natural transformations. Higher compositions give \( \text{Hom}(A_0, A_1) \) the structure of an \( A_\infty \) category [39 10.17], [65 8.1], [99 Section 1d].

(f) (Homotopy natural transformations) Any \( A_\infty \) natural transformation \( T : \mathcal{F}_0 \to \mathcal{F}_1 \) induces a natural transformation of the corresponding homological morphisms \( H(\mathcal{F}_0) \to H(\mathcal{F}_1) \).

(g) (\( A_\infty \) homotopies) Suppose that \( \mathcal{F}_1, \mathcal{F}_2 : A_0 \to A_1 \) are morphisms. A homotopy from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \) is a pre-natural transformation \( T \in \text{Hom}(\mathcal{F}_1, \mathcal{F}_2) \) such that
\[
\mathcal{F}_1 - \mathcal{F}_2 = \mu_{\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)}^1(T).
\]
where \( \mu_{\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)}^1(T) \) is defined in [55]. If a homotopy exists we say that \( \mathcal{F}_1 \) is homotopic to \( \mathcal{F}_2 \) and write \( \mathcal{F}_1 \equiv \mathcal{F}_2 \). As shown in [99 Section 1h], homotopy of \( A_\infty \) morphisms is an equivalence relation.

(h) (Composition of homotopies) Given homotopies \( \mathcal{T}_1 \) from \( \mathcal{F}_0 \) to \( \mathcal{F}_1 \), and \( \mathcal{T}_2 \) from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \), the sum
\[
\mathcal{T}_2 \circ \mathcal{T}_1 := \mathcal{T}_1 + \mathcal{T}_2 + \mu^2(\mathcal{T}_1, \mathcal{T}_2) \in \text{Hom}(\mathcal{F}_0, \mathcal{F}_2)
\]
is a homotopy from \( \mathcal{F}_0 \) to \( \mathcal{F}_2 \).

(i) (Composition of morphisms) Let \( A_0, A_1, A_2 \) be \( A_\infty \) algebras. Given a morphism \( \mathcal{F}_{12} : A_1 \to A_2 \) resp. \( \mathcal{F}_{01} : A_0 \to A_1 \), right composition with \( \mathcal{F}_{12} \) resp.
left composition with \( \mathcal{F}_{01} \) define \( A_\infty \) morphisms
\[
\mathcal{R}_{\mathcal{F}_{12}} : \text{Hom}(A_0, A_1) \to \text{Hom}(A_0, A_2)
\]
\[
\mathcal{L}_{\mathcal{F}_{01}} : \text{Hom}(A_0, A_2) \to \text{Hom}(A_1, A_2).
\]
The action on pre-natural transformations is given as follows [99 Section 1e]: Let \( \mathcal{F}_{01}', \mathcal{F}_{01}'' : A_0 \to A_1 \) be \( A_\infty \) morphisms and \( T_{01} \) a pre-natural transformation from \( \mathcal{F}_{01}' \) to \( \mathcal{F}_{01}'' \). Define
\[
(\mathcal{R}_{\mathcal{F}_{12}}(T_{01}))_{d}(a_1, \ldots, a_d)
\]
\[
= \sum_{r,j} \sum_{i_1+\ldots+i_r=d} (-1)^{\hat{t}} \mathcal{F}_{12}(\mathcal{F}_{01}'(a_1, \ldots, a_{i_1}), \ldots, \mathcal{F}_{01}'(\ldots), T_{01}(a_{i_1+\ldots+i_{j-1}+1}, \ldots, a_{i_1+\ldots+i_j}), \mathcal{F}_{01}''(\ldots), \ldots, \mathcal{F}_{01}''(a_{i_1+\ldots+i_{r-1}+1}, \ldots, a_d)).
\]

(j) (Homotopy equivalence of \( A_\infty \) algebras) We say that \( A_\infty \) algebras \( A_0, A_1 \) are homotopy equivalent if there exist morphisms \( \mathcal{F}_{01} : A_0 \to A_1 \) and \( \mathcal{F}_{10} : A_1 \to A_0 \) such that \( \mathcal{F}_{01} \circ \mathcal{F}_{10} \) and \( \mathcal{F}_{10} \circ \mathcal{F}_{01} \) are homotopic to the respective identities. Homotopy equivalence of \( A_\infty \) algebras is an equivalence relation: Symmetry and reflexivity are immediate. For transitivity note that by the
previous item, if \( F_{01} \) and \( F''_{01} \) are homotopy equivalent then so are \( F_{12} \circ F'_{01} \) and \( F_{12} \circ F''_{01} \); repeating the argument for left composition, if \( F_{01} \) and \( F''_{01} \) are homotopic and also \( F'_{12} \) and \( F'''_{12} \) then \( F'_{12} \circ F'_{01} \) is homotopic to \( F''_{12} \circ F''_{01} \). Hence if \( F_{01} : A_0 \to A_1, F_{10} : A_1 \to A_0 \) and \( F_{12} : A_1 \to A_2, F_{21} : A_2 \to A_1 \) are homotopy equivalences then

\[
(F_{10} \circ F_{21}) \circ (F_{12} \circ F_{01}) = F_{10} \circ (F_{21} \circ F_{12}) \circ F_{01} \\
\cong F_{10} \circ F_{01} \cong \text{Id}
\]

and similarly for \((F_{12} \circ F_{01}) \circ (F_{10} \circ F_{21})\).

The following is a version of the material in [41, Chapter 4], which we will use in order to obtain homotopy invariance of the Maurer-Cartan moduli space. Let \( A_0, A_1 \) be convergent \( A_\infty \) algebras. Let \( F : A_0 \to A_1 \) be an \( A_\infty \) morphism or pre-natural transformation. We write

\[
F^n = \sum_m F^{n,m}, \quad F^{n,m} = \pi_{1-n+m} \circ F^n.
\]

We say that \( F \) is convergent iff \( F^0(1) \in A_{>0} \) and there exists a sequence \( E_m \to \infty \) such that

\[
F^{n,m}(A_0) \subset q^{E_m} A_1, \quad \forall n \geq 0.
\]

**Lemma 3.2.** (Map between Maurer-Cartan moduli spaces) Suppose that \( A_0, A_1 \) are convergent strictly-unital \( A_\infty \) algebras and \( F : A_0 \to A_1 \) is a convergent unital \( A_\infty \) morphism. Then

\[
\phi_F(b_0) = F^0(1) + F^1(b_0) + F^2(b_0, b_0) + F^3(b_0, b_0, b_0) + \ldots
\]

defines a map from \( A_0^+ \) to \( A_1^+ \) and restricts to a map \( \widetilde{MC}(A_0) \to \widetilde{MC}(A_1) \) of the moduli spaces of solutions to the weak Maurer-Cartan equation and descends to a map \( MC(A_0) \to MC(A_1) \). That is: For every \( b_0 \in \widetilde{MC}(A_0), \phi_F(b_0) \in \widetilde{MC}(A_1) \); and

\[
(b_0 \sim b'_0) \implies (\phi_F(b_0) \sim \phi_F(b'_0)), \quad \forall b_0, b'_0 \in \widetilde{MC}(A_0).
\]

Moreover, if \( F_0, F_1 : A_0 \to A_1 \) are unital \( A_\infty \) morphisms that are homotopic by a convergent \( A_\infty \) homotopy \( T : A_0 \to A_1 \) then \( F_0, F_1 \) induce the same map on Maurer-Cartan moduli spaces, that is, \( \phi_{F_0} = \phi_{F_1} \). In particular, if \( F : A \to A \) is convergent and convergent-homotopic to the identity then \( F \) induces the identity on \( MC(A) \). Hence the set \( MC(A) \) is an invariant of the homotopy type of \( A \).

**Proof.** The proof that the sum \( F^0(1) + F^1(b_0) + F^2(b_0, b_0) + F^3(b_0, b_0, b_0) + \ldots \) converges is essentially the same as that for Lemma 2.3 and left to the reader. Regarding gauge invariance, we first notice that for every \( b \sim b' \in \widetilde{MC}(A_0) \) with
$h \in A_0$ such that $b - b' = \mu^{1}_{b,b'}(h)$.

$$
\phi_\mathcal{F}(b) - \phi_\mathcal{F}(b') = \sum_{n_0,n_1} \mathcal{F}^{n_0+n_1+1}(b, \ldots, b, b, b', \ldots, b')
$$

$$
= \sum_{n_0,n_1,n_2,n_3} (-1)^{n_0+n_0|b|} \mathcal{F}^{n_0+n_1+1}(b, \ldots, b, \mu^{n_2+n_3+1}(b, \ldots, b, h, b', \ldots, b', b, b'))
$$

$$
= \mu^{1}_{\phi_\mathcal{F}(b), \phi_\mathcal{F}(b')}(\sum_{n_4,n_5} \mathcal{F}^{n_4+n_5+1}(b, \ldots, b, h, b', \ldots, b'))
$$

so that $\phi_\mathcal{F}(b) \sim \phi_\mathcal{F}(b')$. Note that the third equality above uses the fact that $b, b' \in MC(A_0)$ and the unitality of $\mathcal{F}$.

The second part of the claim follows from the transitivity of $\sim$, the above calculation, plus the fact that $\mathcal{F}_0 - \mathcal{F}_1 = \mu^{1}_{\text{Hom}(\mathcal{F}_0, \mathcal{F}_1)}(\mathcal{T})$ for some unital pre-natural transformation $\mathcal{T}$, then

$$
\phi_{\mathcal{F}_0}(b) - \phi_{\mathcal{F}_1}(b) = \sum_{k \geq 0} (\mu^{1}_{\text{Hom}(\mathcal{F}_0, \mathcal{F}_1)}\mathcal{T})^k(b, \ldots, b)
$$

$$
= \sum_{k \geq 0} \sum_{m} \sum_{i_1 + \ldots + i_m = k} (-1)^{i} \mu_{A_2}^{m}(\mathcal{F}_{0,i}^{i_1}(b, \ldots, b), \mathcal{F}_{0,i}^{i_2}(b, \ldots, b), \ldots)
$$

$$
\mathcal{T}^{i_1}(b, \ldots, b), \mathcal{F}_{1,i}^{i_1+1}(b, \ldots, b), \ldots, \mathcal{F}_{1,i}^{i_m}(b, \ldots, b))
$$

$$
- \sum_{i,e} (-1)^{i+\sum_{j=1}^{i} |b| + |\mathcal{T}| - 1} \mathcal{T}^{i-1}(b, \ldots, b, \lambda e_{A_1}, b, \ldots, b)
$$

Using that $b$ is a weak Maurer-Cartan solution we continue

$$
\mathcal{T}^{i_1}(b, \ldots, b), \mathcal{F}_{1,i}^{i_1+1}(b, \ldots, b), \ldots, \mathcal{F}_{1,i}^{i_m}(b, \ldots, b))
$$

$$
= \sum_{k \geq 0} \sum_{m} \sum_{i_1 + \ldots + i_m = k} (-1)^{i} \mu_{A_2}^{m}(\mathcal{F}_{0,i}^{i_1}(b, \ldots, b), \mathcal{F}_{0,i}^{i_2}(b, \ldots, b), \ldots)
$$

$$
\mathcal{T}^{i_1}(b, \ldots, b), \mathcal{F}_{1,i}^{i_1+1}(b, \ldots, b), \ldots, \mathcal{F}_{1,i}^{i_m}(b, \ldots, b))
$$

$$
= \mu^{1}_{\phi_\mathcal{F}_0(b), \phi_\mathcal{F}_1(b)}(\sum_{k} \mathcal{T}^{k}(b, \ldots, b)).
$$

Since $\mathcal{T}$ is convergent and $b \in A^+ = A_{<0} + \Lambda_{>0} A_{>0}$ the sum

$$
h_{01} := \sum_{k} \mathcal{T}^{k}(b, \ldots, b)
$$
exists in $A$. Furthermore since $\mathcal{T}^k$ has degree $-k$ and $\mathcal{T}^0(1) \in \Lambda_{>0} A$ we have

$$h_{01} \in A^{++} = A_{<0} + \Lambda_{>0} A_{\geq 0}.$$ 

Hence $\phi_{F_0}(b) \sim_{h_{01}} \phi_{F_1}(b)$ as claimed. \hfill \Box

Similarly one has a homotopy invariance property of the cohomology vector bundle introduced in \cite{50}:

**Lemma 3.3.** (Maps of cohomology bundles) Any convergent $A_\infty$ morphism $\mathcal{F} : A_0 \to A_1$ induces a morphism $H(\mathcal{F}) : H(A_0) \to H(A_1)$, that is, a morphism $H(b) \to H(\mathcal{F}(b))$ for each $b \in \widehat{MC}(A_0)$. If $\mathcal{F}_0, \mathcal{F}_1 : A_0 \to A_1$ are convergent morphisms related by a convergent homotopy then $H(\mathcal{F}_0)$ is equal to $H(\mathcal{F}_1)$ up to gauge transformation. In particular, if there exist convergent $A_\infty$ maps $\mathcal{F}_0_1 : A_0 \to A_1$ and $\mathcal{F}_{10} : A_1 \to A_0$ such that $\mathcal{F}_0 \circ \mathcal{F}_{10}$ and $\mathcal{F}_{10} \circ \mathcal{F}_0$ are homotopic to the identities via convergent homotopies then $H(A_0)$ is isomorphic to $H(A_1)$ up to gauge equivalence in the sense that

$$H(b_0) \cong H(\mathcal{F}_0(b_0)), \quad H(b_1) \cong H(\mathcal{F}_{10}(b_1))$$

for any $b_0 \in \widehat{MC}(A_0), b_1 \in \widehat{MC}(A_1)$.

The proof is similar to that of Lemma \cite[2.36]{2} and omitted. Thus having a non-trivial cohomology is an invariant of the homotopy type of a convergent, strictly-unital $A_\infty$ algebra.

3.2. Multiplihedra. The terms in the $A_\infty$ morphism axiom correspond to codimension one cells in a cell complex called the multiplihedron introduced by Stasheff \cite{105}. Stasheff’s definition identifies the $n$-multiplihedron as the cell complex whose vertices correspond to total bracketings of $x_1, \ldots, x_n$, together with the insertion of expressions $f(\cdot)$ so that every $x_j$ is contained in an argument of some $f$. For example, the second multiplihedron is an interval with vertices $f(x_1)f(x_2)$ and $f(x_1x_2)$.

A geometric realization of this polytope was given by Boardman-Vogt \cite{14} in terms of what we will call quilted metric trees. A quilted metric planar tree is a planar metric tree

$$T = (\mathcal{T}, e_0 \in \text{Edge}_{\to}(\mathcal{T}), \ell : \text{Edge}(\mathcal{T}) \to [0, \infty])$$

together with a subset $\text{Vert}^\text{col}(\mathcal{T}) \subset \text{Vert}(\mathcal{T})$ of colored vertices such that every simple path from the root to a leaf meets precisely one colored vertex, satisfying the

(Balanced lengths condition) For any two colored vertices $v_1, v_2 \in \text{Vert}^\text{col}(\mathcal{T})$, \begin{equation}
\sum_{e \in P_+(v_1,v_2)} \ell(e) = \sum_{e \in P_-(v_1,v_2)} \ell(e)
\end{equation}

where $P(v_1, v_2)$ is the (finite length) oriented non-self-crossing path from $v_1$ to $v_2$ and $P_+(v_1, v_2)$ resp. $P_-(v_1, v_2)$ is the part of the path towards resp. away from the root edge, see Ma’u-Woodward \cite{175}.

The set of combinatorially finite resp. semiinfinite edges is denoted $\text{Edge}_-(\mathcal{T})$ resp. $\text{Edge}_-(\mathcal{T})$; the latter are equipped with a labelling by integers $0, \ldots, n$. A quilted tree is stable if each colored vertex has valence at least two and any non-colored vertex has valence at least three. One can consider broken quilted trees as in the
non-quilted case, but requiring that any simple path from the root of the broken tree to a leaf still meets only one colored vertex. There is a natural notion of convergence of quilted trees, in which edges whose length approaches zero are contracted and edges whose lengths go to infinity are replaced by broken edges.

A different realization of the multiplihedron is the moduli space of stable quilted disks in Ma’u-Woodward [75]. In this realization, one obtains Stasheff’s cell structure on the multiplihedron naturally. Namely in [75] a quilted disk was defined as a datum $(S, Q, x_0, \ldots, x_n \in \partial S)$ consisting of a marked complex disk $(S, x_0, \ldots, x_n \in \partial S)$ (the points are required to be in counterclockwise cyclic order) together with a circle $Q \subset S$ (here we take $S$ to be a ball in the complex plane, so the notion of circle makes sense) tangent to the 0-th marking $x_0$. An isomorphism of quilted disks from $(S, Q, x_0, \ldots, x_n)$ to $(S', Q', x'_0, \ldots, x'_n)$ is an isomorphism of holomorphic disks $S \to S'$ mapping $Q$ to $Q'$ and $x_0, \ldots, x_n$ to $x'_0, \ldots, x'_n$. The open stratum may be identified with the set of sequences $0 = x_1 < \ldots < x_n$. A compactification is obtained by allowing bubbles to form either when the points come together, in which case a disk bubble forms, or when the markings go to infinity, in which case one rescales to keep the maximum distance between the markings constant and a collection of quilted disk bubbles form. The combinatorial type of a quilted disk is the tree obtained by collapsing disks. The set of vertices $\text{Vert}(\Gamma)$ has a distinguished subset $\text{Vert}^1(\Gamma)$ of colored vertices corresponding to the quilted components. The path from the root edge of the tree to any leaf is required to pass through exactly one colored vertex. Another way of defining a quilting is to equip the disk with a distinguished affine structure: an isomorphism with a half-space $\phi : D - \{z_0\} \to \mathbb{H}$, where two isomorphisms $\phi, \phi' : D - \{z_0\} \to \mathbb{H}$ are considered equivalent if they differ by a translation of $\mathbb{H}$. In this context the notion of quilted disk admits a natural generalization to the notion of a quilted sphere: a marked sphere $(C, (z_0, \ldots, z_n))$ equipped with an isomorphism $C - \{z_0\} \to \mathbb{C}$ to the affine line $\mathbb{C}$. Again, two such isomorphisms are considered equivalent if they differ by a translation. Given the notion of quilted sphere, there is a natural compactification of the space of quilted disks with interior markings which involves both quilted disks and quilted spheres. Each interior marking defines a leaf of the tree of interior type, and again any path from the root edge to any leaf is required to cross a single colored vertex. As the interior and boundary markings go to infinity, they bubble off onto either quilted disks or quilted spheres. A picture, involving the trees as well, is given below in Figure 17. The case of combined boundary and interior markings is a straightforward generalization of the boundary and interior cases treated separately in [75].

There is a combined moduli space of quilted marked treed disks given by (i) a quilted rooted metric planar tree (ii) for each non-colored vertex of the tree a marked disk or sphere with boundary markings whose number is the valence of the given vertex (iii) for each colored vertex, a marked quilted disk or sphere with number of boundary and interior nodes equal to valence of that of the colored vertex. From this datum one defines a space obtained by attaching the endpoints of the segments of the quilted metric tree to the marked points on the disks corresponding to the vertices. A marked quilted treed disk is stable if its underlying tree is stable and each of its quilted and unquilted disks is stable. Let $\overline{\mathcal{M}}_{n,m,1}$ denote the moduli space
of stable marked quilted treed disks with $n$ boundary leaves and $m$ interior leaves. See Figure 16 for a picture of $\overline{\mathcal{M}}_{2,0,1}$. The quilted disks are those with two shadings; while the ordinary disks have either light or dark shading depending on whether they can be connected to the zero-th edge without passing a colored vertex. The hashes on the line segments indicate nodes connecting segments of infinite length, that is, broken segments. Note that any interior marking now corresponds to a leaf.

**Figure 16.** Moduli space of stable quilted treed disks

Any path from the root edge to that leaf must pass through a colored vertex; this could be either a quilted disk or quilted sphere. See Figure 17 for the combinatorics of the top-dimensional cells in the case of one boundary leaf and one interior leaf; the $s$ indicates a quilted sphere component.

**Figure 17.** Moduli space of stable quilted treed disks with a boundary leaf and an interior leaf

Orientations of the moduli space of quilted treed disks are defined as follows. Each main stratum of $\overline{\mathcal{M}}_{n,m,1}$ can be oriented using the isomorphism of the stratum made of quilted treed disks having a single disk with $\mathbb{R}$ times $\overline{\mathcal{M}}_{n,m}$, the extra factor corresponding to the quilting parameter. The boundary of the moduli space
is naturally isomorphic to a union of moduli spaces:

\[
\partial \mathcal{M}_{n,m,1} \cong \bigcup_{m_1 + m_2 = m} (\mathcal{M}_{n-i+1,m_1,1} \times \mathcal{M}_{i,m_2}) \cup \bigcup_{m_0 + \sum_{j=1}^{r} m_j = m} \left( \mathcal{M}_{r,m_0} \times \prod_{j=1}^{r} \mathcal{M}_{i_j,m_j,1} \right).
\]

By construction, for the facet of the first type, the sign of the inclusions of boundary strata are the same as that for the corresponding inclusion of boundary facets of \( \mathcal{M}_{n,m,1} \), that is, \((-1)^{(n-i-j)+j}\). For facets of the second type, the gluing map is

\[
(0, \infty) \times \mathcal{M}_{r,m_0} \times \bigoplus_{j=1}^{r} \mathcal{M}_{[I_j],[m_j,1]} \to \mathcal{M}_{n,m,1}
\]

given for boundary markings by

\[
(\delta, x_1, \ldots, x_r, (x_{1,j} = 0, x_{2,j}, \ldots, x_{[I_j],[j]}_{j=1}^r)) \mapsto (x_1, x_1 + \delta^{-1} x_{2,1}, \ldots, x_1 + \delta^{-1} x_{[I_1],[1]}, \ldots, x_r, x_r + \delta^{-1} x_{2,r}, \ldots, x_r + \delta^{-1} x_{[I_r],[r]}).
\]

This map changes orientations by \(\sum_{j=1}^{r} (r - j)(|I_j| - 1)\); in case of non-trivial weightings, \(|I_j|\) should be replaced by the number of incoming markings or non-trivial weightings on the \(j\)-th component.

The combinatorial type of a quilted disk is the graph obtained as in the unquilted case by replacing each quilted disk component with its combinatorial tree (now having colored vertices), each unquilted disk or sphere component with its combinatorial tree, and each edge being identified as infinite, semi-infinite, finite non-zero or zero. We also wish to allow disconnected types. In this case, labelling of the unquilted components by \(\{0,1\}\). Morphisms of graphs (Cutting infinite length edges edges, collapsing edges, making lengths/weights finite/non-zero, and forgetting tails) induce morphisms of moduli spaces of stable quilted treed disks as in the unquilted case. In the space case of cutting an edge of infinite length, one of the pieces will be quilted and the other unquilted. The \(\{0,1\}\)-labelling of the unquilted components takes value 0 resp. 1 if component is further away from the root resp. closer to the root than the quilted components with respect to any non-self-crossing path of components. For any combinatorial type \(\Gamma\) of quilted disk there is a universal quilted treed disk \(\mathcal{U}_\Gamma \to \mathcal{M}_\Gamma\) which is a cell complex whose fiber over \(C\) is isomorphic to \(C\), and splits into surface and tree parts \(\mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_{\partial,\Gamma} \cup \mathcal{T}_{\bullet,\Gamma}\), where the last two sets are the boundary and interior parts of the tree respectively.

One can furthermore consider labels and weights on the semi-infinite ends of quilted tree disks as in the case of treed disks. We suppose there is a partition of the boundary semi-infinite edges

\[
\text{Edge}^*(T) \sqcup \text{Edge}^\circ(T) \sqcup \text{Edge}^\dagger(T) = \text{Edge}_{\partial,\to}(T)
\]

into weighted resp. forgettable resp. unforgettable edges as in the unquilted case, except that now the root of the quilted tree with one weighted leaf and no marking is weighted with the same weight as the leaf, see Figure 18. The moduli space with a single weighted leaf and no markings is then a point.
The morphisms of moduli spaces of different type for quilted treed disks are defined as in the unquilted case, except that the (Cutting edges) resp. (Collapsing edges) resp. (Making an edge length or weight finite/non-zero) axiom now allows one to cut resp. collapse resp. change the length of a collection of edges of infinite resp. zero resp. infinite or zero length above a quilted treed disk component. In particular, in the case of making a collection of such edges finite the corresponding morphism is the inclusion of the second union $\mathcal{M}_{r,m_0} \times \prod_{j=1}^{r} \mathcal{M}_{i_j,m_j,1}$ in (61) as a component of $\partial \mathcal{M}_{n,m,1}$.

### 3.3. Quilted holomorphic disks.

To prove that the Fukaya algebra is independent of the perturbation system, one considers two systems of perturbations and extends them to a set of perturbations for the moduli space of quilted treed disks.

In the next few sections we deal only with divisors of the same degree, built from homotopic sections of the same line bundle; the general case is treated in Section 3.6 below. Suppose that $D^0, D^1$ are stabilizing divisors for $L$ with respect to compatible almost complex structure $J^0, J^1$, of the same degree and built from homotopic unitary sections over $L$. Choose a path $J^t$ from $J^0$ to $J^1$. By Lemma 2.22 above, there exists a path of $J^t$-stabilizing divisors $D^t, t \in [0,1]$ connecting $D^0, D^1$, and a path $J^t_{D^t}$ of compatible almost complex structures such that $D^t$ contains no $J^t_{D^t}$-holomorphic spheres.

In order to specify which divisor of the above family we use at a given point of a quilted domain, define a function as follows. For every point $z \in C$ of a quilted treed disk $C$, let

$$d(z) : = \sum_{\ell(e) \in [-\infty, \infty]} \ell(e) \in [-\infty, \infty]$$

be the distance of $z$ to the quilted components of $C$ (with respect to the lengths of the edges) times 1 resp. $-1$ if $z$ is above resp. below the quilted components (that is, further from resp. closer to the root than the quilted components).

Given perturbation data $P^0$ and $P^1$ with respect to metrics $G^0, G^1 \in \mathcal{G}(L)$ over unquilted treed disks for $D^0$ resp. $D^1$, a perturbation morphism $P^{01}$ from $P^0$ to $P^1$ for the quilted combinatorial type $\Gamma$ consists of

(a) a smooth function $\delta^{01}_\Gamma : [-\infty, \infty] \to [0,1]$ (to be composed with $d$)

(b) a smooth domain-dependent choice of metric

$$G^{01}_\Gamma : \mathcal{T}_{c,\Gamma} \to \mathbb{R}$$
constant to $G^0$ resp. $G^1$ on the neighborhood of the points at infinity of the semi-infinite edges

$$\overline{T}_{\ast,\Gamma} - \overline{T}_{\ast,\Gamma}^{cp},$$

for which $d = -\infty$ resp. $d = \infty$,

(c) a domain-dependent Morse function

$$F^{01}_{\Gamma} : \overline{T}_{\ast,\Gamma} \to \mathbb{R}$$

constant to $F^0$ resp. $F^1$ on the neighborhood $\overline{T}_{\ast,\Gamma} - \overline{T}_{\ast,\Gamma}^{cp}$ of the endpoints for which $d = -\infty$ resp. $d = \infty$ and equal to $F^{01}_{\Gamma_0}$ resp. $F^{01}_{\Gamma_1}$ on the (unquilted) treed disks components of type $\Gamma_0, \Gamma_1$ for which $d = -\infty$ resp. $d = \infty$, and

(d) a domain-dependent almost complex structure

$$J^{01}_{\Gamma} : \overline{\mathcal{S}}_\Gamma \to \mathcal{J}_\tau(X)$$

with the property that for any surface component $C_i$ of $C$, $J^{01}_{\Gamma}$ is equal to the given $J$ away from the compact part:

$$J_{\Gamma}|_{\overline{\mathcal{S}}_\Gamma - \overline{\mathcal{S}}_\Gamma^{cp}} = \pi_2^*J$$

where $\pi_2$ is the projection on the second factor in $[84]$, and equal to the complex structures $J^{01}_{\Gamma_0}$ resp. $J^{01}_{\Gamma_1}$ on the (unquilted) treed disks components of type $\Gamma_0, \Gamma_1$ for which $d = -\infty$ resp. $d = \infty$: Let $\iota_k : \overline{\mathcal{S}}_{\Gamma_k} \to \overline{\mathcal{S}}_\Gamma$ denote the inclusion of the unquilted components. Then we require

$$J_{\Gamma}|_{\overline{\mathcal{S}}_{\Gamma_k}} = J^{k}_{\Gamma_k}, \quad k \in \{0, 1\}.$$

(e) One can also require the following invariance property: A perturbation system is quilt-independent if $G^{01}_{\Gamma}, F^{01}_{\Gamma}, J^{01}_{\Gamma}$ are pull-backs under the forgetful morphism forgetting the quilting on the quilted disk components.

To obtain a well-behaved moduli space of quilted holomorphic treed disks we impose a stability condition and quotient by an equivalence relation. Given a quilted treed disk $C$, we obtain a stable quilted treed disk by collapsing unstable surface and tree components. The result may be identified with a fiber of a universal curve of some type $s(\Gamma)$. By pullback we obtain a triple on $C$, still denoted $(\delta^{01}_{\Gamma}, J^{01}_{\Gamma}, F^{01}_{\Gamma})$.

A holomorphic quilted treed disk $u : C \to X$ of combinatorial type $\Gamma$ is a continuous map from a quilted treed disk $C$ that is smooth on each component, $J^{01}_{\Gamma}$-holomorphic on the surface components, $F^{01}_{\Gamma}$-Morse trajectory with respect to the metric $G^{01}_{\Gamma}$ on each boundary tree segment of disk type $e \in \text{Edge}_\ast(T)$, and constant on the tree segments of sphere type $e \in \text{Edge}_\circ(T)$. What this means is that the interior parts of the tree are irrelevant for our purposes; but they do affect the combinatorics of the boundary of the moduli spaces because of the balanced condition, see Figure 17. A quilted disk $u : C \to X$ is stable if

(a) each unquilted disk component on which $u$ is constant has at least three special boundary points or one special boundary point and one interior special point

$$du(C_i) = 0 \quad C_i \text{ unquilted disk} \quad \implies \quad 2\#\{z_k, w_k \in \text{int}(C_i)\} + \#\{w_k \in \partial C_i\} \geq 3;$$
(b) each quilted disk component on which $u$ is constant has at least two special points on the boundary or one interior special point:

$$\text{du}(C_i) = 0, \ C_i \text{ quilted disk} \implies \# \{ z_i, w_j \in \text{int}(C_i) \} + \# \{ w_k \in \partial C_i \} \geq 2$$

(c) each sphere $C_i \subset C$ component on which $u$ is constant has at least three special points:

$$\text{du}(C_i) = 0, \ C_i \text{ sphere} \implies \# \{ z_i, w_j \in C_i \} \geq 3.$$  

As in the unquilted case, the stability condition is not quite the same as requiring no automorphisms, because of the exceptional case in the last item. A stable holomorphic quilted tree disk is adapted if

(a) each sphere component $C_i$ of $C$ that maps to $D^{\delta_{01}^0}_{\phi(C_i)}$ is constant,

(b) each the interior marking $z_i$ maps to $D^{\delta_{01}^0}_{\phi(z_i)}$ and

(c) for each $t \in [0, 1]$, each component of $u^{-1}(D^t \cap (\delta_{01}^0)^{-1}(t))$ contains a marking.

We remark that the condition that $\delta_{01}^0$ is constant on each disk or sphere implies that the union

$$D^{\delta_{01}^i} = \cup_{z \in S_i} \left( \{ z \} \times D^{\delta_{01}^0}_{\phi(z)} \right)$$

is an almost complex submanifold of $S_i \times X$. In particular, the intersection multiplicity of $u : C \to X$ with $D^{\delta_{01}^i}$ at $z_i$ is positive.

In order to obtain a moduli space of holomorphic treed quilted disks we quotient by an equivalence relation. Two stable weighted disks $u_0 : C_0 \to X, u_1 : C_1 \to X$ are isomorphic if there exists an isomorphism $\phi : C_0 \to C_1$ intertwining $u_0$ and $u_1$. Note that if $C_0, C_1$ have a single unmarked quilted disk component and a single leaf, the $u_0, u_1$ equivalence does not involve weightings on the leaf. On the set of isomorphism classes we quotient by a further equivalence relation as follows: Given a non-constant holomorphic quilted treed disk $u : C \to X$ with leaf $e_i \in \text{Edge}^*(\Gamma)$ on which there is a weighting $\rho(e_i) = 0$ resp. $\infty$, we declare $u$ to be equivalent to the holomorphic treed disk $u' : C \to X$ obtained by replacing the asymptotic critical point at $e_i$ by $x_M^*$ resp. $x_M^*$ and adding a constant segment from $x_M^*$ to $x_M^*$ resp. $x_M^*$ above that leaf as in the unquilted case in Figure 11. In other words, forgetting constant infinite segments and replacing them by the appropriate weights gives equivalent holomorphic treed disks. For any combinatorial type $\Gamma$ of quilted disks we denote by $\overline{M}_\Gamma(L, D)$ the compactified moduli space of equivalence classes of adapted quilted holomorphic treed disks.

The moduli space of quilted disks breaks into components depending on the limits along the root and leaf edges. Denote by $\overline{M}_\Gamma(L, D, \overline{z}) \subset \overline{M}_\Gamma(L, D)$ the moduli space of isomorphism classes of stable adapted holomorphic quilted treed disks with boundary in $L$ and limits $\overline{z}$ along the root and leaf edges, where $\overline{z} = (x_0, \ldots, x_n) \in \tilde{T}(L)$ satisfies the requirement:

(a) (Label axiom)

(i) If $x_0 = x_M^*$ resp. $x_0 = x_M^*$, then there is a single leaf reaching $x_M^*$ resp. $x_M^*$ and no interior marking (in which case the moduli space will be a point).
(ii) If $x_i = x^*_M$, resp. $x^*_M$, resp. $x^*_M$, for some $i \geq 1$ then the $i$-th leaf is required to be weighted resp. forgettable resp. unforgettable and the limit along this leaf is required to be $x_M$.

(iii) If $x_i \notin \{x^*_M, x^*_M, x^*_M\}$, then the $i$-th leaf is required to be unforgettable.

(b) (Outgoing edge axiom) The outgoing edge $e_0$ is weighted (resp. forgettable) only if there is a single incoming leaf, which is weighted (resp. forgettable) with the same weight and the configuration has no interior markings (so there is a single quilted disk with no markings.)

In order to obtain moduli spaces with the expected boundary, we introduce a coherence condition. We say that a collection $P^0 = (P^0)$ of perturbation morphisms is coherent if $P^0$ is compatible with the morphisms of moduli spaces as before:

(a) (Cutting edges) If $\Gamma'$ is obtained from $\Gamma$ by cutting an edge or a collection of edges of infinite length, then $P^0$ is the pushforward of $P^0$.

(b) (Collapsing edges/making an edge/weight finite/non-zero) If $\Gamma'$ is obtained from $\Gamma$ by collapsing an edge or edges, then $P^0$ is the pullback of $P^0$.

(c) (Products) If $\Gamma$ is the union of a quilted type $\Gamma_1$ and a non-quilted type $\Gamma_0$, then $P^0$ is obtained from $P^0$ and $P^0$ as follows: Let $\pi_k : M_{\Gamma} \cong M_{\Gamma_1} \times M_{\Gamma_0} \to M_{\Gamma_1}$ denote the projection on the $k$-factor, so that $U_{\Gamma}$ is the union of $\pi_1^{*}U_{\Gamma_1}$ and $\pi_0^{*}U_{\Gamma_0}$. Then we require that $P^1$ is equal to the pullback of $P^0$ on $\pi_1^{*}U_{\Gamma_1}$ and to the pullback of $P^0$ on $\pi_0^{*}U_{\Gamma_0}$.

Similarly, if $\Gamma$ is the union of a non-quilted type $\Gamma_1$ and quilted types $\cup_i \Gamma_{0,i}$ and quilted types with interior markings, then $P^0$ is equal to the pullback of $P^1$ on $\pi_1^{*}U_{\Gamma_1}$ and to the pullback of $P^0$ on $\pi_0^{*}U_{\Gamma_{0,i}}$.

The case of constant quilted types requires special treatment. If any of the types $\Gamma_{0,i}$ have no interior markings and a single weighted incoming leaf then we label the corresponding incoming leaf of $\Gamma_1$ with the same weight, by our (Cutting edges) construction. This guarantees that the moduli spaces are of expected dimension.

(d) (Infinite weights) Whenever a weight parameter $\rho(e_i)$ is equal to infinity, then the $P^0$ is pulled back under the forgetful map forgetting the $e_i$ semi-infinite edge and stabilizing from the perturbation morphism $P^0$ given by (Forgetting tails).

For generic perturbation morphisms close to the given base structure the moduli space of holomorphic adapted quilted treed disks has compactness and transversality properties similar to those for unquilted disks. A perturbation morphism is stabilized if it satisfies a condition analogous to that in Definition (2.24). A perturbation morphism is convergent if it satisfies a condition analogous to (2.26). For a comeager subset of perturbation morphisms extending those chosen for unquilted disks, the uncrowded moduli spaces of expected dimension at most one are smooth and of expected dimension. For sequential compactness, it suffices to consider a sequence $u_\nu : C_\nu \to X$ of quilted treed disks of fixed combinatorial type $\Gamma_\nu$ constant in $\nu$. Coherence of the perturbation morphism implies the existence of a stable limit $u : C \to X$ which we claim is adapted. In particular, the (Marking Property) is justified as follows. For each component $C_i \subset C$, the almost complex structure $J_{\Gamma}|C_i$ is
constant near the almost complex submanifold \( D_{\delta_0^1 \odot (C_i)} \). We suppose that \( C_i \) is a component of the limit of some sequence of components \( C_{i,\nu} \) of \( C_\nu \). Coherence for the parameter \( \delta_0^1 \) implies that \( D_{\delta_0^1 \odot (C_i)} \) is the limit of the divisors \( D_{\delta_{i,\nu}^1 \odot (C_{i,\nu})} \). Then local conservation of intersection degree implies that any component of \( u^{-1}(D_{\delta_0^1 \odot (C_i)}) \) contains a limit point of some markings \( z_{i,\nu} \in u^{-1}(D_{\delta_{i,\nu}^1 \odot (C_{i,\nu})}) \). For types of index at most one, each component of \( u^{-1}(D_{\delta_0^1 \odot (C_i)}) \) is a limit of a unique component of \( u^{-1}(D_{\delta_{i,\nu}^1 \odot (C_{i,\nu})}) \), otherwise the intersection degree would be more than one which is a codimension two condition. Since non-trivial sphere bubbling is a codimension two condition and ghost bubbling is impossible unless two markings come together, this implies that \( u^{-1}(D_{\delta_0^1 \odot (C_i)}) = \{z_i\} \) is also a marking.

The condition (6) implies the following properties of the moduli spaces of constant quilted disks. By the argument in the proof of Proposition 3.7 in the 0-dimensional strata all of the quilted disks are mapped to points (and thus they are unmarked in the domains). The 1-dimensional strata will be of two types: First, there are 1-dimensional families of quilted trajectories where every quilted disk is constant. Second, there may be 1-dimensional families for which only the quilting parameter on a single non-constant quilted disk varies from \(-\infty\) to \(\infty\).

### 3.4. Morphisms of Fukaya algebras.

Given a regular, stabilized, convergent, coherent perturbation morphism \( P^{01} \) from \( P^0 \) to \( P^1 \), define

\[
\phi^n : \widehat{CF}(L; P^0)^{\otimes n} \to \widehat{CF}(L; P^1)
\]

\[
(x_1, \ldots, x_n) \mapsto \sum_{x_0, [u] \in \mathcal{M}_L(L, D, x_0, \ldots, x_n)_0} (-1)^{\nu_{\odot}} (\epsilon([u]) (\sigma([u])!))^{-1} q^{\nu_{\odot}} \text{Hol}_L(u)x_0
\]

where the sum is over quilted disks in strata of dimension zero with \( x_1, \ldots, x_n \) incoming labels.

**Remark 3.4.** (Lowest energy terms) For \( x \in \widehat{\Gamma}(L) \), the element \( \phi^1(x^*_M) \) resp. \( \phi^1(x^*_{\hat{M}}) \) resp. \( \phi^1(x^*_{\hat{M}}) \) has a \( x^*_M \) resp. \( x^*_{\hat{M}} \) resp. \( x^*_{\hat{M}} \) term coming from the count of a quilted treed disk with no interior marking, that is, a treed disk with only one disk that is quilted and mapped to a point. In the latter case, it will be the only term with \( x^*_{\hat{M}} \) output, by the (Label axiom).

**Remark 3.5.** The codimension one strata are of several possible types: either there is one (or a collection of) edge of length infinity, there is one (or a collection of) edge of length zero, or equivalently, boundary nodes, or there is an edge with zero or infinite weight. The case of an edge of zero or infinite weighting is equivalent to breaking off a constant trajectory, and so may be ignored. In the case of edges of infinite length(s), then either \( \Gamma \) is

(a) (Breaking off an uncolored tree) a pair \( \Gamma_1 \sqcup \Gamma_2 \) consisting of a colored tree \( \Gamma_1 \) and an uncolored tree \( \Gamma_2 \) as in Figure 19 necessarily the breaking must be a leaf of \( \Gamma_1 \); or

(b) (Breaking off colored trees) a collection consisting of an uncolored tree \( \Gamma_0 \) containing the root and a collection \( \Gamma_1, \ldots, \Gamma_r \) of colored trees attached to
Figure 19. Breaking off an unquilted treed disk

each of its $r$ leaves as in Figure 20. Such a stratum $\mathcal{M}_\Gamma$ is codimension one because of the (Balanced Condition) which implies that if the length of any edge between $e_0$ to $e_i$ is infinite for some $i$ then the path from $e_0$ to $e_i$ for any $i$ has the same property.

Figure 20. Breaking off a collection of quilted disks

In the case of a zero length(s), one obtains a fake boundary component with normal bundle $\mathbb{R}$, corresponding to either deforming the edge(s) to have non-zero length or deforming the node(s).

**Theorem 3.6.** (A$_\infty$ morphisms via quilted disks) For any coherent, stabilizing, regular, convergent, ghost-marking-independent collection $P_{01}$ of perturbations morphisms from $P_0$ to $P_1$, the collection of maps $\phi = (\phi^n)_{n \geq 0}$ constructed above by counting quilted adapted treed holomorphic disks is a convergent unital A$_\infty$ morphism from $\hat{CF}(L, P_0)$ to $\hat{CF}(L, P_1)$.

**Proof.** By counting the ends of the one-dimensional moduli spaces we obtain the relation (29) but with $\mathcal{T}_{n,m}$ replaced by the set of types $\mathcal{T}_{n,m,1}$ of quilted treed disks with $n$ leaves and $m$ interior markings. The true boundary strata are those described in Remark 3.5 and correspond to the terms in the axiom for A$_\infty$ morphisms (53). The signs for the terms of the type $\phi^{n-d+1}(\ldots, \mu^d(\ldots), \ldots)$ are similar to those for the A$_\infty$ axiom, see (37), and will be omitted. For terms of the second type
\( \mu^r(\phi^i(\ldots), \ldots, \phi^r(\ldots)) \) we verify the sign of the gluing map

\[
\mathbb{R} \otimes \mathcal{M}_{r,m_0}(y_0, \ldots, y_r) \times \bigoplus_{j=1}^{r} \mathcal{M}_{|I_j|, m_j, 1}(y_i, x_{I_j}) \to \mathcal{M}_{n,m,1}(y_0, x_1, \ldots, x_n).
\]

An orientation of the former is determined by an orientation of

\[
\mathbb{R} \oplus T\mathcal{M}_{r,m_0} \oplus TL \oplus T_{y_0}^+ \oplus T_{y_1}^- \oplus \cdots \oplus T_{y_r}^- \oplus \bigoplus_{j=1}^{r} \left( T\mathcal{M}_{|I_j|, m_j, 1} \oplus TL \oplus T_{y_j}^+ \oplus \bigoplus_{k \in I_j} T_{x_k}^- \right)
\]

(where \( T_{y_0}^+ \) denotes \( T_{y_0}^+ W_{y_0}^+ \) etc.). Transposition to

\[
\mathbb{R} \oplus T\mathcal{M}_{r,m_0} \oplus TL \oplus T_{y_0}^+ \oplus \bigoplus_{j=1}^{r} \left( T\mathcal{M}_{|I_j|, m_j, 1} \oplus T_{y_j}^+ \oplus TL \oplus T_{y_j}^- \oplus \bigoplus_{k \in I_j} T_{x_k}^- \right)
\]

changes sign by \((-1)^{\sum_{j=1}^{r}(|I_j| - 1)|y_j|} \). Using \( T_{y_j}^- \oplus T_{y_j}^+ \cong TL \) these three factors disappear. Transposing the factors \( T\mathcal{M}_{|I_j|, m_j, 1} \) and \( T_{x_k}^- \) for \( k < \min I_j \) contributes a number of signs \( \sum_{j=1}^{r} (|I_j| - 1) \left( \sum_{k<\min I_j} |x_k| \right) \). Transposing \( TL \oplus T_{y_0}^+ \) past \( \bigoplus_{j=1}^{r} T\mathcal{M}_{|I_j|, m_j, 1} \) gives \( |y_0| \sum_{j=1}^{r} (|I_j| - 1) \) additional signs. Finally we have a contribution from the signs in the definition of \( \phi^{|I_j|} \) and the sign from the definition of \( \mu^r \), which is \( \sum_{k=1}^{n} k|x_k| + \sum_{j=1}^{r} \sum_{i=1}^{|I_j|} |x_i| \). The gluing map has sign \((-1)^2\). We note the identities

\[
\sum_{k=1}^{n} k|x_k| - \sum_{j=1}^{r} \sum_{i=1}^{I_j} |x_i| = \sum_{j=1}^{r} \left( \sum_{i \in I_j} |I_1| + \ldots + |I_{j-1}| \right) |x_i|
\]

\[
\sum_{j=1}^{r} |I_j| \sum_{k \notin I_j} |x_k| = \sum_{j=1}^{r} |I_j| \left( \sum_{k} |x_k| - \sum_{k \in I_j} |x_k| \right)
\]

\[
= n(|y_0| + n - 2) - \sum_{j=1}^{r} |I_j|(|y_j| + |I_j| - 1)
\]

\[
\sum_{j=1}^{r} \sum_{k \in I_j} (r - j)|x_k| = \sum_{j=1}^{r} (|y_j| + |I_j| - 1)(r - j)
\]

\[
= \sum_{j=1}^{r} (|y_j|r - |y_j|j + (|I_j| - 1)(r - j)).
\]


Using these identities the total sign is $(-1)^\Theta$ where

$$\Theta = \sum_{j=1}^r (|I_j| - 1)|y_j| + \sum_{j=1}^r (|I_j| - 1) \sum_{k < \text{min } I_j} |x_k| + |y_0| \sum_{j=1}^r (|I_j| - 1)$$

$$+ \sum_{k=1}^n k|x_k| + \sum_{j=1}^r \sum_{i=1}^{|I_j|} |x_i| + \sum_{j=1}^r |y_j| + \sum_{j=1}^r (r - j)(|I_j| - 1)$$

$$= \sum_{j=1}^r (|I_j| - 1)|y_j| + \sum_{j=1}^r (|I_j| - 1) \left( \sum_{k < \text{min } I_j} |x_k| \right) + |y_0| \sum_{j=1}^r (|I_j| - 1)$$

$$+ \sum_{j=1}^r |I_j| \left( \sum_{k > \text{max } I_j} |x_k| \right) + \sum_{j=1}^r j|y_j| + \sum_{j=1}^r (r - j)(|I_j| - 1)$$

$$\Theta = -\left( \sum_{j=1}^r \sum_{k < \text{min } I_j} |x_k| \right) + \sum_{j=1}^r (|I_j| - 1)|y_j| + |y_0| \sum_{j=1}^r (|I_j| - 1) + \sum_{j=1}^r j|y_j| + n(|y_0| + n - 2)$$

$$- \sum_{j=1}^r (|I_j|(|y_j| + |I_j| - 1)) + \sum_{j=1}^r (r - j)(|I_j| - 1)$$

$$= -\sum_{j=1}^r (|y_j|(r - |y_j|) + (|I_j| - 1)(r - j)) + \sum_{j=1}^r (|I_j| - 1)|y_j| + |y_0| \sum_{j=1}^r (|I_j| - 1) + \sum_{j=1}^r j|y_j|$$

$$+ n(|y_0| + n - 2) - \sum_{j=1}^r (|I_j|(|y_j| + |I_j| - 1)) + \sum_{j=1}^r (r - j)(|I_j| - 1)$$

$$= (r + 1)(\sum_{j=1}^r |y_j| - |y_0|) + y_0 + n(n - 2) + 2 \sum_{j=1}^r j|y_j|$$

$$= (r + 1)(r + 2) + (n - 2) + \sum_{k=1}^n |x_k| + n(n - 2) + 2 \sum_{j=1}^r j|y_j| \equiv 2 \sum_{k=1}^n |x_k|.$$

It follows that the sign induced by gluing the broken configuration of the second type is the same as that induced by the first type in [37]. The case of breaking off a trivial trajectory is similar to that in the proof the $A_\infty$ axiom Theorem 2.29 and left to the reader.

The assertion on the strict units is a consequence of the existence of forgetful maps for infinite values of the weights. By assumption the $\phi^n$ products involving $x_M^*$ as inputs involve counts of quilted treed disks using perturbation that are independent of the disk boundary incidence points of the lines marked $x_M^*$ asymptotic to $x_M \in X$. Since forgetting that semi-infinite edge gives a configuration of negative expected dimension, if non-constant, the only configurations contributing to these terms must
be the constant maps. Hence
\[ \phi^1(x_M) = x_M, \quad \phi^n(x_M, \ldots, x_M, \ldots) = 0, \quad n \geq 2. \]
In other words, the only regular quilted trajectories from the maximum, considered as \( x_M \), being regular are the ones reaching the other maximum that do not have interior markings (i.e., nonconstant disks). The proof of convergence is similar to that of Proposition 2.39 and left to the reader. \( \square \)

**Proposition 3.7.** Suppose that \( P^0 = P^1 \) is a regular, coherent perturbation datum for treed disks. For each type \( \Gamma \) of quilted treed disk, let \( \Gamma' \) denote the corresponding type of unquilted treed disk obtained by forgetting the quilting and collapsing unstable components. Pulling back \( P^0_{\Gamma',0} = P^1_{\Gamma',1} \) to a perturbation morphism \( P^0_{\Gamma,0} = P^1_{\Gamma,1} \) for quilted disks gives a regular, coherent perturbation morphism for quilted disks such that the corresponding \( A_\infty \) morphism is the identity.

**Proof.** Any regular perturbation system for unquilted disks induces a regular perturbation system for quilted disks by pullback under the forgetful map forgetting the quilting and collapsing unstable components. Given a non-trivial configuration contributing to \( \phi^n \) in the moduli space of expected dimension 0, one obtains a configuration contributing to \( \mu^n \) in the moduli space of expected dimension \(-1\) via the forgetful map. Therefore, the only configurations contributing to \( \phi^n \) are the constant configurations. Hence \( \phi^1 \) is the identity and all other maps \( \phi^n, n > 0 \) vanish. \( \square \)

### 3.5. Homotopies

The morphism of homotopy-associative algebras constructed above is a homotopy equivalence of \( A_{\infty} \) algebras by an argument using twice-quilted disks. A twice-quilted disk is defined in the same way as once-quilted disks, but with two interior circles that are either equal or with the second contained inside the first, say with radii \( \rho_1 < \rho_2 \). The moduli space of twice-quilted treed disks is a cell complex constructed in a similar way to the space of once-quilted treed disks. We denote the moduli space with \( n \) leaves and two quiltings by \( \mathcal{M}_{n,2} \). We show in Figure 21 the moduli space of twice-quilted stable disks \( \mathcal{M}_{n,2} \) without trees in the case \( n = 2 \). The moduli space \( \mathcal{M}_{2,2} \) is a pentagon whose vertices correspond to the expressions
\[ f(g(x_1 x_2)), f(g(x_1)g(x_2)), f(g(x_1))f(g(x_2)), ((fg)(x_1))((fg)(x_2)), (fg)(x_1 x_2). \]
The combinatorial type of a twice-quilted disk is a tree \( \Gamma = (\text{Vert}(\Gamma), \text{Edge}(\Gamma)) \) equipped with subsets \( \text{Vert}^1(\Gamma), \text{Vert}^2(\Gamma) \subset \text{Vert}(\Gamma) \) corresponding to the quilted components; the set \( \text{Vert}^{12}(\Gamma) := \text{Vert}^1(\Gamma) \cap \text{Vert}^2(\Gamma) \) corresponds to the twice-quilted components. The ratios
\[ \lambda_S(v) = \rho_2(v)/\rho_1(v), \quad v \in \text{Vert}^{12}(\Gamma) \]
of the radii of the interior circles with radii \( \rho_2(v), \rho_1(v) \), \( v \in \text{Vert}^{12}(\Gamma) \) are required to be equal for each twice-quilted disk in the configuration, if the configuration has twice-quilted components:
\[ \lambda_S(v_1) = \lambda_S(v_2), \quad \forall v_1, v_2 \in \text{Vert}^{12}(\Gamma). \]
One can also consider a moduli space of twice-quilted disks with interior markings;
these are required to lie on components \( v \in \text{Vert}(\Gamma) \) which are at least as far away from the root edge as the vertices \( v \in \text{Vert}^2(\Gamma) \). That is, if \( v \in \text{Vert}(\Gamma) \) lies on a non-self-crossing path from the root edge to an element in \( \text{Vert}^2(\Gamma) \) then \( v \in \text{Vert}^2(\Gamma) \). Once one allows interior markings, these can bubble off onto twice-quilted spheres, which are marked spheres \((C, (z_0, \ldots, z_n))\) equipped with two isomorphisms \( C - \{z_0\} \to \mathbb{C} \).

As in the quilted case, there is now a moduli space of treed twice-quilted spheres which assigns lengths to the interior and boundary nodes. The lengths to each colored vertex satisfy the balanced condition (60) for each color:

(a) For any two vertices of the same color \( v_1, v_2 \in \text{Vert}^k(\Gamma) \),

\[
\sum_{e \in P_+ (v_1, v_2)} \ell(e) = \sum_{e \in P_- (v_1, v_2)} \ell(e)
\]

where \( P(v_1, v_2) \) is the (finite length) oriented non-self-crossing path from \( v_1 \) to \( v_2 \) and \( P_+ (v_1, v_2) \) resp. \( P_- (v_1, v_2) \) is the part of the path pointing towards resp. away from the root edge, and

(b) for two vertices of different colors \( v_1 \in \text{Vert}^1(\Gamma) \) and \( v_2 \in \text{Vert}^2(\Gamma) \) for which there is a (finite length) oriented non-self-crossing path \( P(v_1, v_2) \) from \( v_1 \) to \( v_2 \), let

\[
\lambda_T(v_1, v_2) = \sum_{e \in P(v_1, v_2)} \ell(e).
\]

Then \( \lambda_T(v_1, v_2) \) is independent of the choice \( v_1 \in \text{Vert}^1(\Gamma) \) and \( v_2 \in \text{Vert}^2(\Gamma) \).

We suppose that divisors and perturbations for unquilted and once-quilted disks have already been chosen. That is, there are given compatible almost complex
structures $J_0, J_1, J_2$, metrics $G_0, G_1, G_2$, divisors $D_0, D_1, D_2$ and perturbation systems $P^0, P^1, P^2$ for unquilted disks. Furthermore, there are given paths $J^t_{01}, J^t_{12}, J^t_{02}$ of compatible almost complex structure from $J_0$ to $J_1$, $J_1$ to $J_2$ and $J_0$ to $J_2$, paths $D^t_{01}$ from $D_0$ to $D_1$, and $D^t_{12}$ from $D_1$ to $D_2$, and $D^t_{02}$ from $D_0$ to $D_2$. (In our application we are particularly interested in the case $D_0 = D_2$ and the constant path $D^t_{02} = D_0 = D_2$.) We suppose there are given perturbation data $P^{ij}$ for once-quilted disks giving rise to morphisms

$$\phi_{ij} : \overline{CF}(L, P^i) \to \overline{CF}(L, P^j), \quad 0 \leq i < j \leq 2.$$ 

We have in mind especially the case that $D_0 = D_2$, $D^t_{01} = D^{1-t}_{12}$, and $D^t_{02}$ is the constant path. In this case one may take $P^{12,t} = P^{12,1-t}$ and $P^{02}$ the perturbation system pulled back by the forgetful map forgetting the quilting as in Proposition 3.7.

We begin by extending the families of stabilizing divisors over the universal twice-quilted disks. A domain-dependent parameter for twice quilted disks is a smooth map

$$\delta^{012} : \triangle \equiv \{(t_1, t_2) \in [−∞, ∞]^2 | \ t_2 \leq t_1\} \to [0, 2].$$

For every point $z \in C$ of a twice quilted treed disk $C$, let $d(z) = (t_1, t_2) \in \triangle$ with $t_1$ being the signed distance of $z$ to the lowest quilted components of $C$ and $t_2$ being the signed distance of $z$ to the highest quilted components of $C$.

**Definition 3.8.** A perturbation $P^{012}_\Gamma$ for twice-quilted treed disks from quilted perturbation systems $L^{01} \times L^{12}$ to $L^{02}$ consists of

(a) a domain-dependent parameter $\delta^{012}_\Gamma$ that agrees
- with $\delta^{01}_{\Gamma_0}$ on $[−∞, ∞] \times \{-∞\}$,
- with $\delta^{12}_{\Gamma_{12}}$ on $\{-∞\} \times [−∞, ∞]$ and
- with $\delta^{02}_{\Gamma_{02}}$ on $\{(t_1, t_2) \in [−∞, ∞]^2 | t_1 = t_2\}$,

where $\Gamma_{ij}, 0 \leq i \leq j \leq 2$ is the corresponding type of once-quilted disk;

(b) a smooth family of metrics $G^{012}_\Gamma$ constant equal to $G^0$ resp. $G^1$ resp. $G^2$ on a neighborhood of the endpoints for which $d = (−∞, −∞)$ resp. $d = (∞, −∞)$ resp. $d = (∞, ∞)$ and that agrees
- with $G^{01}_{\Gamma_0}$ on $[−∞, ∞] \times \{-∞\}$, with $G^{12}_{\Gamma_{12}}$ on $\{+∞\} \times [−∞, ∞]$ and
- with $G^{02}_{\Gamma_{02}}$ on $\{(t_1, t_2) \in [−∞, ∞]^2 | t_1 = t_2\}$,

where $\Gamma_{ij}, 0 \leq i \leq j \leq 2$ is the corresponding type of once-quilted disk;

(c) a domain-dependent Morse function $F^{012}_\Gamma$ equal to $F^0$ resp. $F^1$ resp. $F^2$ on in a neighborhood of the endpoints at $d = (−∞, −∞)$ resp. $d = (∞, −∞)$ and that agrees with $F^{01}_{\Gamma_0}$ resp. $F^{12}_{\Gamma_{12}}$ resp. $F^{02}_{\Gamma_{02}}$ on the once quilted treed disk components of type $\Gamma_0$ resp. $\Gamma_{12}$ resp. $\Gamma_{02}$ containing the root resp. the leaves resp. where the quilting radii coincide,

(d) a domain-dependent almost-complex structure $J^{012}_\Gamma$ such that for every surface component $C_i$, it is equal to $J^{012,od(z)}_{\Gamma_0}$ on $D^{012,od(z)}_{\Gamma_0}$ in a neighborhood of the spherical nodes, the interior markings and on the boundary of $C_i$, that is

- with $J^{01}_{\Gamma_0}$ on $\{−∞\} \times [−∞, ∞]$ and
- with $J^{12}_{\Gamma_{12}}$ on $\{+∞\} \times [−∞, ∞]$ and
- with $J^{02}_{\Gamma_{02}}$ on $\{(t_1, t_2) \in [−∞, ∞]^2 | t_1 = t_2\}$.
• equal to $J_{\Gamma_0}^0$, resp. $J_{\Gamma_1}^1$, resp. $J_{\Gamma_2}^2$ on the unquilted components of type $\Gamma_0$ resp. $\Gamma_1$ resp. $\Gamma_2$ at $d = (\infty, -\infty)$ resp. $d = (\infty, \infty)$ and

• agrees with $J_{\Gamma_{01}}^{01}$ resp. $J_{\Gamma_{12}}^{12}$ resp. $J_{\Gamma_{02}}^{02}$ on the once quilted treed disk components of type $\Gamma_{01}$ resp. $\Gamma_{12}$ resp. $\Gamma_{02}$ containing the root resp. the leaves resp. where the quilting radii coincide.

(e) One can also require the following invariance property: A perturbation datum is quilting-independent if $G_{\Gamma}^{012}$, $F_{\Gamma}^{012}$, and $J_{\Gamma}^{012}$ are pull-backs under the forgetful morphism forgetting the quiltings on each once-quilted or twice-quilted disk.

Given a treed twice-quilted treed disk $C$ of type $\Gamma$, one obtains perturbation data by pull-back from the stabilization of $C$, which may be identified with a fiber of the universal twice-quilted disk. A holomorphic twice-quilted treed disk is a twice-quilted disk $C$ a map $u : C \to X$ that is $J_{\Gamma}^{012}$-holomorphic on surface components, a $F_{\Gamma}^{012}$-Morse trajectory on boundary tree segments with respect to the metrics $G_{\Gamma}^{01}$, $G_{\Gamma}^{12}$, $G_{\Gamma}^{02}$, and constant on the interior part of the tree. Stable and adapted twice-quilted treed disks are defined as in the once-quilted case. In particular, each interior marking $z_i$ maps to the divisor $D_{\delta_{\Gamma}}^{012}(z_i)$. Assuming the perturbations satisfy coherence and stabilized conditions similar to those for quilted disks, the moduli spaces of adapted stable twice-quilted disks are compact for each uncrowded combinatorial type of expected dimension at most one. The property (e) of Definition 3.8 ensures that the latter zero dimensional spaces will not contain non-constant (either once or twice) quilted disks and that the one dimensional strata may involve at most one once quilted disk and if it does, it is a constant family over which the latter quilting radius varies freely.

In order to obtain transversality the fiber products involved in the definition of the universal twice-quilted disks in [64] must be perturbed, using delay functions, as in Seidel [99] and Ma’u-Wehrheim-Woodward [75] which we follow closely. Again, the problem is a variation on the multiple cover problem: if an isolated twice-quilted component bubbles off with ratio given by some $\lambda_i$, then configurations with the same bubble repeated also appear, again with the same ratio. But transversality with the diagonal in (64) implies that at most one such twice-quilted component can be isolated in its moduli space.

We define a map incorporating both the condition on ratios and distances between quilted components as follows. We identify $M_{1,0,2}$ with $[0, \infty]$ as in Figure 22. For $n \geq 1, m \geq 0$ denote by

$$\lambda : M_{n,m,2} \to M_{1,0,2} \cong [0, \infty]$$

the forgetful morphism forgetting all but the first marking; note that on the interval consisting of only once-quilted disks, $\lambda$ is essentially equivalent to the map $\lambda_T$ of [66] while on the interval with twice-quilted disks $\lambda$ is given by the map $\lambda_S$ of [64]. We note that $\lambda$ is also defined in the case $n = 0$, by combining the maps [66] and [64]. This is also true if furthermore $m = 0$ although the latter twice-quilted treed disks are unstable and will not appear in the moduli spaces considered later.
Figure 22. Moduli of treed twice-quilted disks with one leaf

We combine the maps from the previous paragraph as follows. Let $\Gamma$ be a combinatorial type of twice-quilted disks. Define $\overline{\mathcal{M}}_{\Gamma}^{\text{pre}}$ as the product of moduli spaces for the vertices,

$$\overline{\mathcal{M}}_{\Gamma}^{\text{pre}} = \prod_{v \in \text{Vert}(\Gamma)} \mathcal{M}_v.$$ 

Let $k$ denote the number of twice-quilted vertices and

$$\lambda_{\Gamma} : \overline{\mathcal{M}}_{\Gamma}^{\text{pre}} \to \mathbb{R}^k, \quad (r_v) \mapsto \prod_{v \in \text{Vert}(\Gamma)} \lambda(r_v)$$

the map combining the forgetful maps for the twice-quilted components. Then $\mathcal{M}_{\Gamma} = \lambda_{\Gamma}^{-1}(\Delta)$ where $\Delta \subset \mathbb{R}^k$ is the diagonal. A delay function for $\Gamma$ is a collection of smooth functions depending on $r \in \overline{\mathcal{M}}_{\Gamma}^{\text{pre}}$

$$\tau_{\Gamma} = (\tau_e \in C^\infty(\overline{\mathcal{M}}_{\Gamma}^{\text{pre}}))_{e \in \text{Edge}(\Gamma_0)}.$$ 

Letting $\lambda_i := \lambda(r_{v_i})$ where $\lambda(r_{v_i})$ is the ratio of the radii circles for $r_{v_i}$, the delayed evaluation map is

$$\lambda_{\tau_{\Gamma}} : \prod_{v \in \text{Vert}(\Gamma)} \mathcal{M}_v \to \mathbb{R}^k$$

$$(r_v, u_v)_{v \in \text{Vert} \Gamma} \mapsto \left( \lambda_i \exp \left( \sum_{e \in p_i} \tau_e(r) \right) \right)_{i=1,\ldots,k}$$

which is the sum of delays along each path $p_i$ to a twice-quilted disk component. We call $\tau_{\Gamma}$ regular if the delayed evaluation map $\lambda_{\tau_{\Gamma}}$ is transverse to the diagonal $\Delta \subset \mathbb{R}^k$. Given a regular delay function $\tau_{\Gamma}$, we define

$$\overline{\mathcal{M}}_{\Gamma} := \lambda_{\tau_{\Gamma}}^{-1}(\Delta).$$

For a regular delay function $\tau_{\Gamma}$, the delayed fiber product has the structure of a smooth manifold, of local dimension

$$\dim \mathcal{M}_{\Gamma} = 1 - k + \sum_{v \in \text{Vert} \Gamma} \dim \mathcal{M}_v.$$
where \( k \) is the number of twice-quilted disk components. A collection \( \{ \tau^d \}_{d \geq 1} \) of delay functions is \emph{compatible} if the following properties hold. Let \( \Gamma \) be a combinatorial type of twice-quilted disk and \( v_0, \ldots, v_k \).

(a) (Subtree property) Suppose that the root component \( v_0 \) is not a twice-quilted disk. Let \( \Gamma_1, \ldots, \Gamma_{|v_0|-1} \) denote the subtrees of \( \Gamma \) attached to \( v_0 \) at its leaves; then \( \Gamma_1, \ldots, \Gamma_{|v_0|-1} \) are combinatorial types for nodal twice-quilted disks. Let \( r_i \) be the component of \( r \in \mathcal{M}_\Gamma^{\text{pre}} \) corresponding to \( \Gamma_i \). We require that \( \tau_\Gamma(r)|_{\Gamma_i} = \tau_{\Gamma_i}(r_i) \), i.e., for each edge \( e \) of \( \Gamma_i \), the delay function \( \tau_{\Gamma_i,e}(r) \) is equal to \( \tau_{r_i,e}(r_i) \). See Figure 23.

\[ \tau_\Gamma,e(r) = \tau_{\Gamma,e} + \sum_{e'} \tau_{\Gamma',e}(r') \]

(b) (Refinement property) Suppose that the combinatorial type \( \Gamma' \) is a refinement of \( \Gamma \), in other words there is a surjective morphism \( f : \Gamma' \to \Gamma \) of trees; let \( r \) be the image of \( r' \) under gluing. We require that \( \tau_{\Gamma'}|_U \) is determined by \( \tau_{\Gamma'} \) as follows: for each \( e \in \text{Edge}(\Gamma) \), and \( r \in U \), the delay function is given by the formula

\[ \tau_{\Gamma',e}(r) = \tau_{\Gamma,e}(r) + \sum_{e'} \tau_{\Gamma',e'(r')} \]

where the sum is over edges \( e' \) in \( \Gamma' \) that are collapsed under gluing and the \( e \) is the next-furthest-away edge from the root vertex. See Figure 24.
In the case that the collapsed edges connect twice quilted components with unquilted components, this means that the delay functions are equal for both types, as in the Figure 25.

(c) (Core property) If two combinatorial types say $\Gamma$ and $\Gamma'$, have the same core $\Gamma_0$, let $r, r'$ be disks of type $\Gamma$ resp. $\Gamma'$. Then $\tau_{\Gamma,e}(r) = \tau_{\Gamma',e}(r')$. (That is, the delay functions depend only on the region between the root vertex and the bicolored vertices.)

A collection of compatible delay functions is positive if, for every vertex $v \in \Gamma_0$ with $k$ leaves labeled in counterclockwise order by $e_1, \ldots, e_k$, their associated delay functions satisfy $\tau_{e_1} < \tau_{e_2} < \ldots < \tau_{e_k}$.

![Figure 25. The (Refinement property), second case](image)

For each combinatorial type $\Gamma$ we may find regular, positive delay functions. The (Subtree property) implies that all the delay functions in $\tau_{\Gamma} := \tau_{\Gamma}(L)$ except those for the finite edges adjacent to $v_0$, the root component, are already fixed. It remains to find regular delay functions for the finite edges adjacent to the root component of each combinatorial type, in a way that is also compatible with conditions (Refinement property). We ensure compatibility by proceeding inductively on the number of leaves of the root component, as follows. We suppose that we have constructed inductively regular delay functions for types $\Gamma$ corresponding to strata of $\mathcal{M}_{e,m,2}$ for $e < d$, as well as for types $\Gamma'$ appearing in the (Refinement property) for $\Gamma$, and now construct inductively on $n \geq 2$ a collection of regular delay functions $\tau_{\Gamma} := \tau_{\Gamma}(L)$ for all combinatorial types $\Gamma$ whose root component has $2 \leq k \leq n$ leaves. We may assume that $\Gamma$ has no components “beyond the twice-quilted components” since by the (Core property) the delay functions are independent of the additional components. Consider a combinatorial type whose root component has $n$ leaves. There is an open neighborhood $U$ of $\partial \mathcal{M}_{\Gamma}^{\text{pre}}$ in $\overline{\mathcal{M}_{\Gamma}^{\text{pre}}}$ in which the delay functions $\tau_{\Gamma}$ for the leaves adjacent to the root vertex are already determined by the compatibility condition (Refinement property) and the inductive hypothesis. We need to show that we may extend the $\tau_{\Gamma}$ over the interior of $\overline{\mathcal{M}_{\Gamma}^{\text{pre}}}$. To set up the relevant function spaces let $l \geq 0$ be an integer and let $f$ be a given $C^l$ function on $\overline{U}$. Let $C^l_f(\mathcal{M}_{\Gamma}^{\text{pre}})$ denote the Banach manifold of functions with $l$ bounded derivatives on $\mathcal{M}_{\Gamma}^{\text{pre}}$, equal to $f$ on $\overline{U}$. Let $\Gamma_i, i = 1, \ldots, n$ be the trees attached to the root vertex $v_0$. Consider
the evaluation map

\[ \text{ev} : \mathcal{M}_{\Gamma_1} \times \ldots \times \mathcal{M}_{\Gamma_n} \times \mathcal{M}_{v_0} \times \prod_{i=1}^n C_{l_i}(\mathcal{M}_{\Gamma_i}^{\text{pre}}) \to \mathbb{R}^{n-1} \]

\[ ((r_1, u_1), \ldots, (r_n, u_n), (r_0, u_0), \tau_1, \ldots, \tau_n) \mapsto (\lambda_{r_j}(r_j) \exp(\tau_j(r)) - \lambda_{r_{j+1}}(r_{j+1}) \exp(\tau_{j+1}(r)))_{j=1}^{n-1} \]

where \( r = (r_0, \ldots, r_n) \). Note that 0 is a regular value. The Sard-Smale theorem implies that for \( l \) sufficiently large the regular values of the projection \( \Pi : \text{ev}^{-1}(0) \to \prod_{i=1}^n C_{l_i}(\mathcal{M}_{\Gamma_i}^{\text{pre}}) \) form an open dense set. Taking the intersection over \( l \) sufficiently large gives that the set of smooth regular delay functions is comeager. Both the positivity condition and the regularity condition (b) are open conditions given an energy bound. It follows from the monotonicity condition that an energy bound for quilts of dimension zero or one exists, and therefore the set of smooth, positive, compatible, delay functions that are regular for a given energy bound is non-empty and open. Taking the intersection of these sets over all possible energy bounds we obtain a comeager set of smooth regular delay functions for each combinatorial type of twice-quilted \( d + 1 \)-marked disk, and hence a regular compatible collection \( \tau^d \).

Given a collection of regular compatible delay functions and a collection of perturbation data \( P_0, P_1, P_2, P_01, P_12, P_02 \) for unquilted and once-quilted disks, perturbations \( P_{012} = (P_{012}^l) \) for twice-quilted disks are constructed inductively using the gluing construction and the Sard-Smale theorem. For each stratum, we first use the construction of the previous paragraph to find regular delay functions for the boundary strata, then (after replacing the fiber products in the boundary strata by the delayed fiber products of boundary strata using the delay functions chosen) extend the perturbations by the gluing construction. A Sard-Smale argument shows that for perturbations in a comeager subset, the uncrowded moduli spaces of index at most one are regular of expected dimension. We say that a system \( P_{012} \) is convergent if it satisfies \( (52) \).

**Theorem 3.9.** \( (A_\infty \text{ homotopies via twice-quilted disks}) \) Given convergent, regular, coherent, stabilizing, ghost-marking-independent perturbation systems \( P_0^{01}, P_1^{12}, P_2^{02} \) and morphisms \( \phi_{ij} : \overline{CF}(L, P_i^j) \to \overline{CF}(L, P_i^j), \quad 0 \leq i < j \leq 2 \) and \( P_{012}^{012} \) as above, counting treed holomorphic twice-quilted disks defines a convergent \( A_\infty \) homotopy between \( \phi_{02} \) and \( \phi_{01} \circ \phi_{12} \).

**Proof.** Consider the map \( \lambda : \overline{M}_{n,m,2} \to [1, \infty] \) giving the ratio of radii of the inner circles of the twice-quilted disks, after subtracting the delay functions. For generic values of \( \lambda \) the moduli space \( \overline{M}_{n,m,2}^\lambda(L, D) = \psi^{-1}(\lambda) \) is smooth of expected dimension. We let \( \phi_{02}^\lambda \) denote the \( A_\infty \) morphism obtained by counting twice-quilted disks with ratio of radii \( \lambda \in [1, \infty) \). After fixing an energy bound \( E \) and a number of
leaves $n$ we may divide the interval $[0,1]$ into subintervals $[\lambda_i, \lambda_{i+1}], i = 0, \ldots, k - 1$ so that each is sufficiently small so that there is a single singular value $\lambda \in [\lambda_i, \lambda_{i+1}]$, contained in the interior of the interval, for which there exist twice-quilted disks of expected dimension zero of energy at most $E$ and $n$ leaves and no such disks with fewer number of leaves. Define $T^{\lambda_1, \lambda_{i+1}; \leq E}_{02}$ by counting such twice-quilted disks,

\[(70) \quad (T^{\lambda; \leq E}_{02})^n : \widehat{CF}(L; P^0)^{\otimes n} \to \widehat{CF}(L; P^1)\]

\[(x_1, \ldots, x_n) \mapsto \sum_{x_0, [u] \in \mathcal{M}^n_{1, \leq E}(L, D; x_0, \ldots, x_n)} (-1)^{\epsilon([u])}(\sigma([u])!)^{-1}q^{E([u])} \text{Hol}_L(u)x_0\]

where the sum is over combinatorial types $\Gamma$ of twice-quilted disks. The difference

\[(\phi^{\lambda}_{02} - \phi^{\lambda_{i+1}}_{02})(x_1, \ldots, x_n)\]

is a count of configurations either involving an unquilted disk breaking off, or a collection of twice-quilted treed disks with $i_1, \ldots, i_r$ leaves breaking off from an unquilted treed disk, see [24] Section 7. For degree reasons, because of the fiber product with the diagonal exactly one of these twice-quilted disks lies in the moduli space of expected dimension zero, while the rest have index one. We suppose that the twice-quilted configuration in the expected-dimension-zero is the $i + 1$-st twice-quilted treed disk attached to the unquilted treed disk. Using positivity of the delay functions one obtains that the moduli space of twice quilted disks $\lambda_i$ and $\lambda_{i+1}$ and expected dimension zero are cobordant:

\[
\begin{align*}
\mathcal{M}^{\lambda_i + \tau_j}_{ij, m_j, 2}(L, D) & \sim \mathcal{M}^{\lambda_i + \tau_j}_{ij, m_j, 2}(L, D) \quad j > i + 2 \\
\mathcal{M}^{\lambda_i + \tau_j}_{ij, m_j, 2}(L, D) & \sim \mathcal{M}^{\lambda_i + \tau_j}_{ij, m_j, 2}(L, D) \quad j \leq i.
\end{align*}
\]

It follows that

\[\phi^{\lambda_i}_{02} - \phi^{\lambda_{i+1}}_{02} = \mu_{02}^{\lambda_i, \lambda_{i+1}}(T^{\lambda_1, \lambda_{i+1}; \leq E}_{02}).\]

The facets of $\mathcal{M}_{n, m, 2}$ with ratio $\lambda = 1$ or $\lambda = \infty$ correspond to either to terms in the definition of composition of $A_\infty$ maps $\phi_{12} \circ \phi_{01} : \widehat{CF}(L, P^0) \to \widehat{CF}(L, P^2)$, to the components contributing to $\phi_{02} : \widehat{CF}(L, P^0) \to \widehat{CF}(L, P^2)$ or to terms corresponding to the bubbling off of some markings on the boundary which define a homotopy operator for the difference $\phi_{12} \circ \phi_{01} - \phi_{02}$. In case $P^0 = P^2$ composition using [58] produces a homotopy

\[T^{\leq E}_{02} := \mu^2(T^{\lambda_{k-1}, \lambda_k; \leq E}_{02}, \mu^2(\ldots \circ \mu^2(T^{\lambda_2, \lambda_1}_{02}, T^{\lambda_1, \lambda_0; \leq E}_{02}) \ldots))\]

between $\phi_{12} \circ \phi_{01}$ and the identity modulo terms involving powers $q^E$. Taking the limit $E \to \infty$ defines a homotopy

\[T_{02} := \lim_{E \to \infty} T^{\leq E}_{02}\]

between $\phi_{02}$ and $\phi_{12} \circ \phi_{01}$. Convergence is similar to Proposition 2.39. In particular $T^{0}_{02}(1) \in \Lambda_{>0}\widehat{CF}(L, P^0)$ since any contributing configuration must contain a non-trivial disk. \qed
Corollary 3.10. For any two regular, stabilized, coherent, convergent collections of perturbation data \( P^0, P^1 \) the Fukaya algebras \( \widehat{CF}(L, P^0) \) and \( \widehat{CF}(L, P^1) \) are homotopy equivalent via convergent maps and homotopies in the sense of Lemma 3.3.

Proof. Using Theorem 3.9 and taking \( D^0, P^1 \) to be constant and \( P^0, P^1 \) to be pulled back by the map forgetting the quilting, one obtains a homotopy between the composition of morphisms

\[
\phi_{01} : \widehat{CF}(L, P^0) \to \widehat{CF}(L, P^1), \quad \phi_{10} : \widehat{CF}(L, P^1) \to \widehat{CF}(L, P^0)
\]

and the identity morphism as in Proposition 3.7. 

Define the Floer cohomology as the stalks of the cohomology complex of (49):

\[
HF(L) := \bigcup_{b \in \tilde{MC}} HF(L)_b, \quad HF(L)_b := H(\widehat{CF}(L), \partial_b).
\]

We say that a Lagrangian brane \( L \) is Floer non-trivial if some fiber \( HF(L)_b \) is non-zero; that is, the space of solutions to the weak Maurer-Cartan equation is non-empty and for at least one solution the cohomology is non-zero. By the Corollary 3.10 and Lemma 3.3, Floer non-triviality of \( L \) is independent of all choices.

3.6. Stabilization. In this section we complete the proof of homotopy invariance of the Fukaya algebras constructed above in the case that the algebras are defined using divisors are not of the same degree or built from homotopic sections.

For this we need to recall some results about existence of a Donaldson hypersurface transverse to a given one. Recall from [26, Lemma 8.3] that for a constant \( \epsilon > 0 \), two divisors \( D, D' \) intersect \( \epsilon \)-transversally if at each intersection point \( x \in D \cap D' \) their tangent spaces \( T_x D, T_x D' \) intersect with angle at least \( \epsilon \). A result of Cieliebak-Mohnke [26, Theorem 8.1] states that there exists an \( \epsilon > 0 \) such that given a divisor \( D \), there exists a divisor \( D' \) of sufficiently high degree \( \epsilon \)-transverse to \( D \). Moreover, for any \( \theta > 0 \), \( \omega \)-tamed almost complex structures \( \theta \)-close to \( J \) making \( D, D' \) almost complex exist (provided that the degrees are sufficiently large).

We apply the result of the previous paragraph as follows. Suppose that \( D^0, D^1 \) are stabilizing divisors for \( L \), possibly of different degrees. By the previous paragraph, there exists a pair \( D^0', D^1' \) of higher degree stabilizing divisors built from homotopic unitary sections over \( L \) that are \( \epsilon \)-transverse to \( D^0 \) and \( D^1 \), respectively. Let \( P^0_0, P^1_1 \) be perturbation systems for \( D^0', D^1' \). We have already shown that

\[
(71) \quad \widehat{CF}(L, P^0) \cong \widehat{CF}(L, P^1)
\]

are homotopy equivalent. It remains to show:

Theorem 3.11. For any convergent, coherent, regular, stabilized perturbation systems \( P_k, P'_k, k = 0, 1 \) as above, the Fukaya algebras \( \widehat{CF}(L, P_k) \) and \( \widehat{CF}(L, P'_k) \) are homotopy equivalent.

Sketch of proof. Denote by \( J^\ast(X, D^k \cup D^{k'}, J, \theta, E) \) the subset of \( J^\ast(X, D^k, J, \theta, E) \) of almost complex structures close to \( J \) preserving \( TD^{k'} \) as well. By [26, Corollary 8.20], there exists a path-connected, open, dense set in \( J^\ast(X, D^k \cup D^{k'}, J, \theta, E) \) with the property that for any \( J \in J^\ast(X, D^k \cup D^{k'}, J, \theta, E) \), neither \( D^k \) nor
contain any non-constant holomorphic spheres of energy at most $E$, and each holomorphic sphere meets both $D^k$ and $D^{k'}$ in at least three points. Fix such an almost complex structure $J_{D^k,D^{k'}}$, and an associated perturbation system $\mathcal{P}_k'$ using the divisor $D^k$ such that the almost complex structures $J_{D^k,D^{k'}}$ are equal to $J_{D^{k'},D^k}$ on $D^k \cup D^{k'}$. The argument in the previous section (keeping the divisor constant but changing the almost complex structures) shows that the associated Fukaya algebras $\hat{CF}(L,\mathcal{P}_k) \cong \hat{CF}(L,\mathcal{P}_k')$ are homotopy equivalent. Similarly, choose a perturbation system $\mathcal{P}'_k$ using the divisor $D^{k'}$. We claim that the Fukaya algebras $\hat{CF}(L,\mathcal{P}'_k) \cong \hat{CF}(L,\mathcal{P}'_k')$ are homotopy equivalent. To see this, we define adapted stable maps adapted to the pair $(D^k,D^{k'})$: a map is adapted if each interior marking maps to either $D^k$ or $D^{k'}$, and the first $n_k$ markings map to $D^k$ and the last $n'_k$ markings map to $D^{k'}$. A perturbation datum morphism $P = (P_\Gamma)$ is coherent if it is compatible if with the morphisms of moduli spaces as before: (Cutting edges) axiom, (Collapsing edges or Making an edge/weight finite/non-zero) axiom, and satisfies the (Infinite weights) axiom and (Product) axiom, where now on the unquilted components above resp. below the quilted components the perturbation system is required to depend only on the first $n_k$ resp. last $n'_k$ points mapping to $D^k$ resp. $D^{k'}$ (that is, pulled back under the forgetful map forgetting the first $n_k$ resp. last $n'_k$ markings). Then the same arguments as before produce the required homotopy equivalence; taking the perturbations to be $C^2$-small implies that the analog of (52) holds. Putting everything together we have homotopy equivalences
\[ \hat{CF}(L,\mathcal{P}_k) \cong \hat{CF}(L,\mathcal{P}_k') \cong \hat{CF}(L,\mathcal{P}'_k). \]
Applying (71) completes the proof. \(\square\)

**Corollary 3.12.** For any stabilizing divisors $D^0,D^1$ and any convergent, coherent, regular, stabilizing, ghost-marking-independent perturbation systems $\mathcal{P}_0,\mathcal{P}_1$, the Fukaya algebras $\hat{CF}(L,\mathcal{P}_0)$ and $\hat{CF}(L,\mathcal{P}_1)$ are convergent-homotopy-equivalent.

**Proof.** Since homotopy equivalence of $A_\infty$ algebras is an equivalence relation by Definition 3.1 (i), combining Theorems 3.6 and 3.11 gives the result. \(\square\)

### 3.7. Canonical variations.

In general the Fukaya algebra is not invariant under variations of the symplectic form and Lagrangian. However, there is one special case in which the non-triviality of the Floer cohomology is invariant under deformation, which we describe in this section and called *canonical variation*. What this means is that the symplectic class is varied by a multiple of the canonical class, and the variation of the Lagrangian is given by the connection one-form on the canonical bundle. The details are below.

**Definition 3.13.** (a) (Families of Lagrangians) a family of embedded Lagrangian submanifolds in $X$ we mean a submanifold $\tilde{L}$ of $X \times [0,1]$ such that each intersection $L_t = \tilde{L} \cap (X \times \{t\})$ is a Lagrangian submanifold of $X$. The infinitesimal variation of $L_t$ is
\[ \frac{d}{dt}L_t : L_t \to (TX|L_t)/TL_t. \]
Via the identification of $TX$ with its dual using the symplectic form we have

$$\iota \left( \frac{d}{dt} \omega \right)|_{L_t} = \alpha_{L,t}, \quad \alpha_{L,t} \in \Omega^1(L_t).$$

The variation $L_t$ of $L_0$ being Lagrangian is equivalent to the one-form $\alpha_{L,t}$ being closed, $d\alpha_{L,t} = 0$.

(b) (Canonical one-forms) For $t$ fixed choose an almost complex structure $J_t$ on $X_t$ and let $K_t = \Lambda^\text{top}(TX)$, independent up to isomorphism of the choice of $J_t$. Using the

$$\Lambda^\text{top}TL \to \Lambda^\text{top}TX$$

any volume form on $L$ induces a trivialization $\tau_L: (K_t|_{L_t}) \to (L_t \times \mathbb{C})$ of the canonical bundle $K_t$. Choose a connection $\alpha_t$ on the canonical bundle $K_t$ and so that $\text{curv}(\alpha_t) = -\frac{d}{dt}\omega_t$. In this trivialization the connection on $K_t$ is given by a one-form $\tau_{L,t} \alpha_t \in \Omega^1(L_t)$; we call $\tau_{L,t} \alpha_t$ the canonical one-form for $L$ determined by the connection $\alpha$ and volume form on $L$.

(c) (Canonical variations) We say that $L_t$ is a canonical variation if the following holds, similar to the definition of Maslov flow in Lotay-Pacini [73]:

$$\exists \beta_t \in \Omega^0(L_t), \quad \alpha_{L,t} - \tau_{L,t} \alpha_t = d\beta_t$$

and if furthermore the orientations, gradings, and relative spin structures are constant along family.

**Example 3.14.** The following are examples of canonical variations:

(a) (Vanishing curvature) Suppose that the connection $\alpha_t$ on $K_t$ is flat in a neighborhood of $L_t$ and the trivialization of $K_t|_{L_t}$ induced by the orientation is homotopic to a flat section. Then $L_t$ constant in $t$ is a canonical variation.

(b) (Euclidean case) Suppose that $X = \mathbb{C}$, $L$ is a circle of radius $r_0$. Choose a connection on $K$ whose curvature is the standard area form on $\mathbb{C}$. A family of circles whose radius changes by $r(t) = r_0 + 2t$ is a canonical variation.

(c) (Kähler-Ricci flow case) Suppose that $\omega_t$ is given by Kähler-Ricci flow and $L_t$ is given by mean curvature flow, so that the one-form giving the variation of $L_t$ is the mean curvature vector. Then $L_t$ is a canonical variation, see [73, Theorem 5.4].

(d) (Toric case) Suppose that $X$ is a toric manifold, $\omega_t$ a family of symplectic structures with $\frac{d}{dt}[\omega_t] = -c_1(X)$, and $\Phi_t: X \to t^\vee$ a family of moment maps such that $\frac{d}{dt}\Phi_t$ is a canonical moment map for the action of $T$ on the canonical bundle, that is, corresponds to the canonical lift of the action. Then the “constant family” $L_t = \Phi_t^{-1}(\lambda)$ for any $\lambda$ in the interior of the polytope is a canonical family, for as long as it exists.

An important property of canonical families of Lagrangians is the behavior of areas of disks. We denote by $B = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ the unit disk and $\partial B$ its boundary. Given a Lagrangian submanifold $L$ of a symplectic manifold $X$, and a
smooth map $u : (B, \partial B) \to (X, L)$ we denote by
\[
A(u) = \int_B u^* \omega, \quad I(u) = I(u^* TX, (\partial u)^T L)
\]
its area and Maslov index respectively. Both are invariant under isotopy.

**Lemma 3.15.** Let $L_t$ be a canonical family as above, $u_t : (B, \partial B) \to (X_t, L_t)$ be a smooth family of disks in $X_t$ with boundary in $L_t$. Then
\[
\frac{d}{dt} A(u_t) = I(u_t).
\]

**Proof.** Consider the vector field
\[
v_t = \frac{d}{dt} u_t \in \Omega^0(B, u_t^* TX).
\]
Given $u_0$, the family $u_t$ is unique up to homotopy and may be given in the local model by
\[
\frac{d}{dt} u_t = \alpha_{L,t}(u_t).
\]
It follows that the vector field satisfies
\[
\iota_{v_t} u_t^* \omega_t = u_t^* \alpha_{L,t}.
\]
Now let $\alpha_t \in \Omega^1(K_1)$ be a connection one-form on the unit circle bundle $K_1 \subset K$. Since $B$ is contractible, each $u_t$ has a lift $\tilde{u}_t : B \to K_1$ unique up to homotopy. By definition the index of $u_t$ is the winding number of $K_1$ in the trivialization defined by $L_t$, so
\[
I(u_t) = \int_{\partial B} \tilde{u}_t^* (\pi^* \alpha_{L,t} - \alpha_t).
\]
Putting everything together we have
\[
\frac{d}{dt} A(u_t) = \frac{d}{dt} \int_B u_t^* \omega_t = \int_B \iota_{v_t} u_t^* \omega_t + u_t^* \frac{d}{dt} \omega_t
\]
\[
= \int_B \iota_{v_t} u_t^* \omega_t + \tilde{u}_t^* d\alpha_t = \int_{\partial B} \iota_{v_t} u_t^* \omega_t - \tilde{u}_t^* \alpha_t
\]
\[
= \int_{\partial B} \tilde{u}_t^* (\pi^* \alpha_{L,t} - \alpha_t) = I(u_t).
\]

**Proposition 3.16.** Suppose that $X$ is a symplectic manifold with a family $\omega_t$ of symplectic forms with $\frac{d}{dt} [\omega_t] = -c_1(X)$, and $L_t \subset X$ is a family of Lagrangian tori for $\omega_t$ with the following property: There exist a collection of families
\[
u_{i,t} : (B, \partial B) \to (X, L_t), \quad i = 1, \ldots, n
\]
of disks such that the boundaries $\partial \nu_{i,t} \in H_1(L_t)$ span $H_1(L_t)$ and furthermore $\frac{d}{dt} A(\nu_{i,t}) = I(\nu_{i,t})$ for all $i$. Then $L_t$ is a canonical family of Lagrangians.
Proof. Since $L_t$ is Lagrangian with respect to $\omega_t$ the variation is given by a one-form $\alpha'_{L,t}$ with

$$\mathrm{d}\alpha'_{L,t} = \frac{d}{dt} \omega_t|_{L_t} = d\alpha_{L,t}.$$  

The difference $\alpha'_{L,t} - \alpha_{L,t}$ is therefore a closed one-form. Therefore to check that a variation corresponding to $\alpha'_{L,t}$ is canonical it suffices to check that

$$[\alpha'_{L,t} - \alpha_{L,t}] = 0 \in H^1(L_t)$$

and for this it suffices to check on generators. As in the proof of Lemma 3.15, the pairing of $[\alpha'_{L,t} - \alpha_{L,t}] = 0$ with $[\partial u_{i,t}]$ is the difference

$$\frac{d}{dt} A(u_{i,t}) - I(u_{i,t}) = 0, \quad i, \ldots, n$$

hence (72). \hfill \Box

We consider Floer chains with integer grading as follows. For critical points $x \in \mathcal{I}(L)$, the grading is the index of the critical point, that is, the dimension of the unstable manifold:

$$CF^i(L) = \bigoplus_{\dim(W^u_x) = i} \Lambda x.$$  

**Definition 3.17.** The Euler automorphism of $CF(L,D)$ for time $t$ is the automorphism

$$\varphi_t : CF^i(L) \to CF^i(L), \quad b \mapsto q^{t(i-2)}b, \quad b \in CF^i(L)$$

extended linearly to $CF^i(L) = \bigoplus CF^i(L)$.

**Lemma 3.18.** Let $\mu^n_x : CF(L)^n \to CF(L)$ be the higher composition maps in the Fukaya algebra over the Novikov field on $\pi_2(X,L)$ with respect to the completion defined by some classes $[\omega_1], [\omega_2]$. Then the higher composition maps are related by the Euler automorphism:

$$\mu^n_x(a_1, \ldots, a_n) = q^t \varphi_t^{-1} \mu^x \varphi_t(a_1, \ldots, a_n)$$

**Proof.** For each $s \in [0,1]$ choose a divisor $D_s$ for $\omega_s$ and perturbation data $P_{\Gamma,s}$ for $X_s, Y_s, \omega_s, L_s$. Then $P_{\Gamma,s}$ is also regular for $t$ in a small neighborhood, say $(s - 2\epsilon, s + 2\epsilon)$ of $s$. First suppose that the divisor $D_{s\pm\epsilon}$ are chosen to equal $D_s$. Since the moduli spaces for $s \pm \epsilon$ are the same, it suffices to compare the invariants in the formula for the composition maps. By construction the orientations $c_\pm([u])$ and holonomies $\rho_\pm([u])$ are identical. The areas $A_\pm([u])$ differ as in Lemma 3.15. As in the definition of convergent $A_\infty$ algebra, we assume that the Fukaya algebra $CF(L)$ admits a $\mathbb{Z}$-grading which is however not preserved by the $A_\infty$ composition maps. For the extra critical points $x^\epsilon_M, x^\epsilon_M$, the gradings are 0, $-1$ respectively. We may suppose that $a_1, \ldots, a_n$ are homogeneous of degrees $i_1, \ldots, i_n$. The term $\mu^n_x(a_1, \ldots, a_n)_{i_0}$ of degree $i_0$ is a sum of contributions from disks of Maslov index $2 - i_0 + \sum_{j=1}^n (1 - i_j)$. It follows that

$$\mu^n_x(a_1, \ldots, a_n)_{i_0} = q^{t(2 - i_0 + \sum_{j=1}^n (1 - i_j))} \mu^n_x(a_1, \ldots, a_n)_{i_0} = q^t \varphi_t^{-1} \mu^n_x(\varphi_t(a_1), \ldots, \varphi_t(a_n))_{i_0}$$
as claimed.

It remains to show that the homotopy type of the Fukaya algebra $CF(L, D_{s+\epsilon})$ is equal to that of a Fukaya algebra $CF(L, D'_{s+\epsilon})$ where now $D'_{s+\epsilon}$ is chosen dual to a large multiple of the symplectic class $[\omega_{s+\epsilon}]$. By Krestiachine [62], there exists a stabilizing divisor $D'_{s+\epsilon}$ for $(X, \omega_{s+\epsilon})$ representing a arbitrarily large multiple of $[\omega_{s+\epsilon}]$ and transverse to the stabilizing divisor $D_s$. The proof of homotopy equivalence is the same as for two divisors representing a large multiple of the same symplectic class.

\[ \square \]

**Corollary 3.19.** Suppose that $L_t \subset X_t, t \in [t_-, t_+]$ is a canonical family of compact Lagrangian branes. Then $\varphi_t, t = t_+ - t_-$ induces a bijection between the space of solutions to the weak Maurer-Cartan equation $MC(L_{t+})$ and $MC(L_{t-})$ and induces an isomorphism of Floer cohomologies.

**Proof.** Since the weak Maurer-Cartan equation must be satisfied separately in each degree, we have $\mu_b^{0,t_+} (1) \in \Lambda e$ iff $q^t \varphi_t^{-1} \mu_b^{0,t_-} \in \Lambda e$ iff $\mu_b^{0,t_+} (1) \in \Lambda e$. Hence the solutions to the weak Maurer-Cartan equation are in bijection. The argument is similar for the Floer cohomologies. \[ \square \]

## 4. Fukaya Bimodules

Floer cohomology is, in a certain sense we describe next, invariant under Hamiltonian perturbation. Namely there exists an $A_\infty$ bimodule for pairs of Lagrangians equipped with a Hamiltonian perturbation so that the perturbed Lagrangians intersect cleanly, that is isomorphic to the bimodule for the Fukaya algebra if the Lagrangians are identical and whose homotopy type is independent of the choice of Hamiltonian. In particular, if a Lagrangian brane is displaceable by a Hamiltonian diffeomorphism then this implies that the Floer cohomology for any element of the Maurer-Cartan moduli space vanishes.

### 4.1. $A_\infty$ Bimodules

We begin with defining conventions on bimodules. Let $A_0, A_1$ be strictly unital $A_\infty$ algebras. Let $g$ be an even integer. An $\mathbb{Z}_g$-graded $A_\infty$ **bimodule** is a $\mathbb{Z} - g$-graded vector space $M$ equipped with operations

$$\mu^{g|e} : A_0^{\otimes d} \otimes M \otimes A_1^\otimes e \to M[1 - d - e]$$

satisfying the relations among homogeneous elements $a_{i,k}, m$

$$\sum_{i,k} (-1)^{g} \mu_d - i + 1|e (a_{0,1}, \ldots, a_{0,i} (a_{0,k}, \ldots, a_{0,k+i-1}), a_{0,k+i}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,e})$$

$$+ \sum_{j,k} (-1)^{g} \mu_d_{i|e-1} (a_{0,1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, \mu_1 (a_{1,k}, \ldots, a_{1,k+j-1}), \ldots, a_{1,e})$$

$$+ \sum_{i,j} (-1)^{g} \mu_d_{i|e-1} (a_{0,1}, \ldots, a_{0,d-i} \mu_{i,j} (a_{0,d-i+1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,j}), \ldots, a_{1,e}) = 0$$

where we follow Seidel’s convention [18] of denoting by $(-1)^g$ the sum of the reduced degrees to the left of the inner expression, except that $m$ has ordinary (unreduced)
degree. The bimodule is strictly unital if $\mu_{1,0}^e$ and $\mu_{0,1}$ are the identity, and all other operations involving $L_0$ and $L_1$ vanish. A morphism $\phi$ of $A_\infty$-bimodules $M_0$ to $M_1$ of degree $|\phi|$ is a collection of maps

$$\phi^{\ell e}_d : A_0^\otimes d \otimes M_0 \otimes A_1^\otimes e \to M_1[|\phi| - d - e]$$

satisfying a splitting axiom

$$\sum_{i,j} (-1)^{|\phi|} \mu_{i,j}^{\ell e}(a_{0,1}, \ldots, a_{0,d-i}, \phi^{i,j}(a_{0,d-i+1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,j}), a_{1,j+1}, \ldots, a_{1,e})$$

$$+ \sum_{i,j} (-1)^{|\phi|+1} \phi_d^{\ell e}(a_{0,1}, \ldots, a_{0,d-i}, \mu_{0,ij}(a_{0,d-i+1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,j}), a_{1,j+1}, \ldots, a_{1,e})$$

$$+ \sum_{i,j} (-1)^{|\phi|+1} \phi_d^{\ell e}(a_{0,1}, \ldots, a_{0,j-1}, \mu_{0,j}(a_{0,j}, \ldots, a_{0,j+i-1}), \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,e})$$

$$+ \sum_{i,j} (-1)^{|\phi|+1} \phi_d^{\ell e}(a_{0,1}, \ldots, a_{0,e}, m, a_{1,1}, \ldots, \mu_{1,j}(a_{1,j}, \ldots, a_{j+i-1}), \ldots, a_{1,e}) = 0.$$ 

A morphism is strictly unital if all operations involving the identities $e_{L_0}$ and $e_{L_1}$ vanish. Composition of morphisms $\phi : M_0 \to M_1, \psi : M_1 \to M_2$ is defined by

$$\phi \circ \psi^{\ell e}_d(a_{0,1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,e})$$

$$= \sum_{i,j} (-1)^{|\psi|} \phi^{i,j}(a_{0,1}, \ldots, a_{0,i}, \psi_d^{i,j}(a_{0,i+1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,e}), a_{1,e-j+1}, \ldots, a_{1,e}).$$

A homotopy of morphisms $\psi_0, \psi_1 : M_0 \to M_1$ of degree zero is a collection of maps $(\phi^{\ell e})_{d,e \geq 0}$ such that the difference $\psi_1 - \psi_0$ is given by the expression on the left hand side of (73). A homotopy is strictly unital if all operations involving $e_{L_0}$ and $e_{L_1}$ vanish. Any $A_\infty$ algebra $A$ is an $A_\infty$ bimodule over itself with operations

$$\mu^{\ell e}_d(a_{0,1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,e})$$

$$= (-1)^{1+\diamond} \mu_{d+e+1}(a_{0,1}, \ldots, a_{0,d}, m, a_{1,1}, \ldots, a_{1,e}), \quad \diamond = \sum_{j=1}^{e} (|a_{1,j}| + 1),$$

see Seidel [100, 2.9]. Suppose that the $A_\infty$ algebras $A_0, A_1$ are convergent so that there exist Maurer-Cartan moduli spaces $MC(A_k), k \in \{0,1\}$ and for each $b \in MC(A_k)$ a cohomology group $H(\mu_b^{k,1})$. We suppose that the vector space underlying $M$ is a finite rank and free $\Lambda$-module and furthermore admits a $\mathbb{Z}$-grading; we write

$$M = \bigoplus_{d \in [d_-, d_+]} M^d, \quad M^{\leq 0} = \bigoplus_{d \geq 0} M^d, \quad M^{< 0} = \bigoplus_{d < 0} M^d$$

for some integers $d_- \leq d_+$. Define $M^{\geq 0}$ and $M^{> 0}$ similarly. Consider the decomposition

$$\text{Hom}(A_0^\otimes n_0 \otimes M \otimes A_1^\otimes n_1, M) = \bigoplus_m \text{Hom}(A_0^\otimes n_0 \otimes M \otimes A_1^\otimes n_1, M)_m$$
where elements of $\text{Hom}(A_0^{\otimes n} \otimes M \otimes A_1^{\otimes n}, M)_m$ have degree $m$. Write the maps $\mu^{d|e}$ in terms of their components,

$$\mu^{d|e} = \sum_m \mu^{d|e,m}, \quad \mu^{d|e,m} \in \text{Hom}(A_0^{\otimes n_0} \otimes M \otimes A_1^{\otimes n_1}, M)_{1-d-e+m}.$$  

We say that an $A_\infty$ bimodule $M$ as above is convergent iff there exists a sequence $E_m \to \infty$ such that

$$\mu^{d|e,m}(M) \subset q^{E_m}M, \quad \forall d, e \geq 0.$$  

Define for $b_0 \in A_0, b_1 \in A_1$ the maps

$$\mu^{d|e,b_0|b_1} : A_0^{\otimes d} \otimes A_1^{\otimes e} \to M,$$


$$(a_0,1,\ldots,a_0,d,m,a_1,1,\ldots,a_1,e)$$

$$\to \sum_{i_0,\ldots,i_d,j_0,\ldots,j_e} \mu^{d+d|e+e'}(b_0_{i_0}, b_0, a_0,1, b_0,\ldots, b_0, a_1, \ldots, b_0, m, a_1,1, \ldots, b_1, a_1,1, \ldots, b_1,1, b_1).$$

**Proposition 4.1.** For any $b_0 \in A_0, b_1 \in A_1$, the operations $\mu^{d|e,b_0|b_1}$ define the structure of an $A_\infty$ bimodule on $M$. For $b_0 \in \tilde{MC}(A_0)$ and $b_1 \in \tilde{MC}(A_0)$ we have

$$(\mu^{0|0,0|b_0|b_1})^2 = 0$$

and so the cohomology $H(\mu^{0|0,0|b_0|b_1})$ is well-defined.

**Proof.** The element $\mu^{0|0,0|b_0|b_1}$ satisfies

$$(-1)^{|a|} \mu^{1|0}((\mu_0^{0|0|b_0|b_1}(1), a) - \mu^{0|1}(a, \mu_0^{0|0|b_0|b_1}(1)).$$

Since the elements $\mu_0^{b_k}(1)$ are multiples of strict identities, the claim follows. \qed

The notions of convergent morphism of $A_\infty$ bimodules and convergent homotopy of morphisms of $A_\infty$ bimodules are similar.

### 4.2. Treed strips

We introduce notation for moduli spaces of marked strips as follows. A marked strip is a marked disk with two boundary markings. Let $C$ be a connected $(2,n)$-marked nodal disk with markings $\tilde{z}$. We write $\tilde{z} = (z_-, z_+, z_1, \ldots, z_n)$ where $z_\pm$ are the boundary markings and $z_1, \ldots, z_n$ are the interior markings. We call $z_-$ (resp. $z_+$) the incoming (resp. outgoing) marking. Let $C_1, \ldots, C_m$ denote the ordered strip components of $C$ connecting $z_-$ to $z_+$; the remaining components are either disk components, if they have boundary, or sphere components, otherwise. Let $\tilde{w}_i := C_i \cap C_{i+1}$ denote the intermediate node connecting $C_i$ to $C_{i+1}$ for $i = 1, \ldots, m-1$. Let $\tilde{w}_0 = z_-$ and $\tilde{w}_m = z_+$ denote the incoming and outgoing markings. Let $C^\times := C - \{\tilde{w}_0, \ldots, \tilde{w}_m\}$ denote the curve obtained by removing the nodes connecting strip components and the incoming and outgoing markings. Each strip component may be equipped with coordinates

$$\phi_i : C_i^\times := C_i - \{\tilde{w}_i, \tilde{w}_{i-1}\} \to \mathbb{R} \times [0,1], \quad i = 1, \ldots, m.$$
satisfying the conditions that if $j$ is the standard complex structure on $\mathbb{R} \times [0,1]$ and $j_i$ is the complex structure on $C_i$ then
\[
\phi^*_i j = j_i, \quad \lim_{z \to \tilde{w}_i} \pi_1 \circ \phi_i(z) = -\infty, \quad \lim_{z \to \tilde{w}_{i+1}} \pi_1 \circ \phi_i(z) = \infty
\]
where $\pi_i$ denotes the projection on the $i^{th}$ factor. We denote by
\begin{equation}
(78) \quad f : C^\times \to [0,1], \quad z \mapsto \pi_2 \circ \phi_i(z)
\end{equation}
the continuous map induced by the time coordinate on the strip components. The time coordinate is extended to nodal marked strips by requiring constancy on every connected component of $C^\times \setminus \bigcup_i C_i^\times$. The boundary of any marked strip $C$ is partitioned as follows. For $b \in \{0,1\}$ denote
\begin{equation}
(79) \quad (\partial C)_b := f^{-1}(b) \cap \partial C
\end{equation}
so that $\partial C^\times = (\partial C)_0 \cup (\partial C)_1$. That is, $(\partial C)_b$ is the part of the boundary from $z_-$ to $z_+$, for $b = 0$, and from $z_+$ to $z_-$ for $b = 1$. An example of a stable strip is shown in Figure 26.

A treed strip of type $\Gamma$ is a marked strip $(C, z)$ with a metric
\[
\ell : \text{Edge}_{\partial C}(\Gamma) \to [0, \infty]
\]
assigning lengths to the finite edges of the subgraph of $\Gamma$ corresponding to disk components. The combinatorial type is defined as before, except that the subset of edges with infinite, zero or $[0, \infty]$ lengths is recorded as part of the data. Combinatorial types of strips naturally define combinatorial types of treed strips by adding zero metrics on their edges corresponding to boundary nodes. An isomorphism of treed marked strips is an isomorphism of marked strips having the same metric. A stable treed strip is one that has a stable underlying strip.

We introduce the following notation for moduli spaces. Let $M_n$ denote the moduli space of isomorphism classes of connected stable treed strips with $n$ interior markings in addition to the incoming and outgoing markings. For $\Gamma$ a connected type we denote by $M_\Gamma \subset M_n$ the moduli space of stable strips of combinatorial type $\Gamma$ and $\overline{M}_\Gamma$ its closure. Each $\overline{M}_\Gamma$ is naturally a manifold with corners, with local charts obtained by a standard gluing construction. Generally $\overline{M}_n$ is the union of several

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure26}
\caption{A stable marked strip}
\end{figure}
top-dimensional strata. For \( \Gamma \) disconnected, \( \mathcal{M}_\Gamma \) is the product \( \prod_i \mathcal{M}_{\Gamma_i} \) of moduli spaces for the connected components \( \Gamma_i \) of \( \Gamma \).

The universal strip is equipped with the following maps. On the universal strip, the time coordinates \( f \) on the strip components extend to a map
\[
(80) \quad f : \overline{U}_\Gamma \to [0, 1]
\]
On the subset with time coordinate equal to zero or one, we have additional maps measuring the distance to the strip components given by summing the lengths of the connecting edges:
\[
\ell_b : f^{-1}(b) \to [0, \infty], \quad z \mapsto \sum_{e \in \text{Edge}(z)} \ell(e), \quad b \in \{0, 1\}
\]
where \( \text{Edge}(z) \) is the set of edges corresponding to nodes between \( z \) and the strip components. Thus any point on the universal strip \( z \in \overline{U}_\Gamma \) which lies on a disk component has \( f(z) \in \{0, 1\} \). The universal treed strip can be written as the union of one-dimensional and two-dimensional parts
\[
\overline{U}_\Gamma = S_\Gamma \cup T_\Gamma
\]
so that \( S_\Gamma \cap T_\Gamma \) is the set of points on the boundary of the disks meeting the edges of the tree. We denote by
\[
T_b_\Gamma = f^{-1}(b) \cap T_\Gamma
\]
the part of the tree corresponding to the minimum resp. maximum values and
\[
T_{01}_\Gamma = T_\Gamma - T_0_\Gamma - T_1_\Gamma.
\]

4.3. Hamiltonian perturbations. We introduce notations for Hamiltonian perturbations and associated perturbed Cauchy-Riemann operators. Let \( K \in \Omega^1(C, \partial C; C^\infty(X)) \) be a one-form with values in smooth functions vanishing on the tangent space to the boundary, that is, a smooth map \( TC \times X \to \mathbb{R} \) linear on each fiber of \( TC \), equal to zero on \( TC|\partial C \). Denote by \( \tilde{K} \in \Omega^1(C, \partial C; \text{Vect}(X)) \) the corresponding one form with values in Hamiltonian vector fields. The curvature of the perturbation is
\[
R_K = dK + \{K, K\}/2 \in \Omega^2(C, C^\infty(X))
\]
where \( \{K, K\} \in \Omega^2(C, C^\infty(X)) \) is the two-form obtained by combining wedge product and Poisson bracket, see McDuff-Salamon [77, Lemma 8.1.6] whose sign convention is slightly different. Given a map \( u : C \to X \), define
\[
(82) \quad \overline{\partial}_{J,K} u := (du + \tilde{K})^{0,1} \in \Omega^{0,1}(C, u^*TX).
\]
The map \( u \) is \((J,K)\)-holomorphic if \( \overline{\partial}_{J,K} u = 0 \). Suppose that \( C \) is equipped with a compatible metric and \( X \) is equipped with a tamed almost complex structure and perturbation \( K \). The \( K \)-energy of a map \( u : C \to X \) is
\[
E_K(u) := (1/2) \int_C |du + \tilde{K}(u)|_J^2
\]
where the integral is taken with respect to the measure determined by the metric on \( C \) and the integrand is defined as in [77, Lemma 2.2.1]. If \( \overline{\partial}_{J,K} u = 0 \), then the
$K$-energy differs from the symplectic area $A(u) := \int_C u^* \omega$ by a term involving the curvature from $\phi$:

$$E_K(u) = A(u) + \int_C R_K(u).$$

In particular, if the curvature vanishes then the area is non-negative.

The notion of perturbed pseudoholomorphic map is obtained from the notion of perturbed pseudoholomorphic map by specializing to the case of a strip

$$C = \mathbb{R} \times [0,1] = \{(s,t) \mid s \in \mathbb{R}, t \in [0,1]\}.$$Given $H \in C^\infty([0,1] \times X)$ let $K$ denote the perturbation one-form $K = -Hdt$ and let $E_H := E_K$. If $u : \mathbb{R} \times [0,1] \to X$ has limits $x_\pm : [0,1] \to X$ as $s \to \pm \infty$ then the energy-area relation $\phi$ becomes

$$E_H(u) = A(u) - \int_{[0,1]} (x^*_+ H - x^*_H)dt.$$

Let $\bar{\partial}_{J,H} = \bar{\partial}_{J,K}$ be the corresponding perturbed Cauchy-Riemann operator from $\phi$. A map $u : C \to X$ is $(J,H)$-holomorphic if $\bar{\partial}_{J,H} u = 0$. A perturbed pseudo-holomorphic strip for Lagrangians $L_0, L_1$ is a finite energy $(J,H)$-holomorphic map $u : \mathbb{R} \times [0,1] \to X$ with $\mathbb{R} \times \{b\} \subset L_b, b = 0, 1$. An isomorphism of Floer trajectories $u_0, u_1 : \mathbb{R} \times [0,1] \to X$ is a translation $\psi : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ in the $\mathbb{R}$-direction such that $\psi^* u_1 = u_0$. Denote by

$$\mathcal{M}(L_0, L_1) := \left\{ u : \mathbb{R} \times [0,1] \to X \mid u(\mathbb{R} \times \{b\}) \subset L_0, b \in \{0,1\} \quad \overline{\partial}_{J,H} u = 0 \quad E_H(u) < \infty \right\} / \mathbb{R}$$

the moduli space of isomorphism classes of Floer trajectories of finite energy, with its quotient topology.

In order to apply the theory of Donaldson hypersurfaces we will need our Hamiltonian perturbations to vanish, and in order to achieve recall the correspondence between Floer trajectories with Hamiltonian perturbation and trajectories without perturbation with different boundary condition. Let $H \in C^\infty([0,1] \times \mathbb{R}, X)$ be a time-dependent Hamiltonian and let $J \in \text{Map}([0,1], \mathcal{J}_\tau(X, \omega))$ be a time-dependent almost complex structure. Suppose that $L_0, L_1$ are Lagrangians such that $\phi_1(L_0) \cap L_1$ is transversal. There is a bijection between $(J_t, H_t)$-holomorphic Floer trajectories with boundary conditions $L_0, L_1$ and $(\phi_1^{-1})^* J_t$-holomorphic Floer trajectories with boundary conditions $\phi_1(L_0), L_1$ obtained by mapping each $(L_0, L_1)$ trajectory $(s,t) \mapsto u(s,t)$ to the $(\phi_1(L_0), L_1)$-trajectory given by $(s,t) \mapsto \phi_1(u(s,t))$. Discussion after (7).

More generally, we consider surfaces with strip-like ends. A surface with strip like ends consists of a surface with boundary $\Sigma$ equipped with a complex structure $j : T\Sigma \to T\Sigma$, and a collection of embeddings

$$\kappa_e : \pm(0, \infty) \times [0,1] \to \Sigma, \quad i = 0, \ldots, n$$

such that $\kappa^*_e j$ is the standard almost complex structure on the strip, and the complement of the union of the images of the maps $\kappa_e$ is compact. Any such surface has a
canonical compactification $\Sigma$ with the structure of a compact surface with boundary obtained by adding a point at infinity along each strip like end and taking the local coordinate to be the exponential of $\pm 2\pi i n$. Given a surface with strip-like ends consider $E := \Sigma \times X$ as fiber bundle over $\Sigma$ with fiber $X$. Following \cite{77} (8.1.3), let $\pi_X : E \to X$ denote the projection on the fiber. In local coordinates $s,t$ on $\Sigma$ define $K_s,K_t$ by $K = K_s ds + K_t dt$. Let $\omega_E = \pi_X^* \omega - \pi_X^* dK_s \wedge ds - \pi_X^* dK_t \wedge dt + (\partial_t K_s - \partial_s K_t) ds \wedge dt$.

The form $\omega_E$ is closed, restricts to the two-form $\omega$ on any fiber, and defines the structure of a symplectic fiber bundle on $E$ over $\Sigma$. Consider the splitting $TE \cong \pi_X^* TX \oplus (\Sigma \times \mathbb{R}^2)$. Let $j_{\Sigma} : T\Sigma \to T\Sigma$ denote the standard complex structure on $\Sigma$. Define an almost complex structure on $E$ by

$$J_E : TE \to TE, \quad (v,w) \mapsto ((J\hat{K} - \hat{K} j_{\Sigma}) w + Jv, j_{\Sigma} w)$$

where $\hat{K} \in \Omega^1(\Sigma, \text{Vect}(X))$ is the Hamiltonian-vector-field-valued one-form associated to $K$. A smooth map $u : \Sigma \to X$ is $(J,K)$-holomorphic if and only if the associated section $(\text{id} \times u) : \Sigma \to E$ is $J_E$-holomorphic \cite{77} Exercise 8.1.5].

Let $\tilde{L}_i = (\partial \Sigma)_i \times L_i$ the fiber-wise Lagrangian submanifolds of $E$ defined by $L_i$. Then $u : \Sigma \to X$ has boundary conditions in $(L_i,i = 1,\ldots,m)$ if and only if $\text{id} \times u : \Sigma \to E$ has boundary conditions in $(\tilde{L}_i,i = 1,\ldots,m)$. Thus, we have re-formulated Hamiltonian-perturbed holomorphic maps with Lagrangian boundary conditions as holomorphic sections.

We recall some terminology for clean intersections. A pair $L_0,L_1 \subset X$ of submanifolds intersect cleanly if $L_0 \cap L_1$ is a smooth manifold and $T(L_0 \cap L_1) = TL_0 \cap TL_1$. Floer homology for clean intersections was constructed in Pozniak \cite{88} and also Schmäschke \cite{92} Section 7] under certain monotonicity assumptions. In this section we show how to extend the monotone results to the case that the union of the cleanly-intersecting Lagrangians is rational using stabilizing divisors. We have in mind especially the case that the two Lagrangians are rational and equal. In the particular case of diagonal boundary conditions, we show that the Floer cohomology is the singular cohomology with Novikov coefficients.

The definition of Floer cohomology in the clean intersection case is a count of configurations of holomorphic strips, disks, spheres, and Morse trees as in, for example, Biran-Cornea \cite{13} Section 4]. We suppose $L_k,k \in \{0,1\}$ are Lagrangian branes equipped with Morse-Smale pairs $(F_k,G_k)$, and perturbation systems $P_k$ giving rise to moduli spaces of holomorphic treed disks with boundary in $L_k$. Let $H_{01} \in C^\infty(X \times [0,1])$ be a Hamiltonian with flow $\phi_t$ so that $\phi_1(L_0) \cap L_1$ is clean and $\phi_1(L_0) \cup L_1$ is rational; that is, some power of the line-bundle-with-connection $\tilde{X}$ is trivializable over $L_0 \cup L_1$. For example, if $\phi_1(L_0) = L_1$ is rational then the intersection is clean and the union is rational. Let $F_{01} : \phi_1(L_0) \cap L_1 \to \mathbb{R}$ be a Morse function. By the Morse lemma, the critical set

$$\mathcal{I}(L_0,L_1;H_{01}) := \text{crit}(F_{01}) = \{ l \in L_0 \cap L_1 \mid \text{d}F_{01}(l) = 0 \}$$

is necessarily finite. Choose a generic metric $G_{01}$ on $L_0 \cap L_1$, and let $\phi_t$ be the time $t$ flow of $-\text{grad}(F) \in \text{Vect}(L_0 \cap L_1)$. Denote the stable and unstable manifolds of
\[ F: \quad W^\pm_x = \left\{ l \in L_0 \cap L_1 \mid \lim_{t \to \pm \infty} \phi_t(l) = x \right\}. \]

The critical set admits a natural grading map
\[ i : \mathcal{I}(L_0, L_1; H_{01}) \to \mathbb{Z}_g \]

obtained by adjusting the Maslov index of paths from \( T_x L_0 \) to \( T_x L_1 \) in \( T_x X \) for any \( x \in \text{crit}(F_{01}) \) by the index \( \dim(W^-_x) \) of the critical point \( x \). The space of Floer cochains is then as before
\[ CF(L_0, L_1; H_{01}) = \bigoplus_{x \in \mathcal{I}(L_0, L_1; H_{01})} \Lambda^{< x>}. \]

\( \dim(W^-_x) \) of the critical point \( x \) with \( \mathbb{Z}_g \)-grading induced by the grading on \( \mathcal{I}(L_0, L_1; H_{01}) \).

The structure maps for the Fukaya bimodule count configurations containing perturbed holomorphic strips and gradient segments for the Morse function on the intersection. We assume that \((F_{01}, G_{01})\) is Morse-Smale, that is, the stable and unstable manifolds meet transversally
\[ T_x W_x^+ + T_y W_y^- = T_1(\phi_1(L_0) \cap L_1), \quad \forall l \in W_x^+ \cap W_y^-, \quad x, y \in \text{crit}(F). \]

Choose a compatible almost complex structure \( J \) on \( X \). Given a stable strip \( C_0 \) with boundary markings \( z_-, z_+ \) let \( w_1, \ldots, w_k \in C_0 \) denote the nodes appearing in any non-self-crossing path between \( z_- \) and \( z_+ \). Define a topological space
\[ C = C_0 \sqcup \bigcup_{i=1}^{k} [0, \ell(w_i)]/ \sim \]

by replacing each node \( w_i \) by a segment \( T_i \cong [0, \ell(w_i)] \) of length \( \ell(w_i) \). Denote by
\[ T = T_1 \cup \ldots \cup T_k \quad S = C - \overline{T} \]

the tree resp. surface part of \( C \). A perturbed pseudoholomorphic strip is then a map from \( C = S \cup T \) that is \( J \)-holomorphic on the surface part and \( a F_{01} \) resp. \( F_0 \) resp. \( F_1 \) gradient trajectory on each segment in \( T \). See Figure 27.

![Figure 27. A treed strip with Lagrangian boundary conditions](image)

We fix thin parts of the universal curves: a neighborhood \( \mathcal{T}_\Gamma^{\text{thin}} \) of the endpoints and a neighborhood \( \mathcal{S}_\Gamma^{\text{thin}} \) of the markings and nodes. In the regularity construction, these neighborhoods must be small enough so that either a given fiber is in a neighborhood of the boundary, where transversality has already been achieved, or otherwise each segment and each disk or sphere component in a fiber meets the
complement of the chosen thin parts. For an integer $l \geq 0$ a \textit{domain-dependent perturbation} of $F$ of class $C^l$ is a $C^l$ map

$$F_\Gamma : \mathcal{T}_\Gamma \times (L_0 \cap L_1) \to \mathbb{R}$$

equal to the given function $F$ away from the endpoints:

$$F_{\Gamma,01}\mathcal{T}_\Gamma^{\text{thin},01} = \pi_2^* F_{01}, \quad F_{\Gamma,k}\mathcal{T}_\Gamma^{\text{thin},k} = \pi_2^* F_k, \quad k \in \{0, 1\}$$

where $\pi_2$ is the projection on the second factor in (84). A \textit{domain-dependent almost complex structure} of class $C^l$ for treed disks of type $\Gamma$ is a map from the two-dimensional part $\mathcal{S}_\Gamma$ of the universal curve $\overline{U}_\Gamma$ to $J_\tau(X)$ given by a $C^l$ map

$$J_\Gamma : \mathcal{S}_\Gamma \times X \to \text{End}(TX)$$

equal to the given $J_D$ away from nodes and boundary:

$$J_\Gamma|\mathcal{S}_{\text{thin}}^{\text{thin}} = \pi_2^* J_D$$

and equal to the given stabilizing almost complex structure $J_D$ on the boundary. A \textit{perturbed pseudoholomorphic strip} for the pair $(L_0, L_1)$ consists of a treed disk $C$ and a map $u : C = S \cup T \to X$ such that

(Boundary condition) The Lagrangian boundary condition holds $u((\partial C)_b) \subset L_b$ for $b \in \{0, 1\}$.

(Surface equation) On the surface part $S$ of $C$ the map $u$ is $J$-holomorphic for the given domain-dependent almost complex structure: if $j$ denotes the complex structure on $S$ then

$$J_{\Gamma,u(z),z} \ du_S = du_j.$$

(Boundary tree equation) On the boundary tree part $T \subset C$ the map $u$ is a collection of gradient trajectories:

$$\frac{d}{ds} u|_{T_b} = -\text{grad}_{F_{\Gamma,b,(s,u(s))}}(u|_{T_b})$$

where $s$ is a local coordinate with unit speed and $b$ is one of the symbols 0, 1 or 01. Thus for each edge $e \in \text{Edge}_-(\Gamma)$ the length of the trajectory is given by the length $u|_{e \subset T}$ is equal to $\ell(e)$.

Given a stabilizing divisor $D \subset X - (L_0 \cup L_1)$, a stable strip $u : C \to X$ is \textit{adapted} if and only if

(Stable domain property) $C$ is a stable marked strip; and

(Marking property) Each interior marking lies in $u^{-1}(D)$ and each component of $u^{-1}(D)$ contains an interior marking.

Let $\overline{\mathcal{M}}(L, D)$ denote the set of isomorphism classes of stable $D$-adapted Floer trajectories to $X$, and by $\mathcal{M}_\Gamma(L, D)$ the subspace of combinatorial type $\Gamma$. Compactness and transversality properties of the moduli space of Floer trajectories in the case of clean intersection, including exponential decay estimates, can be found in [110] and [92]. The necessary gluing result can be found in Schmäschke [92, Section 7]. For
any \( u \in \mathcal{M}_0(L, D, x_+, x_-) \) we denote by \( \text{Hol}_{L_0, L_1}(u) \in \Lambda_x \) the parallel transport of the local systems along the boundary. Define

\[
\text{(85)} \quad CF(L_0, L_1; H_{01}) = \bigoplus_{l \in I(L_0, L_1; H_{01})} \Lambda_l,
\]

\[
\partial <x_+> = \sum_{[u] \in \mathcal{M}_0(L, D, x_+, x_-)} \epsilon([u]) q^{E([u])} \sigma([u]) <x_->.
\]

Counting Floer trajectories defines operations

\[
\mu_{d|e} : \widetilde{CF}(L_0) \otimes \widetilde{CF}(L_0, L_1; H_{01}) \otimes \widetilde{CF}(L_1)^{\otimes e} \to CF(L_0, L_1; H_{01})
\]

by

\[
\text{(86)} \quad <x_0, 1> \otimes \ldots \otimes <x_0, d> \otimes <x> \otimes <x_{1, 1}> \otimes \ldots \otimes <x_{1, e}>
\]

\[
\mapsto \sum_{[u] \in \mathcal{M}_{d|e,n}(x_0, 1, \ldots, x_0, d, x, x_{1, 1}, \ldots, x_{1, e}, y)} (-1)^{\diamond} \epsilon(u) \text{Hol}_{L_0, L_1}(u) q^{A(u)} <y>
\]

where \( \diamond \) is defined in \( (75) \).

**Theorem 4.2.** For coherent, regular, stabilizing, convergent, ghost-marking-independent collections \( \mathcal{P} = (P_1) \) of perturbation data the maps \( \mu_{d|e}^{d|e}_{d,e} \) induce on \( CF(L_0, L_1; H_{01}) \) the structure of a strictly unital, convergent \( A_\infty \) \((\widetilde{CF}(L_0), \widetilde{CF}(L_1))\)-bimodule.

The \( A_\infty \) bimodule axiom follows from the compactness and transversality properties of the one-dimensional moduli spaces of treed strips and a sign computation similar to that for \( A_\infty \) algebras, left to the reader.

### 4.4. Morphisms

In the remainder of this section we show that the Fukaya bimodule is independent of all choices, including the choice of Hamiltonian perturbation. In particular, if a Lagrangian is Hamiltonian displaceable then its Fukaya bimodule is homotopy equivalent to the trivial bimodule.

The necessary morphisms of \( A_\infty \) bimodules are given by counting *parametrized treed strips*. If \( C \) is a disk with markings \( z_-, z_+ \) on the boundary then a parametrization is a holomorphic isomorphism

\[
\phi : C - \{z_-, z_+\} \to \mathbb{R} \times [0, 1].
\]

A \((d|e, n)\)-marking of a parametrized strip is a collection of \( d \) resp. \( e \) resp \( n \) markings on \( \mathbb{R} \times \{0\} \) resp. \( \mathbb{R} \times \{1\} \) resp. in \( \mathbb{R} \times (0, 1) \). The moduli space of \((d|e, n)\)-marked treed parametrized strips has a natural compactification by stable treed parametrized strips, in which unparametrized strips are allowed to bubble off the ends. Let \( \overline{\mathcal{M}}_{d|e,n,1} \) denote the moduli space of stable parametrized strips of stable treed \( d, e \)-marked parametrized treed strips. This moduli space is equipped with a continuous map \( \overline{\mathcal{M}}_{d|e,n,1} \to \overline{\mathcal{M}}_{d|e,n} \) forgetting the parametrization, between quilted strip components. The fibers of the forgetful map are canonically oriented so that the positive orientation corresponds to moving the quilting to the left, and so the orientation of \( \overline{\mathcal{M}}_{d|e,n} \) induces an orientation on \( \overline{\mathcal{M}}_{d|e,n,1} \).
We define holomorphic parametrized strips as in the unparametrized case. Our
regularization of the space of holomorphic parametrized treed strips uses a divisor
in the total space of the fibration $\mathbb{R} \times [0,1] \times X$. Suppose that $J_e \in J(X,\omega)$
are compatible almost complex structures stabilizing for $\phi_{H_e,1}(L_e,\cdot) \cup L_e,\cdot_+$. Let $\phi^{*}_{e,1-t}J_e$ denote the corresponding time-dependent almost complex structures, and
$\sigma_{e,k,b} : X → X^k$ are asymptotically $J_e$-holomorphic, uniformly transverse sequence
of sections with the property that $D_e = \sigma_{e,k,1}^{-1}(0)$ are stabilizing for $\phi_{H_e}(L_e,\cdot) \cup L_e,\cdot$ for $k$ sufficiently large. The pull-backs $\phi^{*}_{e,1-t}J_e$ are then $\phi^{*}_{e,1-t}J_e$-holomorphic, and any $(J_e,H_e)$-holomorphic strip with boundary in $(L_{e,-},L_{e,\cdot})$ meets $\phi^{*}_{e,1-t}(\Sigma \times D_e)$ in at least one point. Denote by $\tilde{E} → E$ the pull-back of $\tilde{X} → X$ to the fibration,
equipped with the almost complex structure induced by the given almost complex structure on $E$. Recall the following from Charest-Woodward [25]:

**Lemma 4.3.** Let $\Sigma$ be a surface with strip-like ends, let $L_i \subset X$ be rational La-
grangians associated to the boundary components $(\partial \Sigma)_i$, and suppose that stabiliz-
ing divisors $D_e$ for the ends $e = 1,\ldots,n$ of $\Sigma$ have been chosen as zero sets of
asymptotically holomorphic sequences of sections $\sigma_{e,k}$ for $k$ sufficiently large.
There exists an asymptotically $J_E$-holomorphic, uniformly transverse sequence $\sigma_k : E → E^k$ with the property that for each end $e$, the pull-back $\kappa^{*}_e\sigma_k(\cdot,s,\cdot)$ converges in $C^\infty$ uniformly on compact subsets to $\phi^{*}_{e,1-t}\sigma_{e,k}$ as $s → ±\infty$. The zero set $D_E = \sigma_{e,1}^{-1}(0)$ is approximately holomorphic for $k$ sufficiently large, asymptotic to $(1 \times \phi^{*}_{e,1-t})^{-1}(\mathbb{R} \times [0,1] \times D_e)$ for each end $e = 1,\ldots,n$, and the intersection of $D_E$ with each boundary fiber $\pi^{-1}(\pm z)$, $z ∈ \partial \Sigma$ is stabilizing.

**Proof.** We include the proof for completeness. Let $\mathbb{H} = \{z ∈ \mathbb{C} \mid \text{Im}(z) ≥ 0\}$ denote
complex half-space. Let $\tilde{E}$ be the fiber bundle over $\tilde{\Sigma}$ with fiber $X$ defined by gluing
together $U_0 = \Sigma × X$ and $U_e = \mathbb{H} × X, e = 1,\ldots,n$ using the transition maps
$\kappa_e × \phi_{H_e,1-t}$ on $\mathbb{H} \setminus \{0\} ≃ \mathbb{R} × [0,1]$ from $U_0$ to $U_e$. Denote the projections over
$U_e$ to $X$ by $\pi_{X,e}$. The two-forms $\omega_E$ on $U_0$, and $\pi_{X,e}^{*}\omega$ on $U_e$ glue together to a
two-form $\omega_E$ on $E$, making $\tilde{E}$ into a symplectic fiber bundle.

The fiber-wise symplectic form above may be adjusted to an honest symplectic
form by adding a pull-back from the base, and furthermore adjusted so that the
boundary conditions have rational union. For the first claim, since $\omega_E$ is fiber-wise
symplectic, there exists a symplectic form $\nu ∈ \Omega^2(\Sigma)$ with the property that $\omega_E + \pi^{*}\nu$ is symplectic, where $\pi : E → \Sigma$ is the projection. The almost complex structure $J_E$ is
compatible with $\omega_E + \pi^{*}\nu$, and equal to the given almost complex structures $J_B ⊕ J_e$ on the ends. For the second claim, let $\Sigma → \Sigma$ be a line-bundle-with-connection
whose curvature is $\nu ∈ \Omega^2(\Sigma)$. Let $\tilde{E} → \tilde{\Sigma}$ be a line-bundle-with-connection whose
curvature is $\omega_E + \pi^{*}\nu$. Denote by $\tilde{L}_i$ the closure of the image of $(\partial \Sigma)_i × L_i$ for
$i = 0,\ldots,m$. Fix trivializations of $\tilde{E}$ over $\tilde{L}_{e,-} ∩ \tilde{L}_{e,\cdot} ≃ L_{e,-} ∩ L_{e,\cdot}$ for each end $e$.
By assumption, the line bundle $\tilde{E}$ is trivializable over $L_e$, hence also $\tilde{L}_e$ by parallel
transport along the boundary components. Let $\epsilon_k,k = 0,1$ be ends connected by a
connected boundary component labelled by $L_i$. For any $p_k ∈ \phi_{H_{\epsilon_k}}(L_{\epsilon_k,-}) ∩ L_{\epsilon_k,\cdot}$ the parallel transport $T(p_0,p_1) ∈ U(1)$ from $p_0$ to $p_1$ is independent of the choice of
path. Indeed, any two paths differ up to homotopy by a loop in \( L_i \), which has trivial holonomy by assumption. After perturbation of the connection and curvature on \( \tilde{E} \), we may assume that the parallel transports \( T(p_0, p_1) \) are rational for all choices of \((p_0, p_1)\). After taking a tensor power of \( \tilde{E} \), we may assume that the parallel transports \( T(p_0, p_1) \) are trivial, hence \( \tilde{E} \) admits a covariant constant section \( \tau \) over the union \( \tilde{L}_e, e = 1, \ldots, n \).

Donaldson’s construction \[33\] implies the existence of a symplectic hypersurface in the total space of the fibration. We show that the hypersurface may be taken to equal the pullback of one of the given ones over a neighborhood of the boundary of the base, as follows. Let \( \sigma, L \) sequence of sections concentrating on \( \tau \) of asymptotically holomorphic sections asymptotic to the pull-backs of \( \sigma \) of a compact set. The resulting sequence \( \sigma \) of asymptotically holomorphic sections asymptotic to the given trivializations on the Lagrangians themselves.

For each point \( z \in \Sigma \), let \( \sigma_{z,k} : \Sigma \to \tilde{\Sigma}^k \) denote the Gaussian asymptotically holomorphic sequence of sections of \( \Sigma \) concentrated at \( z \) as in (87). We may assume that the images of the strip-like ends are disjoint. Let \( V_i \) be disjoint open neighborhoods \( (\partial \Sigma)_i - \cup_e \text{Im}(\kappa_e) \) in \( \Sigma \). For each \( p = (z, x) \in \tilde{L}_e \), let \( \sigma_{x,k} \) be either equal to \( \sigma_{e,k} \), for \( z \in \text{Im}(\phi_e) \) or otherwise equal to \( \sigma_{i,k} \) if \( b \) lies in \( V_i \). Let \( P_k \) be a set of points in \( \partial \Sigma \) such that the balls of \( g_\kappa \)-radius 1 cover \( \partial \Sigma \) and any two points of \( P_k \) are at least distance \( 2/3 \) from each other, where \( g_\kappa \) is the metric defined by \( k \nu \).

The desired asymptotically-holomorphic sections are obtained by taking products of asymptotically-holomorphic sections on the two factors: Write \( \theta(z, x) \tau(z, x) = \sigma_{x,k}(x) \boxtimes \sigma_{z,k}(z) \) so that \( \theta(x, z) \in \mathbb{C} \) is the scalar relating the two sections. Define

\[
(87) \quad \sigma_k = \sum_{p \in P_k} \theta(z, x)^{-1} \sigma_{x,k} \boxtimes \sigma_{z,k}.
\]

Then by construction the sections (87) are asymptotically holomorphic, since each summand is.

It remains to achieve the uniformly transverse condition. Recall from \[33\] that a sequence \((s_k)_{k \geq 0}\) is uniformly transverse to 0 if there exists a constant \( \eta \) independent of \( k \) such that for any \( x \in X \) with \(|s_k(x)| < \eta\), the derivative of \( s_k \) is surjective and satisfies \(|\nabla s_k(x)| \geq \eta\). The sections \( \sigma_k \) are asymptotically \( J_\epsilon \)-holomorphic and uniformly transverse to zero over \( \partial \Sigma \times X \), since the sections \( \sigma_{i,k} \) and \( \sigma_{e,k} \) are uniformly transverse to the zero section. Hence \( \sigma_k \) is also uniformly transverse over a neighborhood of \( \partial \Sigma \). Pulling back to \( E \) one obtains an asymptotically holomorphic sequence of sections of \( \tilde{E}|\Sigma \) that is informally transverse in a neighborhood of infinity, that is, except on a compact subset of \( E \). Donaldson’s construction \[33\] although stated only for compact manifolds, applies equally well to non-compact manifolds assuming that the section to be perturbed is uniformly transverse on the complement of a compact set. The resulting sequence \( \sigma_{E,k} \) is uniformly transverse and consists of asymptotically holomorphic sections asymptotic to the pull-backs of \( \sigma_{e,k} \) on the ends. The divisor \( D_E = \sigma_{E,k}^{-1}(0) \) is approximately holomorphic for \( k \) sufficiently
large and equal to the given divisors $\pm (0, \infty) \times D_e$ on the ends, by construction, and concentrated at $L_i$, over each boundary component $(\partial \Sigma)_i$. □

A perturbation scheme similar to the one for Floer trajectories makes the moduli spaces transverse. Choose a tamed almost complex structure $J_E \in \mathcal{J}_\tau(\mathcal{L}, \omega + \pi_2 \nu)$ making $\mathcal{D}_E$ holomorphic, so that $\mathcal{D}_E$ contains no holomorphic spheres, each holomorphic sphere meets $\mathcal{D}_E$ in at least three points, and each disk with boundary in $L_{e-} \cup L_{e+}$ meets $\mathcal{D}_E$ in at least one point. Since $\mathcal{D}_E$ is only approximately holomorphic with respect to the product complex structure, the complex structure $J_E$ will not necessarily be of split form, nor will the projection to $\Sigma$ necessarily be $(\bar{J}_E, j_\Sigma)$-holomorphic away from the ends. Furthermore, choose domain-dependent perturbations $F_\tau$ of the Morse functions $F_e$ on $\phi_{H_e,1}(L_{e-}) \cap L_{e+}$, so that $F_\tau$ is a perturbation of $F_\tau$ on the segments that map to $\phi_{H_e,1}(L_{e-}) \cap L_{e+}$.

Domain-dependent perturbations give a regularized moduli space of stable adapted treed strips. These are maps to $E$, homotopic to sections, with the given Lagrangian boundary conditions and mapping the positive resp. negative end of the strip to the positive resp. negative end of $E$. Let $\mathcal{M}(L, D, P)$ denote the moduli space of adapted stable treed strips in $E$ of this type. For generic domain-dependent perturbations $P_\tau, E = (F_\tau, E, J_\tau, E)$ on $E$, the moduli space $\mathcal{M}(L, D, P, (x_e))$ of perturbed maps to $E$ with boundary in $(L_{e-0}, L_{e+1})$ and limits $(x_e)$ has zero and one-dimensional components that are compact and smooth (as manifolds with boundary) with the expected boundary. In particular the boundary of the one-dimensional moduli spaces $\mathcal{M}_1(L, D, P, (x_e))$ are 0-dimensional strata $\mathcal{M}_1(L, D, P, (x_e))$ corresponding to either a perturbed pseudoholomorphic strip bubbling off on end, or a disk bubbling off the boundary.

Given two Hamiltonian perturbations $H'_{01}, H''_{01}$ such that the intersections $\phi'_1(L_0) \cap L_1$ and $\phi''_1(L_0) \cap L_1$ are clean, we construct a morphism of $A_\infty$ bimodules

$$CF(L_0, L_1; H'_{01}) \to CF(L_0, L_1; H''_{01})$$

via a continuation argument which counts parametrized treed holomorphic strips. Let $H \in C^\infty_c((0,1) \times X)$ be a regular perturbation for the pair $(L, L)$. Given a parametrized strip we consider a Hamiltonian perturbation $H = H_d ds + H_t dt$ equal to $H'_{01} dt$ for $s \gg 0$ and to $H''_{01}$ for $s \ll 0$. We define coherent perturbations $\mathcal{P}(X; H)$ as before, consisting of domain-dependent almost complex structures and domain-dependent metrics making the moduli spaces of parametrized trajectories of dimension at most one compact and regular. Define operations

$$(88) \quad \phi^{de} : \overline{CF}(L_0)^{\otimes d} \otimes CF(L_0, L_1; H'_{01}) \otimes \overline{CF}(L_1)^{\otimes e} \to CF(L_0, L_1; H''_{01})$$

$$\langle z_{0,1} \rangle \otimes \ldots \otimes \langle z_{0,d} \rangle \otimes \langle z \rangle \otimes \langle z_{1,1} \rangle \otimes \ldots \otimes \langle z_{1,e} \rangle$$

$$\mapsto \sum_{[u] \in \mathcal{M}_{d+1}(z_{0,1}, \ldots, z_{0,d}, z, z_{1,1}, \ldots, z_{1,e}, y)} (-1)^{d+1} \epsilon(u) \text{Hol}_L(u) q^{A(u)} \langle y \rangle$$

where the cochain groups involved are defined using $\Lambda$ coefficients; the values $A(u)$ are not necessarily positive because of the additional term in the energy-area relation.
Proposition 4.4. For regular, coherent collections of perturbation data, the collection \( \phi = (\phi_{d,e})_{d,e \geq 0} \) is a morphism of \( A_\infty \) bimodules from \( CF(L_0, L_1; H^0_{01}) \) to \( CF(0, L_1; H^0_{01}) \).

Proof. The boundary components of \( \overline{\mathcal{M}}_{d,e,1}(z_0, z_1, y) \) consist of configurations where a treed strip has broken off or a treed disk has broken off. The former case corresponds to one of the first two terms in (73) while the latter corresponds to the last two terms. The first term in (73) (in which \( \mu \) appears before \( \phi \)) has an additional sign from coming from the definition of the orientation on \( \overline{\mathcal{M}}_{d,e,1} \), so that the orientation of the fiber corresponds to composing parametrization with a translation in the positive direction; this means that the positive orientation on the gluing parameter for the boundary components corresponding to terms of the first type becomes identified with the negative orientation on these fibers, giving rise to the additional sign. The degree of the morphism is zero, which causes those contributions to the sign in (73) to vanish. This leaves the contributions from \( \aleph \), which are similar to those dealt with before and left to the reader. \( \square \)

4.5. Homotopies. In this section we show that the morphisms of bimodules introduced above satisfy a gluing law of the type found in topological field theory. The construction uses a moduli space of twice parametrized treed strips defined as follows

If \( C \) is a disk with markings \( z_-, z_+ \) on the boundary then a twice-parametrization is pair of holomorphic isomorphisms

\[
\phi_k : C - \{ z_-, z_+ \} \to \mathbb{R} \times [0, 1], \quad k \in \{ 0, 1 \}.
\]

In the compactification, parametrized or unparametrized strips can now bubble off on either end. Let \( \overline{\mathcal{M}}_{d,e,n,2} \) denote the moduli space of twice parametrized strips with \( d \) markings resp. \( e \) markings on the first resp. second boundary component, and \( n \) markings in the interior.

Now suppose that we have chosen Hamiltonian perturbations \( H^a_{01}, a = 0, 1, 2 \) so that the intersections \( \phi^t_1(L_0) \cap L_1 \) are clean where \( \phi^0_t \) denotes the time one flow of \( H^a_{01} \). We extend the Hamiltonian perturbation over the universal twice-parametrized strip and choose generic domain-dependent almost complex structures and

\[
(89) \quad \tau^{d,e} : \overline{CF}(L_0)^{\otimes d} \otimes CF(L_0, L_1; H^0_{01}) \otimes \overline{CF}(L_1)^{\otimes e} \to CF(L_0, L_1; H^2_{01})[-1]
\]

\[
\sum_{[u] \in \overline{\mathcal{M}}_{d,e,2}(z_0, \ldots, z_d, z, \ldots, z_1, \ldots, z_1, y),} (-1)^{\sigma + \phi(u)} \text{Hol}_{L_0, L_1}(u) q^{A(u)} <y>.
\]

Theorem 4.5. Suppose that coherent, stabilizing, regular, convergent, ghost-marking-independent perturbation data have been chosen giving rise to morphisms of \( A_\infty \) bimodules \( \phi : CF(L_0, L_1; H^0_{01}) \to CF(L_0, L_1; H^0_{01}) \) resp. \( \psi : CF(L_0, L_1; H^0_{01}) \to CF(L_0, L_1; H^0_{01}) \) resp. \( \chi : CF(L_0, L_1; H^0_{01}) \to CF(L_0, L_1; H^0_{01}) \). Then for coherent any stabilizing, regular, convergent collection extending the given collection over the
universal curves over moduli spaces of twice-parametrized strips, the resulting operations \((\tau^d)_{d,e} \geq 0\) define a homotopy of morphisms of \(A_\infty\) bimodules from \(\psi \circ \phi\) to \(\chi\) in \(\text{Hom}(CF(L_0, L_1; H^0), CF(L_0, L_1; H^2))\).

**Proof.** The components of the boundary of \(\mathcal{M}_{d,e}(z_0,1, \ldots, z_0,d, z, z_1,1, \ldots, z_1,e, y)\) which do not involve the parametrized strip breaking, or the shaded region disappearing, correspond to the terms on the left-hand-side of (73). The components where the parametrized strip breaks correspond to the contributions to the composition \(\psi \circ \phi\), while the components where the parametrized strip vanishes give the identity morphism of \(A_\infty\) modules. \(\square\)

For any \(b_0 \in \tilde{MC}(L_0), b_1 \in \tilde{MC}(L_1)\) we denote by \(HF(L_0, L_1; H_{01}; b_0, b_1)\) the cohomology of the operator \(\mu^{1}_{b_0, b_1}\) the cohomology of the operator \(\mu^{1}_{b_0, b_1}\). It follows as in Lemma 3.3 that any morphism of \(A_\infty\) bimodules induces a map of cohomology groups, functorially. As a result, \(HF(L_0, L_1; H_{01}; b_0, b_1)\) is independent of the choice of Hamiltonian perturbation. On the other hand, for the case \(L_0 = L_1 = L\) the \(A_\infty\) bimodule \(CF(L_0, L_1; 0)\) is equal to \(CF(L)\) itself considered as an \(A_\infty\) bimodule over \(\hat{CF}(L) \times CF(L)\). This shows the following:

**Corollary 4.6.** If a compact Lagrangian brane \(L\) is Hamiltonian displaceable, then \(HF(L)_b\) vanishes for every \(b \in MC(L)\).

## 5. Minimal model surgeries

In this section begin the study of Floer-non-trivial Lagrangians associated to flips and blow-ups, by reviewing some aspects of the minimal model program for birationally-Fano varieties introduced by Mori and others, see [59]. The goal of the minimal model program is to classify algebraic varieties by finding a minimal model in each birational equivalence class. In the birationally-Fano case, the hoped minimal model is a Mori fibration: a fibration with Fano fiber. In general, singularities play an important role in the minimal model program but here we are interested in the case that the variety admits a smooth (or at least orbifold) running of the minimal model program. While this case is considered somewhat trivial by algebraic geometers, it includes a number of smooth projective varieties whose symplectic geometry is poorly understood. We introduce a symplectic version of the minimal model program which is a path (with singularities corresponding to surgeries) of symplectic manifolds. We recall that by a suggestion of Song-Tian and others [102], an mmp running is conjecturally equivalent to running Kähler-Ricci flow with surgery; the paths in our symplectic mmp are essentially a weak version of the Kähler-Ricci flow.

### 5.1. Symplectic mmp runnings.

We begin with a description of the transitions in the minimal model program. Each transition is a special kind of birational transformation, for which we review the definition: Let \(X_\pm\) be normal projective varieties. A rational map from \(X_+\) to \(X_-\) is a Zariski dense subset \(U \subset X_+\) and a morphism \(\phi : U \rightarrow X_-\). A rational map is a birational equivalence if it has a rational inverse, that is, a rational map \((V \subset X_-, \psi : V \rightarrow X_+)\) to \(X_+\) such that the compositions \(\phi \circ \psi|_{\phi^{-1}(U)}, \psi \circ \phi|_{\phi^{-1}(V)}\) are the identity on the domains of definition.
Definition 5.1. A minimal model transition of $X_+$ is a birational transformation of one of the following types, see for example Hacon-McKernan [50]:

(a) (Divisorial contractions) A morphism $\tau : X_+ \to X_-$ that is the contraction of a Cartier divisor (codimension one subvariety); a typical example is a blow-down.

(b) (Flips) Let $\tau_+ : X_+ \to X_0$ be a birational morphism. The morphism $\tau_+$ is small iff $\tau_+$ does not contract a divisor. The morphism $\tau_+$ is a flip contraction iff the relative Picard number of $\tau_+$ is one. The flip of $X_+$ is another small birational morphism $\tau_- : X_- \to X_0$ of relative Picard number one.

(c) (Mori fibrations) A fibration $\tau : X_+ \to X_-$ with Fano fiber, that is, a fiber whose anticanonical bundle is well-defined and very ample.

Here the relative Picard number is the difference in Picard numbers, that is, the difference in dimensions in the moduli spaces of line bundles. The relative Picard number is one iff every two curves contracted by $\tau_-$ are numerical multiples of each other, that is, define proportional linear functions on the space of degree two cohomology classes.

Definition 5.2. (a) A running of the minimal model program (mmp) is a sequence of smooth projective varieties $X = X_0, \ldots, X_{k+1}$ such that for $i = 0, \ldots, k$, $X_{i+1}$ is obtained from $X_i$ by a divisorial contraction or a flip, and $X_{k+1}$ is obtained from $X_k$ by a fibration with Fano fiber (sometimes called Mori fibration). The variety $X_k$ may be called the minimal model of $X_0$. The different minimal models which occur are related by the Sarkisov program [51].

(b) An extended running of the mmp is a sequence of smooth projective varieties $X_0, \ldots, X_{k_1}, X_{k_1+1}, \ldots, X_{k_l+1}$, where $k_1, \ldots, k_l$ is an increasing sequence of integers, such that

(i) $X_{i+1}$ is obtained from $X_i$ by a divisorial contraction or flip for $k \neq k_i$ and

(ii) $X_{k_i+1}$ is obtained from $X_{k_i}$ by a Mori fibration.

It is expected that runnings of the mmp exist for all smooth projective varieties (in the case considered here, birationally Fano). This is proved up to dimension three [59].

Example 5.3. (Toric surfaces) The simplest example of the minimal model program occurs for compact toric surfaces. Recall that any compact toric surface $X$ is projective and corresponds to a convex polygon $P \subset \mathbb{R}^2$ whose edge vectors at any vertex give a lattice basis. Each edge vector corresponds to an invariant prime divisor in $P$. Elementary combinatorics shows that if $P$ has more than four edges then there always exists some invariant prime divisor $D_i$ with $D_i^2 = -1$; blowing down $D_i$ one eventually reaches a Hirzebruch surface or a projective plane, which is the minimal model, see Audin [7, Theorem VIII.2.9]. A different way of obtaining a running, which allows orbifold singularities, is discussed below in Section 5.2. Note that toric surfaces often admit several runnings of the toric mmp, depending on the order in which the $-1$-curves are blown down.
Example 5.4. (del Pezzo surface) Let $dP_n$ denote the blow up of $\mathbb{P}^2$ at $n - 1$ generic points. For $1 \leq n \leq 9$, $dP_n$ is Fano and admits a running of the mmp with (in our notation) one transition, since $dP_n$ is itself a Mori fibration over a point. However, clearly $dP_n$ for $n < 8$ admits multiple mmp runnings. For example, for $dP_4$ one can blow down the exceptional curves in any order; on the other hand, one may view $dP_4$ as the thrice-blow-up of the dual projective plane via the Cremona transformation, and blowing down these curves in any order gives another six runnings of the mmp.

Of course, $dP_4$ is toric with moment polytope a hexagon, and the 12 runnings above correspond to the ways of “restoring the corners” to make a triangle. See Figure 28; the points giving Floer non-trivial tori in Theorem 1.1 are darkly shaded in the figure. The fiber over the medium-shaded point is also Floer-non-trivial.

![Figure 28. Blowing down $dP_4$ to $\mathbb{P}^2$](image)

Runnings of the minimal model program can often be obtained from geometric invariant theory as follows, see Reid [90] and Thaddeus [108].

Remark 5.5. (Mmp runnings via git) Let $G$ be a complex reductive group and $X$ a smooth projective $G$-variety. Recall that a linearization of $X$ is an ample $G$-line bundle $\mathcal{L}$. Given such a linearization the geometric invariant theory quotient $X//G$ is the quotient of the semistable locus, defined as the set of points where an invariant section is non-vanishing,

$X^{ss} := \{x \in X \mid \exists k > 0, s \in H^0(\mathcal{L}^k)^G, s(x) \neq 0\},$

by the orbit equivalence relation

$x_1 \sim x_2 \iff Gx_1 \cap Gx_2 \neq 0.$

The git quotient depends only on the ray generated by $\mathcal{L}$ in the equivariant Picard group, that is, tensor powers of $\mathcal{L}$ define the same git quotient. In particular, git quotients are defined for ample elements of the rational Picard group $\text{Pic}_Q(X) = \text{Pic}(X) \otimes \mathbb{Q}$. Varying the linearization $\mathcal{L}$ by rational powers of the anti-canonical bundle $K^{-t}$ produces a family of rational linearizations $\mathcal{L}_t = \mathcal{L} \otimes K^{-t} \in \text{Pic}_Q(X), t \in \mathbb{Q}$.

Let

$X//_t G := X_t^{ss} / \sim$
be the corresponding family of geometric invariant theory quotients. By results of Thaddeus \cite{108} and others, $X//_tG$ undergo a sequence of divisorial contractions, flips, and fibrations each obtained as follows: The \textit{transition times} are the set of values of $t$ where the quotient $X//_tG$ has stabilizer groups of positive dimension:

$$T := \{ t \in \mathbb{R}, \exists x \in X^s_t, \#G_x = \infty \}.$$ 

Thaddeus \cite{108} reduces the study of the wall-crossings to the case that $G \cong \mathbb{C}^\times$ is a one-parameter subgroup by replacing $X$ with the \textit{master space} $X_{t_1, t_2} := \mathbb{P}(K^{-t_1} \oplus K^{-t_2})//G$ for some $t_1, t_2$, which may be taken to be integers after suitably large tensor power. The residual $\mathbb{C}^\times$-action has git quotients $X_{t_1, t_2}//\mathbb{C}^\times \cong X//_tG$ for $t \in (t_1, t_2)$.

Having reduced to the case of a circle action, one can now check that variation of quotient produces a flip. In the circle group case $G = \mathbb{C}^\times$, let $F \subset X^s_t$ be a component of the fixed point set which is stable at time $t$. Let $\mu_1, \ldots, \mu_k \in \mathbb{Z}$ denote the weights of $\mathbb{C}^\times$ on the normal bundle $N$ to $F$, and $N_i$ the weight space for weight $\mu_i, i = 1, \ldots, k$. Let

$$N_\pm := \bigoplus_{\pm \mu_i > 0} N_i$$

denote the positive resp. negative weight subbundle. For $t_- < t < t_+$ with $t_\pm$ close to $t$ the Hilbert-Mumford criterion and Luna slice theorem imply that the semistable locus changes by replacing a variety isomorphic to $N_-$ with one isomorphic to $N_+$, see \cite{108}. Hence $X//_{t_\pm}G$ is obtained from $X//_{t}G$ by replacing the (weighted)-projectivized bundle $N_-//G$ of with $N_+//G$:

$$(X//_{t_-}G)\backslash(N_-//G) \cong (X//_{t_+}G)\backslash(N_+//G).$$

The definition of the anticanonical bundle implies that the sum of the weights on the anticanonical bundle at any fixed point are positive, and the morphisms are relatively $K$-ample resp. $-K$-ample over the center. Thus, in the absence of singularities, the spaces $X//_tG$ yield a smooth running of the minimal model program.

The symplectic story can be described as follows in terms of Morse theory of the moment map. We suppose that we have a Hamiltonian $U(1)$-action on a symplectic manifold with proper moment map $\Phi : X \to \mathbb{R}$. Let

$$\text{Crit}(\Phi) = \{ x \in X | d\Phi(x) = 0 \}, \quad \text{Critval}(\Phi) = \Phi(\text{Crit}(\Phi))$$

denote the set of critical points resp. critical values of $\Phi$. Given a critical value $c \in \text{Critval}(\Phi)$, we denote by $c_\pm \in \mathbb{R}$ regular values on either side of $c$, so that $c$ is the unique critical value in $(c_-, c_+)$: $\text{Critval}(\Phi) \cap (c_-, c_+) = \{ c \}$. We suppose for simplicity that $\Phi^{-1}(c)$ contains a unique critical point $x_0 \in X$: $\text{Crit}(\Phi) \cap \Phi^{-1}(c) = \{ x_0 \}$. By
the equivariant Darboux theorem, there exist Darboux coordinates \( z_1, \ldots, z_n \) near \( x_0 \) and weights \( \mu_1, \ldots, \mu_n \in \mathbb{Z} \) so that
\[
\Phi(z_1, \ldots, z_n) = c - \sum_{j=1}^{n} \mu_j |z_j|^2 / 2.
\]

In particular, \( \Phi \) is Morse and we denote by \( W^\pm_{x_0} \) the stable and unstable manifolds of \( -\text{grad}(\Phi) \). The gradient flow \( \phi_t : X \to X \) of \( -\text{grad}(\Phi) \) induces a diffeomorphism between level sets on the complement of the stable unstable manifolds:
\[
\Phi^{-1}(c_+) \backslash W^+_{x_0} \to \Phi^{-1}(c_-) \backslash W^-_{x_0}, \quad x \mapsto \phi_t(x), \quad t \text{ such that } \Phi(\phi_t(x)) = c_+.
\]

Assuming the gradient vector field is defined using an invariant metric one obtains an identification of symplectic quotients except on the symplectic quotients of the stable and unstable manifolds:
\[
(X//c_- U(1)) \backslash (W^+_{x_0} // c_- U(1)) \to (X//c_+ U(1)) \backslash (W^-_{x_0} // c_+ U(1)).
\]

By the description from equivariant Darboux, one sees that the symplectic quotients of the stable and unstable manifolds are weighted projective spaces with weights given by the positive resp. negative weights:
\[
(W^+_{x_0} // c_+ U(1)) \cong \mathbb{P}[\pm \mu_i, \pm \mu_i > 0].
\]

Thus \( X//c_- U(1) \) is obtained from \( X//c_+ U(1) \) by replacing one weighted projective space with another.

The change in the symplectic class under variation of symplectic quotient is described by Duistermaat-Heckman theory \([35]\). Let \( c_0 \in \mathbb{R} - \text{Critval}(\Phi) \) be a regular value of \( \Phi \). Consider the product \( \Phi^{-1}(c_0) \times [c_- , c_+] \) for \( c_+ \) close to \( c_0 \). Let \( \pi_C, \pi_R \) be the projections on the factors of \( \Phi^{-1}(c_0) \times [c_- , c_+] \). Choose a connection one-form and denote its curvature
\[
\alpha \in \Omega^1(\Phi^{-1}(c_0))^{U(1)}, \quad \text{curv}(\alpha) \in \Omega^2(X//c_0 U(1)).
\]

Define a closed two-form on \( \Phi^{-1}(c_0) \times [c_- , c_+] \) by \( \omega_0 = \pi_C^* \omega_c + d(\alpha, \pi_R - c_0) \). For \( c_- , c_+ \) sufficiently close to \( c_0 \), \( \omega_0 \) is symplectic and has moment map given by \( \pi_R \).

By the coisotropic embedding theorem, for \( c_- , c_+ \) sufficiently small there exists an equivariant symplectomorphism \( \psi : U \to \Phi^{-1}(c_0) \times [c_- , c_+] \) of a neighborhood \( U \) of \( \Phi^{-1}(c_0) \) in \( X \) with \( \Phi^{-1}(c_0) \times [c_- , c_+] \). Hence the symplectic quotients \( X//c U(1) \) for \( c \) close to \( c_0 \) are diffeomorphic to \( X//c_0 C \), with symplectic form \( \omega_c + (c - c_0) \text{curv}(\alpha) \).

This ends the remark.

The following describes a symplectic version of an mmp running.

**Definition 5.6.** (a) (Simple symplectic flips) Two symplectic manifolds \( V_+ , V_- \) are related by a *simple symplectic flip* if there exists a Hamiltonian vector space \( \tilde{V} \) with moment map \( \Psi : \tilde{V} \to \mathbb{R} \) and a unique critical component mapping to \( 0 \) such that the sum of the weights is positive, \( \sum \mu_i > 0 \), at least two weights are positive, and at least two weights are negative: \( n_\pm := \{ \mu_i > 0 \} \geq 2 \) and
\[
V_\pm := \tilde{V} // \pm U(1)
\]
the symplectic quotients at \( \pm 1 \). In particular, \( V_+ \) is obtained from \( V_- \) by replacing a projective space of dimension \( n_- - 1 \) with a projective space of dimension \( n_+ - 1 \), as in Remark 5.5.

(b) (Symplectic flips) Let \( X_{\pm} \) be non-empty symplectic manifolds. We say that \( X_+ \) is obtained from \( X_- \) by a symplectic flip if \( X_+ \) is obtained locally by a simple symplectic flip: that is, there exist symplectic manifolds \( U, V_{\pm}, V_+ \), open embeddings

\[
U \to X_-, U \to X_+, V_- \to X_-, V_+ \to X_+
\]
such that the following hold:

(i) Each \( X_{\pm} \) is covered by \( U, V_{\pm}, \) that is, \( X_{\pm} = U \cup V_{\pm} \).
(ii) The manifold \( U \) admits a family of symplectic forms \( \omega_{U,t} \in \Omega^2(U), t \in [-\epsilon, \epsilon] \) and embeddings \( i_{\pm} : U \to X_{\pm} \) such that \( i_{\pm}^* \omega_{\pm} = \omega_{U, \pm \epsilon} \).
(iii) The manifolds \( V_{\pm} \) are related by a simple symplectic flip as in part 5.6.
(iv) The difference between symplectic classes on \( X_+ \) and \( X_- \) is a positive multiple of the first Chern class in the following sense: Under the canonical identification \( H^2(X_-) \to H^2(X_+) \) induced by the diffeomorphism in codimension at least four we have for some \( \epsilon > 0 \)

\[
[\omega_+] - [\omega_-] = \epsilon c_1(X_{\pm}).
\]

(c) (Symplectic divisorial contraction) Symplectic divisorial contractions are defined in the same way as symplectic flips, but in this case all but one weight is positive. So that there exists a projection \( \pi : X_+ \to X_- \) with exceptional locus \( E \subset X_+ \) a (weighted)-projective bundle. We require

\[
[\omega_+] = \pi_+^*[\omega_-] + \epsilon (\pi_+^* c_1(X_-) + [E]^v)
\]

where \( [E]^v \in H^2(X_+) \) is the Poincaré dual of the exceptional divisor. Note that this differs from the usual definition of symplectic blow-down because of the presence of the additional change in symplectic class \( c_1(X_-) \).

(d) (Symplectic Mori fibration) By a symplectic Mori fibration we mean a symplectic fibration \( \pi : X_+ \to X_- \) such that the fiber \( \pi^{-1}(x), x \in X_- \) is a monotone symplectic manifold ( the definition of Guillemin-Lerman-Sternberg [18] is more general.) In all our examples flips \( X_- \) will be obtained from \( X_+ \) by a variation of symplectic quotient using a global Hamiltonian circle action, and symplectic Mori fibrations will be Mori fibrations in the usual sense.

(e) (Symplectic mmp running) A symplectic mmp running we mean a sequence \( X = X_0, X_1, \ldots, X_k \) together with, for each \( i = 0, \ldots, k \), a path of symplectic forms \( \omega_{i,t} \in \Omega^2(X_i), t \in [t_i^-, t_i^+] \) such that

\[
\frac{d}{dt} [\omega_{i,t}] = c_1(X_i)
\]

and each \( (X_i, \omega_{i,-}) \) is obtained from \( (X_{i-1}, \omega_{i-1}^+) \) by a symplectic mmp transition for \( i = 1, \ldots, k \).

Remark 5.7. (Exceptional piece) Suppose that \( X_+ \) is obtained from \( X_- \) by a reverse minimal model transition of flip or blow-up type, replacing a projective bundle
\( \mathbb{P}(N_-) \to Z \) with a projective bundle \( \mathbb{P}(N_+) \to Z \). By the constant rank embedding theorem \([68]\), a neighborhood \( U \) of \( X_+ \) is symplectomorphic to a neighborhood \( V \) of the zero section in a symplectic vector bundle \( E_+ \) over \( \mathbb{P}(N_+) \). Any fiber \( \pi^{-1}(z) \) of \( \mathbb{P}(N_+) \to Z \) has a canonical family of isotropic tori,

\[
\pi^{-1}(z) = \cup_{n \in \pi^{-1}(z)} T[n]
\]
given by orbits of a maximal torus \( T \) of the group \( \text{Aut}((N_+)_z) \) of unitary transformations of the fiber \( (N_+)_z \). Similarly, the fibers of \( E_+ \) have canonical family of Lagrangian tori given by the orbits of the maximal torus of the unitary group of its fibers.

By construction \( X_+ \) contains a neighborhood \( U \) of the exceptional locus \( \mathbb{P}(N_+) \) which is a toric fibration over \( \mathbb{P}(N_+) \). The boundary of the neighborhood \( U \) may be taken to be fibered coisotropic by, for example, taking the neighborhood to be a ball in each fiber in the symplectic local model. Collapsing the coisotropic boundary of \( U \) yields as in Lerman \([66]\) a fibration with compact toric fibers denoted \( \overline{U} \).

We introduce a class of Lagrangians associated to minimal model transitions which we call regular; these will later be shown to be Floer non-trivial.

**Definition 5.8.** Let \( X \) be obtained by a reverse flip or blow-up with center \( Z \). Let \( \mathbb{P}(N_+) \to Z \) denote the projective bundle produced by the reverse flip or blow-up, and \( E_+ \to \mathbb{P}(N_+) \) the normal bundle of \( \mathbb{P}(N_+) \) in \( X \) as above.

(a) A Lagrangian \( L \subset \mathbb{P}(N_+) \) is toric if the restriction of \( \pi : \mathbb{P}(N_+) \to L \) has image a submanifold \( L_Z \) of \( Z \) with fiber a standard Lagrangian (in the fiber) torus \( L_F \subset \mathbb{P}(N_+) \) in a fiber of \( \mathbb{P}(N_+) \).

(b) A Lagrangian \( L \) in \( E_+ \) is toric if the restriction of \( \pi : E_+ \to \mathbb{P}(N_+) \) has image a toric Lagrangian in \( \mathbb{P}(N_+) \) with fiber a torus \( L_F \) in the fiber of \( E_+ \).

To specify which Lagrangians are regular we need to recall a few facts about disks in toric varieties.

**Remark 5.9.** (Toric manifolds as symplectic/git quotients) Any smooth toric Deligne-Mumford stack with projective coarse moduli space has a presentation as a geometric invariant theory quotient \([28, \text{Section 3.1}]\). Take \( V \) be a Hermitian vector space of dimension \( k \) with an action of a torus \( G \). Denote by \( \mu_1, \ldots, \mu_k \) the weights of the action in the weight lattice \( \mathfrak{g}_G^\vee \subset \mathfrak{g}_Z \). Choose an identification \( V \to \mathbb{C}^k \) so that the weight on the \( i \)-th factor is \( \mu_i \). A linearization of \( V \) is determined by an equivariant Kähler class \( \omega_{V,G} \in H^2_{\text{et}}(V) \cong \mathfrak{g}_Z \). The geometric invariant theory \( V//G \) is, if locally free, the quotient of the semistable locus

\[
V^{ss} = \{ (v_1, \ldots, v_k) \in V, \quad \text{span}\{\mu_k|v_k \neq 0\} \ni \omega_{V,G} \}
\]

by the action of \( G \). Suppose \( G \) is contained in a maximal torus \( H \) of the unitary group of \( V \); then the residual torus \( T = H/G \) acts on \( X = V//G \) making \( X \) into a toric variety.

The moment polytope for the action of the residual torus on the quotient can be computed from the moment polytope for the original action. We assume that \( \omega_{V,G} \)
has an equivariant extension to $H = (\mathbb{C}^*)^k$, written in terms of the standard basis vectors $e_i^\vee \in \mathfrak{h}^\vee$ as

$$\omega_{V,H} = \sum c_i e_i^\vee \in \mathfrak{h}^\vee \cong H^2_H(V).$$

Choose coordinates $z_1, \ldots, z_k$ on $V$ so that the symplectic form is $(1/2i) \sum_{j=1}^k dz_j \wedge d\bar{z}_j$. A moment map for the $H$-action is given by the formula $(z_i)_{i=1}^k \mapsto (c_i - |z_i|^2/2)_i$. Let $\nu_i$ denote the image of the $i$-standard basis vector $-\epsilon_i \in \mathfrak{h}$ under the projection $\mathfrak{h} \to \mathfrak{g}$. The residual action of the torus $T = H/G$ on $X = V/G$ has moment image

$$P = \{\lambda \in t^\vee \mid \langle \lambda, \nu_j \rangle \geq c_j, \quad j = 1, \ldots, k\}.$$

**Remark 5.10.** (Primitive disks) The description of a toric manifold as a git quotient in the previous remark leads to the following description of disks with boundary in a torus orbit. Let $X$ be a compact symplectic toric manifold equipped with the action of a torus $T$, realized as a symplectic quotient of a vector space $V \cong \mathbb{C}^k$ by the action of a torus $G$, and let $L \subset V$ be a Lagrangian orbit of $T$. Let $\tilde{L} \cong (S^1)^k \subset V$ denote the lift of $L$ to $V$, given as the orbit of a point $(\tilde{\mu}_1, \ldots, \tilde{\mu}_k)$ under the standard torus action:

$$\tilde{L} = \{ (e^{i\theta_1} \tilde{\mu}_1, \ldots, e^{i\theta_k} \tilde{\mu}_k) \mid \theta_1, \ldots, \theta_k \in \mathbb{R} \}.$$

Let $C = \{z \in \mathbb{C} \mid |z| \leq 1\}$ denote the unit disk. For each $i = 1, \ldots, k$ there is a family of disks $\tilde{u}_i : (C, \partial C) \to (\tilde{V}, \tilde{L})$ of the form

$$\tilde{u}_i(z) = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{i-1}, \tilde{\mu}_i z, \tilde{\mu}_{i+1}, \ldots, \tilde{\mu}_k).$$

By passing to the quotient we obtain a collection of disks $u_i : (C, \partial C) \to (X, L)$ called the **primitive disks**.

More generally, disks in toric varieties are classified by a result of Cho-Oh [29]: a **Blaschke product** of degree $(d_1, \ldots, d_n)$ is a map from the disk $C := \{|z| \leq 1\} \subset \mathbb{C}$ to $\mathbb{C}^n$ with boundary in a toric Lagrangian prescribed by coefficients $a_{i,j} \in \mathbb{C}$ with $|a_{i,j}| < 1$ for $i \leq n$ and $j \leq d_i$.

$$u : C \to \mathbb{C}^n, \quad z \mapsto \left( \prod_{j=1}^{d_i} \frac{z - a_{i,j}}{1 - za_{i,j}} \right)_{i=1, \ldots, n}.$$

As in Cho-Oh [29], the products (91) are a complete description of holomorphic disks with boundary in $L$. Since the image of $\tilde{u}(z)$ is disjoint from the semistable locus, the Blaschke products descend to disks $u : (C, \partial C) \to (X, L)$. We compute their Maslov index using the splitting $\tilde{u}^* TV \cong u^* TX \oplus \mathfrak{g}$. Since the Maslov index is additive, and the second factor has Maslov index zero, the Maslov index of the disk is given by $I(u) = \sum_{i=1}^k 2d_i$. Thus $I(u)$ is twice the sum of the intersection number with the canonical divisor (the disjoint union of prime invariant divisors.)

Returning to the case of primitive disks, each primitive disk $u_j$ intersects the $j$-th prime boundary divisor once, and is disjoint from the remaining divisors. Furthermore, the area of this disk is given by $A(u_j) = \langle \lambda, \nu_j \rangle - c_j$ by a standard computation in Darboux coordinates.
Remark 5.11. (Primitive disks in the toric piece) Let us return to the case of a symplectic manifold $X$ obtained by a reverse blow-up or flip with point center. Let $U \subset X$ be the corresponding toric piece from 5.7. A toric Lagrangian $L \subset U$ is the boundary of $\dim(X) + 1$ primitive disks of Maslov index two contained in $U$; we call these the primitive disks in $X$. Of course there are additional disks that lie in the compactification $\overline{U}$.

Example 5.12. (Blow-up of a product of projective lines) The disks in the case of blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ are shown in Figure 29. The image of each disk is one-dimensional, since the angular direction in each disk is tangent to the level sets of the moment map. On the other hand, the image of each disk (shown roughly as dotted lines in the figure) is non-linear since the map from $V$ to $X = V//G$ is non-linear. In Figure 29 the areas of the three primitive disks of smallest area are all equal.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure29}
\caption{Representation of Maslov index two disks}
\end{figure}

These disks play a role in the following definition:

Definition 5.13. Let $X$ be a symplectic manifold obtained by a symplectic reverse flip or blow-up. A Lagrangian $L \subset X$ is regular if

(a) $L$ is a toric Lagrangian in the toric piece $U \subset X$ of 5.7.
(b) the primitive disks $u : (D, \partial D) \to (X, L)$ contained in $U$ all have equal area $A_0$ and
(c) the primitive disk $u : (D, \partial D) \to (\overline{U}, L)$ meeting the complement $U \setminus U$ has area greater than $A_0$.

5.2. Mmp runnings for toric manifolds. In this and the following subsections we give a sequence of examples of mmp runnings. Each of these examples is in some sense obtained by variation of quotient as in Remark 5.5.

The minimal model program for toric varieties was established by Reid [89]. Here we describe mmp runnings given by shrinking the moment polytope, see González-Woodward [47] and Pasquier [85]. Let $X = V//G$ be given as the symplectic quotient of a symplectic vector space $V \cong \mathbb{C}^k$ by the action of a torus $G \subset H \cong U(1)^k$. The equivariant first Chern class $c_1^H(V) \in H^2_V(V)$ of $V$ is represented by the element
(1, 1, ..., 1) in $H^2_{	ext{H}}(V) \cong \mathbb{h}^k \cong \mathbb{R}^k$. Variation of symplectic quotient in this directions produces a running of the mmp, by Remark 5.5. Hence if the constants $c_j$ from (1) are generic, a running of the mmp is given by the sequence of toric varieties $X_t$ corresponding to the sequence of polytopes

$$P_t = \{ \mu \in t^\vee \mid \langle \mu, \nu_j \rangle \geq c_j + t, \ j = 1, \ldots, k \}.$$

Here we assume that each $P_t$ defines a smooth toric variety or toric Deligne-Mumford stack, so that $X_t$ is the toric stack whose fan is dual to $P_t$. The transition times are the set of times

$$T := \{ t \mid \exists \mu \in t^\vee, \nu_j \mid \langle \mu, \nu_j \rangle = c_j + t \} \text{ is linearly dependent} \}.$$

For generic choices of constants, one obtains an mmp running in which every stage is an orbifold. If the facets stay the same, the transition is a flip; if a facet disappears then the transition is a divisorial contraction (where the divisor is the preimage of the disappearing face). For example, in Figure 1 there are two divisorial contractions, occurring at the dots shaded in the diagram. Finally if a generic point $\mu$ in $P_{t_0}$ satisfies $\langle \mu, \nu_j \rangle = c_j + t_0$ then $X_t$ undergoes a Mori fibration at $t_0$ with base given by the toric variety $X_{t_0}$ with polytope $P_{t_0}$; one can then continue the running with $X_{t_0}$ to obtain an extended running.

The regular Lagrangians are described as follows. Let $\mu \in P$ and $t(\mu) = \min_j \langle \mu, \nu_j \rangle - c_j$. The real number $t(\mu)$ is the time at which $\mu$ “disappears” under the mmp. Suppose the set

$$N(\mu) := \{ \nu_j \mid \langle \mu, \nu_j \rangle - c_j = t \}$$

is linearly dependent. Then $L(\mu) := \Phi^{-1}(\mu)$ satisfies the first two parts of the definition of regularity in Definition 5.13. To see this we first compute the areas of disks with boundary on the Lagrangian. Suppose that $X$ is realized as the git quotient of a vector space $V \cong \mathbb{C}^k$ by a torus $G$. We may assume that $\dim(X) > 1$ so that the real codimension of the unstable locus is at least four. Let $\tilde{L}_\mu$ denote the preimage of $L_\mu$ in $\mathbb{C}^k$. The Lagrangian $\tilde{L}_\mu$ is a Lagrangian torus orbit of the group $U(1)^k$ acting on $\mathbb{C}^k$, that is, $\tilde{L}_\mu = U(1)^k(\tilde{\mu}_1, \ldots, \tilde{\mu}_k)$ for some constants $(\tilde{\mu}_1, \ldots, \tilde{\mu}_k)$. Each map to a holomorphic disk to $\mathbb{C}^k$ with boundary in $\tilde{L}_\mu$ corresponds to a collection of maps from disks to $\mathbb{C}$ with boundary on $U(1)\tilde{\mu}_j$ as in Remark 5.10. It follows that if $L_\mu$ is a regular Lagrangian then the minimal areas of the holomorphic disks are $t$, these correspond to disks in the components corresponding to facets distance $t$ from $\mu$, and each of these disks has Maslov index two. The last assumption in Definition 5.13 holds if the other facets are sufficiently far away. For example, in Figure 1 we have two regular Lagrangians, given as the inverse images of the shaded dots in the Figure under the moment map.

5.3. Mmp runnings for moduli spaces of polygons. The moduli space of polygons is the quotient of a product of two-spheres by the diagonal action of the group of Euclidean rotations in three-space. This moduli space is often used as one of the primary examples of geometric invariant theory/ symplectic quotients, see for example Kirwan [57], and is famous as an “almost toric” example in a sense we will
describe. It is also a special case of the moduli space of flat bundles on a Riemann surface, which appears in a number of constructions in mathematical physics.

First recall the Hamiltonian structure of the two-sphere via its realization as a coadjoint orbit. Let $S^2 \subset \mathbb{R}^3$ the unit two-sphere equipped with the $SO(3)$-invariant symplectic form with area one. The action of $SO(3)$ on $S^2$ is naturally Hamiltonian. Viewing $SO(3)$ as a coadjoint orbit in $\mathfrak{so}(3)^\vee \cong \mathbb{R}^3$, the moment map is the inclusion of $S^2$ in $\mathbb{R}^3$, as a special case of the Kirillov-Kostant-Souriau construction \[104\].

Starting with scaled two-forms on a collection of spheres we form a symplectic manifold of higher dimension by taking products. Let $n \geq 1$ be an integer, and $\lambda_1, \ldots, \lambda_n > 0$ a sequence of positive real numbers. The product

$$\tilde{X} = (S^2 \times \ldots \times S^2, \lambda_1 \pi_1^* \omega + \ldots + \lambda_n \pi_n^* \omega)$$

is naturally a symplectic manifold of dimension $2n$. The group $SO(3)$ acts diagonally on $\tilde{X}$ with moment map

$$\Psi : \tilde{X} \to \mathbb{R}^3, \quad (v_1, \ldots, v_n) \mapsto v_1 + \ldots + v_n.$$ 

The symplectic quotient $X = \tilde{X} \, / \, SO(3)$ is the moduli space of $n$-gons

$$P(\lambda_1, \ldots, \lambda_n) = \left\{(v_1, \ldots, v_n) \in (\mathbb{R}^3)^n \mid \|v_i\| = \lambda_i, \forall i, \sum_{i=1}^n v_i = 0 \right\}.$$ 

The moduli space may be alternatively realized from the geometric invariant theory perspective as a quotient by the complexified group. We view each $v_i$ as a point in the projective line $\mathbb{P}^1$. A tuple $(v_1, \ldots, v_n) \in \mathbb{P}^1$ is semistable iff for each $w \in \mathbb{P}^1$, the slope inequality

$$\sum_{v_i = w} \lambda_i \leq \sum_{v_i \neq w} \lambda_i$$

holds \[82\]. Then $P(\lambda_1, \ldots, \lambda_n)$ is the quotient of the semistable locus by the action of $SL(2, \mathbb{C})$ by a special case of the Kempf-Ness theorem \[56\].

A running of the mmp for the moduli space of polygons is given by varying the lengths in a uniform way. The first Chern class of the product of spheres $\tilde{X}$ is the class of the form $\sum_{j=1}^n \pi_j^* \omega_j$ where $\pi_j$ is projection onto the $j$-th factor. It follows from Remark \[5,5\] that the sequence of moduli spaces $P(\lambda_1 - t, \ldots, \lambda_n - t)$ is a minimal model program for $P(\lambda_1, \ldots, \lambda_n)$. Transitions occur whenever there are one-dimensional polygons. That is,

$$\mathcal{T} := \left\{ t \mid \exists \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}, \sum_{j=1}^n \epsilon_j (\lambda_j - t) = 0 \right\}.$$ 

We may assume that the number of positive signs is greater at least the number of minus signs, by symmetry. Equivalently, $\mathcal{T}$ is the set of times for which there exist polystable configurations where for some $w_+ \in \mathbb{P}^1$, each $v_i \in \{ w_+, w_- \}$ and

$$\sum_{v_i = w_+} (\lambda_i + t) = \sum_{v_i = w_-} (\lambda_i + t).$$
For example, if the initial configuration is $\lambda = (10, 10, 12, 13, 14)$ then there are three transitions, at $t = 5, 7, 9$, corresponding to the equalities $5 + 5 + 7 = 8 + 9, 3 + 3 + 6 = 5 + 7, 1 + 1 + 5 = 3 + 4$. There is a final transition when the smallest edge acquires zero length.

Each flip or blow-down replaces a projective space of dimension equal to the number of plus signs, minus one, with a projective space of dimension equal to the number of minus signs, minus one. One can explicitly describe the projective spaces involved in the flips as follows. For $\pm \in \{+, -\}$ let $I_{\pm} = \{i, \epsilon_i = \pm 1\}$ denote the set of indices with positive resp. negative signs. Let $t_+ > t$ resp. $t_- < t$ and let

$$S_{\pm} = \{(v_1, \ldots, v_n) \in P(\lambda_1 - t_{\pm}, \ldots, \lambda_n - t_{\pm}) \mid \mathbb{R}_{>0}v_i = \mathbb{R}_{>0}v_j, \forall i, j \in I_{\pm}\}$$

denote the locus where the vectors with indices in $I_{\pm}$ point in the same direction. (Since $\pm$ is a variable, this is a requirement for $I_+$ or $I_-$ but not both.) Thus $S_{\pm}$ is the symplectic quotient of a submanifold $\tilde{S}_{\pm}$ of the product of $S^2$’s with only positive resp. negative weights for the circle action. Thus $S_+$ is a projective space and the flip replaces $S_+$ with $S_-$:

$$
\begin{array}{ccc}
P(\lambda_1 - t_-, \ldots, \lambda_n - t_-) & \rightarrow & P(\lambda_1 - t_+, \ldots, \lambda_n - t_+) \\
S_- & \rightarrow & \text{pt} \\
\uparrow & & \uparrow \\
S_+ & \leftarrow & 
\end{array}
$$

For example, in the case of lengths $10, 10, 12, 13, 14$ for the transition at $t = 5$, the configurations with edge lengths $5 + \epsilon, 5 + \epsilon, 7 + \epsilon, 8 + \epsilon, 9 + \epsilon$ with $9 + \epsilon, 8 + \epsilon$ edges approximately colinear are replaced with configurations with edges with lengths $5 - \epsilon, 5 - \epsilon, 7 - \epsilon$ approximately colinear. The first set of configurations is a two-sphere, corresponding to a moduli space of quadrilaterals, while the latter set of configurations is a point, corresponding to a moduli space of triangles. It follows that the corresponding transition is a blow-down.

There are two situations in which one obtains a Mori fibration. First, in the case that one of the $\lambda_i$’s becomes zero, say $\lambda_i - t$ is very small in relation to the other weights, there is a fibration

$$P(\lambda_1 - t, \ldots, \lambda_n - t) \rightarrow P(\lambda_1 - t, \ldots, \lambda_{i-1} - t, \lambda_{i+1} - t, \ldots, \lambda_n - t).$$

Symplectically, this is a special case of the results of Guillemin-Lerman-Sternberg [4S, Section 4] while from the algebraic point of view, in this case the value of $v_i$ does not affect the semistability condition, and forgetting $v_i$ defines the fibration. In the case that the moduli space becomes empty before one of the $\lambda_i$’s reaches zero, the moduli space is a projective space at the last stage, by the same discussion as in the case of flips. For example, in the case of lengths $10, 10, 12, 13, 14$ one obtains a fibration $P(1, 1, 2, 1, 3, 1, 4, 1) \rightarrow P(2, 3, 4)$ when $t = 10$ over the moduli space $P(2, 3, 4)$ which is a point. Therefore, the moduli space at $t = 10 - \epsilon$ is a product of two-spheres: $P(\epsilon, \epsilon, 2 + \epsilon, 3 + \epsilon, 4 + \epsilon) \cong S^2 \times S^2$. The conclusion is that $P(10, 10, 12, 13, 14)$ is a thrice-blow-up of $S^2 \times S^2$, that is, a del Pezzo surface.
The second way that one may obtain a Mori fibration is that one can reach a chamber in which the moduli space is empty, because one of the lengths, say $\lambda_i - t$ is so long compared to the others that the sum of the remaining lengths is smaller than the first length:

$$\sum_{j \neq i} \lambda_j - t < \lambda_i - t.$$ 

In this case, the last non-empty moduli space is a fiber bundle over the space of reducible polygons corresponding to the transition, which form a point. The fiber is a projective space, by a repeat of the arguments above. See Moon-Yoo [81] for more details.

A natural family of Lagrangian tori is generated by the bending flows studied in Klyachko [58] and Kapovich-Millson [80]. Fix a subset $I \subset \{1, \ldots, n\}$ of the edges of the polygon and define a diagonal length function

$$\tilde{\Psi}_I : S^2 \times \ldots \times S^2 \to \mathbb{R}_{\geq 0}, \quad (v_1, \ldots, v_n) \mapsto \|v_I\|,$$

$$v_I := \sum_{i \in I} v_i.$$ 

The diagonal length is smooth on the locus where it is positive.

The Hamiltonian flow of the diagonal length function is given by rotating part of the polygon around the diagonal. The following is, for example, explained in [80, Section 3]:

**Lemma 5.14.** The function $\tilde{\Psi}_I$ generates on $\tilde{\Psi}_I^{-1}(\mathbb{R}_{>0})$ a Hamiltonian circle action given by rotating the vectors $v_i, i \in I$ around the axis spanned by $v_I := \sum_{i \in I} v_i$:

$$v_i \mapsto R_{\theta, v_I} v_i, \quad i = 1, \ldots, n$$

where $R_{\theta, v_I}$ is rotation by angle $\theta$ around the span of $v_I$. Furthermore if $\mathcal{T}$ is a collection of subsets such that for all $I, J \in \mathcal{T}$ either $I \subseteq J$ or $J \subseteq I$ then the associated flows commute.

**Proof.** We provide a proof for completeness. By the symplectic cross-section theorem [49, Theorem 26.7], the inverse of the interior of the positive Weyl chamber $\mathbb{R}_{>0}$ under the moment map $\Psi$ is a symplectic submanifold $\Psi^{-1}(\mathbb{R}_{>0} \times \{0\} \times \{0\}) \subset \tilde{X}$. This inverse image is the locus

$$\tilde{X}_I = \left\{ (v_1, \ldots, v_n) \in \tilde{X} \mid v_I \in \mathbb{R}_{>0} \times \{0\} \times \{0\} \subset \mathbb{R}^3 \right\}.$$ 

The flow-out of $\tilde{X}_I$ is $SO(3) \tilde{X}_I = SO(3) \times_{SO(2)} \tilde{X}_I$. Any $SO(2)$-equivariant Hamiltonian diffeomorphism of $\tilde{X}_I$ extends uniquely to a Hamiltonian diffeomorphism of $SO(3) \tilde{X}_I$ which is equivariant with respect to the $SO(3)$-action. Now on $\tilde{X}_I$, the function $\tilde{\Psi}_I$ is the first component of the moment map and so the flow of $\tilde{\Psi}_I$ is rotation around the first axis. It follows that the flow of $\tilde{\Psi}_I$ is rotation around the line spanned by $\sum_{i \in I} v_i$, as long as this vector is non-zero. In particular the flow of $\tilde{\Psi}_I$ is $SO(3)$-equivariant and so descends to a function $\Psi_I$ generating a circle action on a dense subset of $P(\lambda_1, \ldots, \lambda_n) = \tilde{X} / SO(3)$.

Next we explain why these circle actions combine to a torus action. We assume that the ordering of vectors $v_1, \ldots, v_n$ is such that $I, J$ consist of adjacent indices. If
$I \subset J$ then the vectors $v_I$ and $v_J$ break the polygon into three pieces, and the flows of $\Psi_I$ and $\Psi_J$ are rotation of the first and third pieces around the diagonals $v_I, v_J$ respectively. In particular, these flows commute. □

![Triangulated polygon](image)

**Figure 30.** Triangulated polygon

Given a triangulation we associate a moment map for a densely-defined torus action as follows. Fix a triangulation of the abstract $n$-gon with edges $v_1, \ldots, v_n$ whose diagonals are the sums of vectors in subsets $I_1, \ldots, I_n \subset \{1, \ldots, n\}$, such that each $I_j \subset I_{j+1}$. See Figure 30. The triangulation gives rise to a map

$$\Psi : P(\lambda_1, \ldots, \lambda_n) \to \mathbb{R}^{n-3}_{\geq 0}, \quad [v_1, \ldots, v_n] \to (\|v_{I_j}\|)_{j=1}^{n-3}$$

given by taking the edge lengths of the diagonals. The discussion above shows that the map $\Psi$ is, where smooth, a moment map for the action of an $n-3$-dimensional bending torus $T \cong U(1)^{n-3}$ which acts as follows: Let $(\exp(i \theta_1), \ldots, \exp(i \theta_{n-3})) \in T$ and $[v_1, \ldots, v_n]$ be an equivalence class of polygons. For each diagonal $v_I$, divide the polygon into two pieces along $v_I$, and rotate one of those pieces, say $(v_i)_{i \in I}$ around the diagonal by the given angle $\theta_I$. The resulting polygon is independent of the choice of which piece is rotated, since polygons related by an overall rotation define the same point in the moduli space.

The regular Lagrangian tori are described as follows as fibers of the Goldman map for which all triangles have the same “looseness”. Suppose that $P(\lambda_1 - t, \ldots, \lambda_n - t)$ is an mmp running for $P(\lambda_1, \ldots, \lambda_n)$. As noted in Example 5.2, mmp transitions correspond to partitions

$$\{1, \ldots, n\} = I_+ \cup I_-,$$

$$\sum_{i \in I_+} \lambda_i - t = \sum_{i \in I_-} \lambda_i - t.$$

For each triangle $T$ in the triangulation with labels $\mu_1, \mu_2, \mu_3$ we denote by $l(T)$ the looseness of the triangle

$$l(T) := \min_{i,j,k \text{ distinct}} \frac{(\mu_i + \mu_j - \mu_k)}{2}.$$
In other words, the looseness measures the failure of the triangle to be degenerate. The looseness is a tropical version of the area of the triangle given by Heron’s formula:

\[
A(T) = \frac{1}{4} \sqrt{(\mu_1 + \mu_2 + \mu_3) \prod_{i \neq j \neq k} (\mu_i + \mu_j - \mu_k)}.
\]

A labelling \( \mu \in \mathbb{R}^{n-3} \) is called regular if \( l(T) \) is independent of \( T \) and greater than \( \min_i \lambda_i \). For example, if the edge lengths are 2, 3, 4, 7, and the triangulation separates the first two edges from the last two, then a regular triangulation is obtained by assigning 4 to the middle edge, so that the looseness of each triangle is \( 1 = 2 + 3 - 4 = 4 + 4 - 7 \).

**Figure 31.** A regular triangulation of a quadrilateral

**Lemma 5.15.** If a labelling \( \mu \in \mathbb{R}^{n-3} \) is regular for \( \lambda \) with small looseness \( l(T) \), \( T \in T \) then the corresponding Lagrangian \( \Psi^{-1}(\mu) \) is regular in the sense of Definition 5.13.

**Proof.** A local toric structure is given by choosing a triangulation compatible with the partition into positive and negative edges, and the action of the bending torus \( T \) above. Let \( v \in P(\lambda_1 - t, \ldots, \lambda_n - t) \) denote the one-dimensional polygon corresponding to the transition time. Each triangle in the triangulation is degenerate for \( v \) and so for each \( T, \mu_i + \mu_j = \mu_k \) for some edges \( i, j, k \) of \( T \). The inequalities defining \( \Psi_T(T) \) near \( v \) are of the form \( \mu_i + \mu_j \geq \mu_k \) as \( i, j, k \) range over all possible indices. It follows that the polytope defining the image of the map \( \Psi \) is given locally by the triangle inequalities, \( l(T) \geq 0 \) for each of the \( n - 2 \) triangles in the triangulation:

\[
\Psi_T(P(\lambda_1, \ldots, \lambda_n)) = \left\{ (\mu_1, \ldots, \mu_{n-3}) \in \mathbb{R}^{n-3}_{\geq 0} \mid \forall T \in T, (T = \{v_i, v_j, v_k\}) \implies \mu_i + \mu_j \geq \mu_k \right\}.
\]

The regular condition is then the same as the condition for the Lagrangian to be regular in the toric case. \(\square\)

See Nishinou-Nohara-Ueda [83] and Nohara-Ueda [84] for another approach to Floer-non-triviality of Lagrangian tori in these spaces.
5.4. Mmp runnings for moduli spaces of flat bundles. The moduli space of flat bundles on a Riemann surface is an example of an infinite-dimensional symplectic quotient, and studied in for example Atiyah-Bott [6]. Let Σ be a compact Riemann surface and G a compact Lie group. The trivial G-bundle $P = \Sigma \times G$ has space of connections $\mathcal{A}(P)$ canonically identified with the space of $g$-valued one-forms which are $G$-invariant and induce the identity on the vertical directions:

$$\mathcal{A}(P) := \{ \alpha \in \Omega^1(\Sigma \times G), \quad \alpha(\xi_P) = \xi, \forall \xi \in g \}$$

where $\xi_P \in \text{Vect}(P)$ is the corresponding vector field. The space $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(\Sigma, g)$, and has a natural symplectic structure given by

$$\Omega^1(\Sigma, g)^2 \to \mathbb{R}, \quad (a_1, a_2) \mapsto \int_{\Sigma} (a_1 \wedge a_2).$$

Here $(a_1, a_2) \in \Omega^2(\Sigma)$ is the result of composition

$$\Omega^1(\Sigma, g)^{\otimes 2} \to \Omega^2(\Sigma, g^{\otimes 2}) \to \Omega^2(\Sigma, g^{\otimes 2}) \to \Omega^2(\Sigma, \mathbb{R})$$

where the latter is induced by an invariant inner product $g \times g \to \mathbb{R}$. The action of the group $G(P)$ of gauge transformations on $\mathcal{A}(P)$ by pullback is Hamiltonian with moment map given by the curvature:

$$\mathcal{A}(P) \to \Omega^2(\Sigma, P(g)), \quad A \mapsto F_A.$$  

The symplectic form on $\mathcal{A}(P)$ descends to a closed two-form on the symplectic quotient

$$\mathcal{R}(\Sigma) := \{ A \in \mathcal{A}(P) \mid F_A = 0 \}/G(P)$$

the moduli space of flat connections on the trivial bundle. The tangent space to $\mathcal{R}(\Sigma)$ at the isomorphism class of a connection $A$ has a natural identification

$$T_{[A]}\mathcal{R}(\Sigma) \cong H^1(d_A)$$

with the cohomology $H^1(d_A)$ of the associated covariant derivative $d_A$ in the adjoint representation. The Hodge star furnishes a Kähler structure on the moduli space.

Extensions to the case with boundary are given in, for example, Mehta-Seshadri [78]. Suppose $\Sigma$ is a compact oriented surface of genus $g$ with $n$ boundary components. That is, $\Sigma$ is obtained from a closed compact oriented surface of genus $g$ by removing $n$ disjoint disks as in Figure 32. Let $Z_k \subset \Sigma$ be the $k$-th boundary circle, and $[Z_k] \in \pi_1(\Sigma)$ the class defined by a small loop around the $k$-th boundary component for $k = 1, \ldots, n$. Let $G = SU(2)$ denote group of special unitary $2 \times 2$ matrices. The space of conjugacy classes $G/\text{Ad}(G)$ is naturally parametrized by an interval:

$$[0, 1/2] \to G/\text{Ad}(G), \quad \lambda \mapsto \text{diag}(\exp(\pm 2\pi i \lambda)).$$

Let $\lambda_1, \ldots, \lambda_n \in [0, 1/2]$ be labels attached to the boundary components. Choose a base point in $\Sigma$ and let $\pi_1(\Sigma)$ denote the fundamental group of homotopy classes of based loops. Each loop $Z_k$ defines an element $[Z_k] \in \pi_1(\Sigma)$, by connecting $Z_k$ to a base point, which is well-defined up to conjugacy. For numbers $\mu_1, \mu_2$ we denote by $\text{diag}(\mu_1, \mu_2)$ the diagonal $2 \times 2$ matrix with diagonal entries $\mu_1$ and $\mu_2$. Let $\mathcal{R}(\lambda_1, \ldots, \lambda_n)$ denote the moduli space of isomorphism classes flat bundles with holonomy around the boundary circles given by $\exp(2\pi i \text{diag}(\lambda_k, -\lambda_k)), k = 1, \ldots, n.$
Since any flat bundle is described up to isomorphism by the associated holonomy representation of the fundamental group, we have the explicit description

\[ R(\lambda_1, \ldots, \lambda_n) = \left\{ \varphi \in \text{Hom}(\pi_1(\Sigma), SU(2)) \mid \varphi([Z_k]) \in SU(2) \exp(2\pi i \text{diag}(\lambda_k, -\lambda_k)) \right\} / SU(2). \]

By Mehta-Seshadri \[78], the moduli space of flat bundles may be identified with the moduli space of parabolic bundles with weights \(\lambda_1, \ldots, \lambda_n\). Here a parabolic bundle means a holomorphic bundle \(E \to \Sigma\) on closed Riemann surface \(\Sigma\) with markings \(z_1, \ldots, z_n \in \Sigma\) with the additional datum of one-dimensional subspace \(L_i \subset E_{z_i}\) in the fiber \(E_{z_i}\) at each marking, together with the weights \(\lambda_1, \ldots, \lambda_n\).

In the case of rank two bundles there is a simple interpretation of these moduli spaces in terms of spherical polygons. Namely \(\pi_1(\Sigma)\) is generated by homotopy classes of paths \(\gamma_1, \ldots, \gamma_n : S^1 \to \Sigma\) with the single relation \([\gamma_1] \cdots [\gamma_n] = 1 \in \pi_1(\Sigma)\). Thus a representation of the fundamental group corresponds to a tuple \(g_1, \ldots, g_n \in SU(2), \ g_1 \cdots g_n = e\) where \(e \in SU(2)\) is the identity. Consider the polygon in \(SU(2)\) with vertices \(e, g_1, g_1g_2, \ldots, g_1 \cdots g_n = e\). Choose a metric on \(SU(2) \cong S^3\) invariant under the left and right actions. Because the metric on \(SU(2)\) is invariant under the right action, the distance between the \(j-1\)-th and \(j\)-th vertices is the distance between \(e\) and \(g_j\). Using invariance again it suffices to assume that \(g_i = \text{diag}(2\pi i(\lambda_j, -\lambda_j))\) in which case the distance is \(\lambda_j\), once the metric is normalized so the maximal torus has volume one. Via the identification of \(S^3\) with \(SU(2)\), any representation gives rise to a polygon in \(S^3\) with edge lengths \(\lambda_1, \ldots, \lambda_n\). Conversely, any closed polygon determines a representation assigning the edge elements to the generators of \(\pi_1(\Sigma)\), and this correspondence is bijective up to isometries of the three-sphere.

For generic weights the moduli space of flat bundles has a smooth running of the mmp given by varying the labels in a uniform way. First, a result of Boden-Hu [10] and Thaddeus [108, Section 7] shows that varying the labels leads to a generalized flips in the sense that all conditions are satisfied except the condition that the morphisms to the singular quotient are relatively ample. For this the variation of Kähler class should be in the canonical direction as we now explain.
In the case without boundary, the anticanonical class was computed by Drezet-Narasimhan [34] and in the case of parabolic bundles by Biswas-Raghavendra [12]; see Meinrenken-Woodward [70] for a symplectic perspective. The anticanonical class is expressed in terms of the symplectic class and the line bundles $L_j$ associated to the eigenspaces of the holonomy around the boundary components by

$$c_1(\mathcal{R}(\lambda_1, \ldots, \lambda_n)) = 4[\omega_{\mathcal{R}(\lambda_1, \ldots, \lambda_n)}] - \sum_{j=1}^{n}(4\lambda_j - 1)2c_1(L_j).$$

In particular, if all weights $\lambda_i = 1/4$ then the moduli space is Fano.

**Proposition 5.16.** The moduli space $\mathcal{R}(\lambda_1, \ldots, \lambda_n)$ has a smooth running of the mmp given by the sequence of moduli spaces

$$(94) \quad \mathcal{R} \left( \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right).$$

**Proof.** This family can be produced as a variation of symplectic quotient using the construction of [70] as follows: Let $LG$ denote the loop group of $G = SU(2)$, $\mathcal{R}$ denote the moduli space of flat $G$-connections with framings on the boundary equipped with its natural Hamiltonian action of $LG^n$, and $O_{\lambda_1}, \ldots, O_{\lambda_n}$ the $LG$-coadjoint orbits corresponding to $\lambda_1, \ldots, \lambda_n$. Then $\mathcal{R}(\lambda_1, \ldots, \lambda_n)$ has a realization as a symplectic quotient

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n) = (\mathcal{R} \times O_{\lambda_1} \times \ldots \times O_{\lambda_n})/LG^n.$$

Consider the product of anticanonical bundles

$$(95) \quad K_{\mathcal{R}}^\vee \boxtimes K_{O_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{O_{\lambda_n}}^\vee \to \mathcal{R} \times O_{\lambda_1} \times \ldots \times O_{\lambda_n}$$

in the sense of [70]. Its total space minus the zero section has closed two form given by

$$\pi^*\omega_{\mathcal{R} \times O_{\lambda_1} \times \ldots \times O_{\lambda_n}} + d(\alpha, \phi) \in \Omega^2((K_{\mathcal{R}}^\vee \boxtimes K_{O_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{O_{\lambda_n}}^\vee) - \{0\})$$

where $\alpha$ is a connection one-form and $\phi$ is the logarithm of the norm on the fiber. This two-form is non-degenerate on the region defined by $(\lambda_i - \phi)/(1 - 4\phi) \in (0, 1/2)$ for each $i$. The action of $S^1$ by scalar multiplication in the fibers is Hamiltonian with moment map $\phi$ and the quotient

$$\tilde{\mathcal{R}}(\lambda_1, \ldots, \lambda_n) := (K_{\mathcal{R}}^\vee \boxtimes K_{O_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{O_{\lambda_n}}^\vee - \{0\})/LG^n$$

has a residual action of $S^1$ whose quotients are the family given above:

$$\tilde{\mathcal{R}}(\lambda_1, \ldots, \lambda_n)/tS^1 = R \left( \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right).$$

At any fixed point the action of $S^1$ on the anticanonical bundle has positive weight, by definition. Note that the case $\lambda_1 = \ldots = \lambda_n = 1/4$ has a trivial mmp. One should think of the markings as moving away from the “center” 1/4 of the Weyl alcove $[0, 1/2]$ under the mmp. □
The flips or blow-downs occur at transition times at which there are reducible (abelian) bundles. More precisely, the set of transition times

\[ T = \left\{ t \mid \exists \epsilon_1, \ldots, \epsilon_n \in \{0, 1\}, \sum_{i=1}^n \epsilon_i \frac{\lambda_i - t}{1 - 4t} \in \mathbb{Z}/2 \right\}. \]

The projective bundles involved in the flip can be explicitly described as follows using the description of the moduli space as loop space quotient in [69]; we focus on the genus zero case and omit the proofs. Let \( I_\pm = \{ i, \epsilon_i = \pm 1 \} \). Fix a decomposition of the curve \( \Sigma \) into Riemann surfaces with boundary \( \Sigma_+, \Sigma_- \) such that \( \Sigma_\pm \) contains the markings in \( I_\pm \). Let \( S_\pm \) denote the moduli space of bundles that are abelian on \( \Sigma_\pm \):

\[ S_\pm = \left\{ [A] \in \mathcal{R} \left( \frac{\lambda_1 - t_\pm}{1 - 4t}, \ldots, \frac{\lambda_n - t_\pm}{1 - 4t} \right) \mid \dim(\mathcal{G}_A|\Sigma_\pm) = 1 \right\}. \]

Then the flip replaces \( S_+ \) with \( S_- \):

\[ \mathcal{R} \left( \frac{\lambda_1 - t_-}{1 - 4t}, \ldots, \frac{\lambda_n - t_-}{1 - 4t} \right) \quad \quad \mathcal{R} \left( \frac{\lambda_1 - t_+}{1 - 4t}, \ldots, \frac{\lambda_n - t_+}{1 - 4t} \right) \]

\[ S_- \quad \quad \mathcal{R}^{ab} \left( \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right) \quad \quad S_+ \]

where \( \mathcal{R}^{ab} \left( \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right) \) is the moduli space of abelian representations. Thus a projective bundle over the Jacobian gets replaced with another projective bundle.

As in the case of polygon spaces, there are two ways of obtaining Mori fibrations: First, fibrations with \( \mathbb{P}^1 \)-fiber occur whenever one of the markings \( \lambda_i - t_\) reaches 0 or 1/2, with base the moduli space of flat bundles with labels

\[ \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_i - t}{1 - 4t}, \frac{\lambda_{i+1} - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \]

resp.

\[ \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_i - t}{1 - 4t}, \frac{1/2 - t}{1 - 4t}, \frac{\lambda_{i+1} - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \]

if the marking \( \lambda_i \) reaches 0 resp. 1/2. Second, in genus zero the moduli space can become empty before any of the markings reach 0 or 1/2. By a special case of Agnihotri-Woodward [5], proved earlier by Treloar [107] we have

\[ \mathcal{R}(\lambda_1, \ldots, \lambda_n) = \emptyset \iff \exists I = \{i_1 \neq \cdots \neq i_{2k+1}\}, \sum_{i \in I} \lambda_i > k + \sum_{i \notin I} \lambda_i. \]

Thus in the last stage there is either a fibration over a moduli space with one less parabolic weight, with two-sphere fiber, or in genus zero one can also have a projective space at the last stage if the moduli space becomes empty. One can then continue with the base to obtain an extended running. The minimal model program of this moduli space is discussed in greater detail in Moon-Yoo [81] as well as Boden-Hu [16] and Thaddeus [108, Section 7].
The analog of the bending flow was introduced by Goldman [45]. First one constructs a densely defined circle action on the moduli space of bundles associated to any circle on the surface. Given any circle $C \subset \Sigma$ disjoint from the boundary, the holonomy $\varphi(C)$ of the flat bundle $P$ around $C$ is given by an element $\exp(\text{diag}(\pm 2\pi i \mu))$ up to conjugacy. After gauge transformation, the holonomy is exactly $\exp(\text{diag}(\pm 2\pi i \mu))$. Given an element $\exp(2\pi i \tau) \in U(1)$, one may construct a bundle $P_\tau$ by cutting $\Sigma$ along $C$ into pieces $\Sigma_+, \Sigma_-$ and gluing back the restrictions $P|\Sigma_+, P|\Sigma_-$ together using the transition map $e(\tau) := \text{diag}(\exp(2\pi i \tau))$:

$$P_\tau := (P|\Sigma_+) \bigcup_{e(\tau)} (P|\Sigma_-).$$

See Figure 33.

**Figure 33.** Twisting a bundle along a circle

The automorphism given by $\text{diag}(\exp(2\pi i \tau))$ commutes with the holonomy so the resulting bundle has a canonical flat structure, whose holonomies around loops $\Sigma_+, \Sigma_-$ are equal, but parallel transport from $\Sigma_+$ to $\Sigma_-$ is twisted by $\text{diag}(\exp(2\pi i \tau))$. Let $\mathcal{R}(\lambda_1, \ldots, \lambda_n)^C$ denote the locus where $\mu \notin \{0, 1/2\}$, for which the construction $[P] \mapsto [P_\tau]$ is well-defined and independent of all choices. The map

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n)^C \rightarrow \mathcal{R}(\lambda_1, \ldots, \lambda_n)^C, \quad [P] \mapsto [P_\tau]$$

defines a circle action. By Goldman [45], see also Meinrenken-Woodward [69] the action has moment map given by

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n)^C \rightarrow (0, 1/2), \quad [\varphi] \mapsto \mu$$

where $\varphi(C) = \text{diag}(\exp(\pm 2\pi i \mu))$ up to conjugacy. Furthermore, if $C_1, C_2$ are disjoint circles then the circle actions defined above commute on the common locus

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n)^{C_1} \cap \mathcal{R}(\lambda_1, \ldots, \lambda_n)^{C_2}.$$

Recall that a pants decomposition of a surface is a decomposition into three-holed spheres. Any compact oriented Riemann surface with boundary admits a finite pants decomposition, by choosing sufficiently many separating surfaces so that each piece has Euler characteristic one. Choose a pants decomposition $\mathcal{P}$ that refines the
decomposition into pieces $\Sigma_+, \Sigma_-$. Given a pants decomposition, one repeats the construction for each interior circle in the pants decomposition to obtain a map

$$\Psi_P : R(\lambda_1, \ldots, \lambda_n) \to [0, 1/2]^{n-3}.$$ 

In the genus zero case, the generic fibers are Lagrangian tori. For each pairs of pants $P$ in the pants decomposition with labels $\mu_1, \mu_2, \mu_3$, define the looseness of $P$ by

$$l(P) := \min \left( \min_{i \neq j \neq k} (\mu_i + \mu_j - \mu_k), 1 - \mu_1 - \mu_2 - \mu_3 \right).$$

As before, this is a tropical version of L'Huilier's generalization of Heron's formula:

$$\tan(A(T)/4) = \sqrt{\tan(2\pi(1 - \mu_1 - \mu_2 - \mu_3)) \prod_{i \neq j \neq k, i < j} \tan(2\pi(\mu_i + \mu_j - \mu_k)).}$$

We say that a labelling $\mu \in [0, 1/2]^{n-3}$ is regular if the looseness $l(P)$ is the same for each pair of pants $P \in P$,

$$\#\{l(P)|P \in P\} = 1$$

and if the first fibration in the running occurs at a time greater than $l(P)$. See Figure 2 for two examples in the case $n = 5$.

The regular Lagrangians are described as follows. Suppose that $R \left( \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right)$ is an mmp running as above. The transition times $T$ are the times $t$ for which there is an abelian representation. Given such a representation with holonomies $\text{diag}(\exp(\pm \epsilon_j \mu_j))$ define a partition of the surface $\Sigma$ into pieces $\Sigma_+, \Sigma_-$ containing the markings $\mu_j$ for which $\epsilon_j$ is positive resp. negative. We claim that if $\mu$ is regular and $l(\mu)$ is sufficiently small then the Goldman fiber $L_{\mu} := \Psi^{-1}_P(\mu)$ is regular. The Goldman bending flow induces a toric structure on $R(\lambda_1, \ldots, \lambda_n)$. The image of the Goldman map (2) is given by

$$\Psi_P(R(\lambda_1, \ldots, \lambda_n)) = \{ \mu \in [0, 1/2]^{n-3} \mid \forall P \in P, l(P) \geq 0 \};$$

that is, for each pair of pants in the pants decomposition the looseness is non-negative. It follows that if $l$ is a regular labelled pants decomposition then $\Psi^{-1}(l)$ is toric and there are $n - 2$ homotopy classes of disks with Maslov index two and boundary in $\Psi^{-1}(l)$, of equal area.

5.5. Mmp runnings for flag varieties. Flag varieties admit mmp runnings given by fibrations over partial flag varieties. Let $X$ be the variety of complete flags in a vector space of dimension $n$ with polarization corresponding to a dominant weight $\lambda$. The space $X$ has a natural transitive action of the unitary group which induces a diffeomorphism $X \cong U(n)/U(1)^n$. We identify the Lie algebra with $\mathbb{R}^n$, the weight lattice with $\mathbb{Z}^n$ and let $\epsilon_1, \ldots, \epsilon_n \in \mathbb{Z}^n$ denote the standard basis of weights for $U(1)^n$. The tangent bundle of $X$ is the associated fiber bundle

$$TX \cong U(n) \times U(1)^n \bigoplus_{1 \leq i < j \leq n} C_{\epsilon_i - \epsilon_j}$$
where $C_{i-j}^\epsilon$ is the space on which $U(1)^n$ acts with weight $\epsilon_i-\epsilon_j$. Hence the canonical bundle of $X$ is

$$\Lambda_{\text{top}}^TX \cong U(n) \times_{U(1)^n} \mathbb{C}_{2\rho}$$

where $\mathbb{C}_{2\rho}$ denotes the one-dimensional representation of $U(1)^n$ with weight

$$2\rho := ((n-1)\epsilon_1 + (n-3)\epsilon_2 + \ldots + (1-n)\epsilon_n).$$

An extended running of the minimal model program is the sequence of partial flag varieties obtained as follows. Consider the piecewise linear path $\lambda_t$ starting with $\lambda_t = \lambda - \rho t$ and continuing as follows: whenever $\lambda_t$ hits a wall $\sigma$ of the positive chamber $\lambda^i_t \in \sigma$ one continues with the path $\lambda_t = \lambda^i_t - (t-t^i)_\pi \sigma \rho$, where $\pi_\sigma$ is the projection onto $\sigma$. Each transition is a Mori fibration with Grassmann fiber and base the partial flag variety corresponding to the element $\lambda^i_t$.

A simple example is the variety of complex flags in a three-dimensional complex vector space which admits the structure of a Mori fibration in two ways. For example, let $X = \text{Fl}(\mathbb{C}^3) := \{ V_1 \subset V_2 \subset \mathbb{C}^3 \mid \dim(V_k) = k, k = 1, 2 \}$ be the variety of full flags in $\mathbb{C}^3$. Equip $X$ with the symplectic class such that the $\mathbb{P}^1$-fiber of the natural fibration $X \to \mathbb{P}^2$, $(V_1, V_2) \mapsto V_1$ has sufficiently small symplectic area. A running of the minimal model program is given by $X, \mathbb{P}^2, \text{pt.}$ There are no flips or divisorial contractions in this case, so no regular Lagrangians.

6. Broken Fukaya algebras

In this section we introduce a version of the Fukaya algebra for Lagrangians in broken symplectic manifolds along the lines of symplectic field theory as introduced by Eliashberg-Givental-Hofer [36]. The symplectic manifold is degenerated, through neck stretching, to a broken symplectic manifold. A sequence of pseudoholomorphic curves with respect to the degenerating almost complex structure converges to a broken pseudoholomorphic map: a collection of pseudoholomorphic curves in the pieces as well as a maps to the neck region. In the version that we consider here introduced by Bourgeois [21], these components are connected by gradient trajectories of possibly finite or infinite length.

6.1. Broken curves. First we describe the kind of domains that appear in the particular kind of broken limit we will consider, following Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [20]. The domains are nodal curves, except that the nodes are replaced with line segments and additionally segments are attached to markings on the boundary.

**Definition 6.1.** Let $n, m, s \geq 0$ be integers.

(a) (Level) A level of a broken curve with $n$ boundary markings, $m$ interior markings, and $s$ sublevels consists of

(i) a sequence $C = (C_1, \ldots, C_s)$ of treed nodal curves with boundary, called sublevels; in our situation only the first piece $C_1$ is allowed to have non-empty boundary: $\partial C_1 = \ldots = \partial C_s = \emptyset$.

(ii) interior markings $x^\pm_i \subset C_i$, $i = 1, \ldots, s$. 

(iii) a collection of (possibly broken) finite and semi-infinite edges attached to boundary points intervals $I_1, \ldots, I_m$ connecting boundary points in components of $C_i$ for $i = 1, \ldots, s$ with lengths $\ell(I_k)$; since in our situation only $C_1$ is allowed to have non-empty boundary, these edges only connecting components of $C_1$; and

(iv) a sequence of intervals $I^1_{i_1}, \ldots, I^s_{i(s)}$ attached to interior points in $C_i$ with finite lengths $\ell(I^k_{i_k})$;

Out of the data above one constructs a topological space $C$ by removing the nodes and gluing in the intervals.

Figure 34. A broken disk

(b) (Broken curve) A broken curve with $k$ levels is obtained from $k$ curves at a single level $C_1, \ldots, C_k$ where the first $C_1$ has only outgoing edges and the last only incoming edges, by gluing together the endpoints at infinity. The combinatorial type of a broken curve $C$ is the graph $\Gamma$ whose

(i) vertices $\text{Vert}(\Gamma)$ are irreducible components of the level curves $C_i$, together with the segments connecting the levels, and

(ii) edges $\text{Edge}(\Gamma)$ are boundary or interior nodes or joining points of the segments or incoming or outgoing markings or boundary markings.

A broken curve is stable if the marked curve obtained by collapsing the intervals is stable.

(c) (Broken disk) A broken disk is defined similarly to a broken curve but the first level $C_1$ is a disjoint union of treed disk and sphere components with segments attached, while the other pieces have surface parts that are spheres; furthermore, the combinatorial type is a tree.

From the description above one sees the following description into one and two-dimensional components: Each broken disk $C$ is the union of a surface part $S$, a tree part $T$ which further decomposes into segments $T_\circ$ connecting boundary components of $C$ and segments $T_\bullet$ connecting interior points of $C$:

$$C = S \cup T, \quad T = T_\bullet \cup T_\circ.$$
Let $\mathcal{M}_{n,m,k}$ denote the moduli space of stable weighted treed broken disks with $n$ boundary markings, $m$ interior markings and $k$ levels. The space $\mathcal{M}_{n,m,k}$ is naturally a Hausdorff stratified space with a stratification by combinatorial type

$$\mathcal{M}_{n,m,k} = \bigcup_{\Gamma \in \mathcal{T}_{n,m,k}} \mathcal{M}_{\Gamma}$$

where $\mathcal{T}_{n,m,k}$ denotes the set of possible combinatorial types. Each stratum fibers over a stratum in the moduli space of curves with boundary without tree parts, with fibers given by the possible assignments of lengths corresponding to nodes connecting curves at different levels. Note that the curve part of any broken curve has two kinds of nodes, those joining sublevels and those joining levels.

For each combinatorial type $\Gamma$ of broken curve we denote by $\mathcal{M}_{\Gamma}$ the closure of the moduli space $\mathcal{M}_{\Gamma}$ and by $U_{\Gamma}$ the universal broken (weighted, treed, stable) disk, whose fiber over $[C] \in \mathcal{M}_{\Gamma}$ is isomorphic to $C$. We have a decomposition into curve and tree parts

$$U_{\Gamma} = S_{\Gamma} \cup T_{\Gamma}$$

and a further decomposition of the tree parts into parts connected to the boundary of disk and those parts connecting to the interior of the curve:

$$T_{\Gamma} = T_{\Gamma,0} \cup T_{\Gamma,\ast}$$

6.2. Broken maps. A broken map is a map from a broken curve into a broken symplectic manifold, defined as follows.

**Definition 6.2.**

(a) (Broken symplectic manifold) Let $X$ be a compact rational symplectic manifold. Let $Z \subset X$ be a coisotropic hypersurface separating $X \setminus Z$ into components with cylindrical ends $X^c_\circ, X^c_\triangleright$. Suppose that $Z$ has null foliation which is a circle fibration over a symplectic manifold $Y = Z/U(1)$. By a construction of Lerman [66] the unions

$$X_C := X^c_\circ \cup Y, \quad X_D := X^c_\triangleright \cup Y$$

have the structure of symplectic submanifolds. Let $N_{\pm} \to Y$ denote the normal bundle of $Y$ in $X_C, X_D$, and $N_{\pm} \oplus \mathbb{C}$ the sums with the trivial bundle $\mathbb{C} = Y \times \mathbb{C}$. Denote by $\mathbb{P}(N_+ \oplus \mathbb{C}) \cong \mathbb{P}(N_- \oplus \mathbb{C})$ the projectivized normal bundle, where the isomorphism is induced from

$$\mathbb{P}(N_+ \oplus \mathbb{C}) \cong \mathbb{P}((N_+ \oplus \mathbb{C}) \otimes N_-) = \mathbb{P}(N_- \oplus \mathbb{C}).$$

The broken symplectic manifold arising from the triple $(X_C, X_D, Y)$ is the topological space $X = X_C \cup_Y X_D$.

(b) (Multiply broken symplectic manifold) For an integer $m \geq 1$ define the $m-1$-broken symplectic manifold

$$X[m] = X_C \cup_Y \mathbb{P}(N_+ \oplus \mathbb{C}) \cup_Y \mathbb{P}(N_+ \oplus \mathbb{C}) \cup_Y \ldots \cup_Y X_D$$

where there are $m-2$ copies of $\mathbb{P}(N_+ \oplus \mathbb{C})$ called broken levels. Define

$$X[m]_0 = X_C, \quad X[m]_1 = \mathbb{P}(N_+ \oplus \mathbb{C}), \quad \ldots, \quad X[m]_m = X_D.$$
There is a natural action of the complex torus \((\mathbb{C}^\times)^{l-2}\) on \(\mathbb{P}(N \oplus \mathbb{C})\) given by scalar multiplication on each projectivized normal bundle:
\[
\mathbb{C}^\times \times \mathbb{P}(N_\pm \oplus \mathbb{C}) \to \mathbb{P}(N_\pm \oplus \mathbb{C}), \quad (z, [n, w]) \mapsto z[n, w] := [zn, w].
\]
The fixed points of the \(\mathbb{C}^\times\) action are the divisors at 0 and \(\infty\):
\[
\mathbb{P}(N_\pm \oplus \mathbb{C}) = \{[n, 0]\} \cup \{[0, w]\}
\]
where \(n\) reps. \(w\) ranges over vectors in \(N_\pm\) resp. \(Y \times \mathbb{C}\).

(c) An almost complex structure \(J \in J(\mathbb{R} \times Z)\) is of cylindrical form if there exists an almost complex structure \(J_Y\) on \(Y\) such that the projection \(\pi_Y: \mathbb{R} \times Z \to Y\) is almost complex and \(J\) is invariant under the \(\mathbb{C}^\times\)-action on \(\mathbb{R} \times Z\) induced from the embedding in \(\mathbb{P}(N_\pm \oplus \mathbb{C})\) given by
\[
\mathbb{C}^\times \times \mathbb{P}(N_\pm \oplus \mathbb{C}) \to \mathbb{P}(N_\pm \oplus \mathbb{C}), \quad s \exp(it)(s_0, z) = (s_0 + s, \exp(it)z).
\]
That is,
\[
D\pi_Y J = J_Y D\pi_Y, \quad J \in J(\mathbb{R} \times Z)^{\mathbb{C}^\times}.
\]
We denote by \(J^{\text{cy}l}(\mathbb{R} \times Z)\) the space of tamed almost complex structures of cylindrical form, and by
\[
J(X) = J(X^\circ) \times_{J^{\text{cy}l}(\mathbb{R} \times Z)} J(\mathbb{R}^\circ)
\]
the fiber product consisting of tamed almost complex structures of cylindrical form on the ends. (Note that this definition differs from the one in [20], which does not suffice for our purposes.)

We now define broken maps. We first treat the unperturbed case.

**Definition 6.3.** (Broken maps) Let \(X\) be a broken symplectic manifold as above, and \(L \subset X_\subset\) a Lagrangian disjoint from \(Y\). Let \(J \in J(X)\) be an almost complex structure on \(X\) of cylindrical form, \(H\) be a Morse function on \(Y\) and \((F, G)\) a Morse-Smale pair on \(L\). A **broken map** to \(X\) with boundary values in \(L\) consists of:

(a) (Broken curve) a broken curve \(C = (C_0, \ldots, C_l)\);
(b) (Broken map) a map \(u: C \to X\), that is, collection of maps (notation from [151])
\[
u_k: C_k \to X[p], \quad k = 0, \ldots, p
\]
satisfying the following non-linear partial differential equations:

(i) (Pseudoholomorphicity) On the two-dimensional part \(S \subset C\), the map \(u\) is \(J\)-holomorphic, that is, \(\overline{\partial}_J(u_S) = 0\).

(ii) (Gradient flow in the Lagrangian) On the one-dimensional part \(T_0 \subset C\) connecting boundary nodes, \(u\) is a segment of a gradient trajectory on each interval component for the Morse function \(F\) on \(L\):
\[
\left(\frac{d}{dt} - \text{grad}_F\right)(u|T_0) = 0.
\]

(iii) (Intersection multiplicity) If a pseudoholomorphic map \(u: C \to X\) has isolated intersections with an almost complex codimension two submanifold \(Y \subset X\) then at each point \(z \in u^{-1}(Y)\) there is a positive
intersection multiplicity \( s(u, z) \in \mathbb{Z}_{>0} \) describing the winding number of a small loop counterclockwise around \( Y \):

\[
s(u, z) = [\exp(2\pi i \theta) \mapsto u(z + r \exp(2\pi i \theta))] \in \pi_1(U - (U \cap Y)) \cong \mathbb{Z}
\]

where \( U \) is a contractible open neighborhood of \( z \) and \( r \) is sufficiently small so that \( u(z + r \exp(2\pi i \theta)) \in U \) for all \( \theta \in [0, 1] \).

(iv) (Gradient flow in the manifold) On the one-dimensional part \( T_0 \subset C \) connecting interior nodes, \( u \) is a segment of a gradient trajectory on each interval component:

\[
\left( \frac{d}{dt} - \text{grad}_H \right)(u|_{T_0}) = 0.
\]

and satisfying the following:

(i) (Matching condition) For the nodes of the domain on either end of a Morse trajectory, the intersection multiplicities \( s_{i,j}^\pm \) with the hypersurface \( Y \) are equal.

We denote by \( \ell(s) \) the number of contact points with the divisor \( Y \).

(c) (Isomorphisms of broken maps) An isomorphism between broken curves \( u_i : C_i \to X[k], i = 0, 1 \) is an automorphism of the domain \( \phi : C_0 \to C_1 \) together with an element \( g \in (C^\times)^{k-1} \) such that \( u_1 \circ \phi = gu_0 \), and the automorphism is trivial on any infinite segment with one weighted end and one unweighted end. A broken map is stable if it has only finitely many automorphisms, except for automorphisms of infinite length segments with one weighted end and one unweighted end. This means in particular at least one component at each level is not a trivial cylinder.

The combinatorial type of a broken map is the combinatorial type of the underlying curve, but with the additional data of the homotopy class of each component (as a labelling of the vertices) and the intersection multiplicities with the stabilizing divisor (for each semi-infinite edge corresponding to an interior marking or edge connecting a non-ghost component to a ghost component.) Denote by \( \overline{M}_\Gamma(X, L) \) the moduli space of stable weighted treed broken disks to \( X \) of type \( \Gamma \). Let \( \Gamma \) be a type with \( n \) leaves (corresponding to trajectories of the Morse function on the Lagrangian) and \( l \) broken Morse trajectories on the degenerating divisor. An admissible labelling for a \( \Gamma \) is a collection \( \ell \in \tilde{\mathcal{I}}(L)^{n+1} \) such that whenever the corresponding label is \( x_M^* \) resp. \( x_M^- \) resp. \( x_M^\ast \) or the corresponding leaf has weight 0 resp. \([0, \infty)\] resp. \( \infty \), Given an admissible labelling we denote by \( \overline{M}_\Gamma(X, L, \ell) \subset \overline{M}_\Gamma(X, L) \) the locus of maps with limits given by \( \ell \) (replacing each \( x_M^* \), \( x_M^- \), \( x_M^\ast \) with \( x_M \)) along the semi-infinite edges.

Broken maps may be viewed as pseudoholomorphic maps of curves with cylindrical ends; this leads to a natural notion of convergence in which the moduli space of broken maps of any given combinatorial type is compact. This is essentially a special case of compactness in symplectic field theory \[20], \[1\], although the particular setup we use here has not been considered before. First we recall terminology for the type of cylindrical ends we consider. First we introduce notation for the symplectic manifolds with cylindrical ends: Let \( X_0^\pm \) denote the manifold obtained by removing
the divisor \( Y \), or more generally, for the intermediate pieces \( \mathbb{P}(N_\pm \oplus \mathbb{C})^0 \cong R \times \mathbb{Z} \) the manifold obtained by removing the divisors at zero and infinity, isomorphic to \( Y \).

We identify a neighborhood of infinity in \( \mathbb{P}(N_\pm \oplus C) \) with \( \mathbb{R}_{>0} \times \mathbb{Z} \) with the almost complex structure induced from a connection on \( Z \) and the given almost complex structure on \( Y \).

Recall that the notion of Hofer energy makes sense for stable Hamiltonian structures. A Hamiltonian structure on a manifold \( Z \) is a closed two-form \( \omega \in \Omega^2(Z) \) of rank \( \dim(Z) - 1 \). A stable Hamiltonian structure is a one-form \( \alpha \in \Omega^1(Z) \) with the property that \( \ker(d\alpha) \subseteq \ker(\omega) \), \( \ker(\alpha) \cap \ker(\omega) = \{0\} \). The second condition means that \( \alpha \) is non-vanishing on the non-zero vectors in \( \ker(\omega) \subset TX \). Any circle-fibered coisotropic submanifold of a symplectic manifold has a stable Hamiltonian structure by taking \( \alpha \) to be a connection on the circle bundle, and \( \omega \) to be the restriction of the symplectic form.

Recall that the notion of Hofer energy makes sense for stable Hamiltonian structures. A Hamiltonian structure on a manifold \( Z \) is a closed two-form \( \omega \in \Omega^2(Z) \) of rank \( \dim(Z) - 1 \). A stable Hamiltonian structure is a one-form \( \alpha \in \Omega^1(Z) \) with the property that \( \ker(d\alpha) \subseteq \ker(\omega) \), \( \ker(\alpha) \cap \ker(\omega) = \{0\} \). The second condition means that \( \alpha \) is non-vanishing on the non-zero vectors in \( \ker(\omega) \subset TX \). Any circle-fibered coisotropic submanifold of a symplectic manifold has a stable Hamiltonian structure by taking \( \alpha \) to be a connection on the circle bundle, and \( \omega \) to be the restriction of the symplectic form.

For stable Hamiltonian structures a suitable notion of energy is introduced in [20]. We recall the definitions of action and energy of holomorphic curves in \( \mathbb{R} \times \mathbb{Z} \), where \( Z \) is equipped with Hamiltonian structure \( \omega_Z \) and connection form \( \alpha \);

**Definition 6.4. (Action and energy)**

(a) (Horizontal energy) The horizontal energy of a holomorphic map \( u = (\phi, v) : (C, j) \to (\mathbb{R} \times \mathbb{Z}, J) \) is (20, 5.3)

\[ E^h(u) = \int_C v^* \omega_Z. \]

(b) (Vertical energy) The vertical energy of a holomorphic map \( u = (\phi, v) : (C, j) \to (\mathbb{R} \times \mathbb{Z}, J) \) is (20, 5.3)

\[ E^v(u) = \sup_{\zeta} \int_C (\zeta \circ \phi) d\phi \wedge v^* \alpha \]

where the supremum is taken over the set of all non-negative \( C^\infty \) functions \( \zeta : \mathbb{R} \to \mathbb{R} \) having compact support and satisfying \( \int_\mathbb{R} \zeta(s) ds = 1 \).

(c) (Hofer energy) The Hofer energy of a holomorphic map \( u = (\phi, v) : (C, j) \to (\mathbb{R} \times \mathbb{Z}, J) \) is (20, 5.3) is the sum

\[ E(u) = E^h(u) + E^v(u). \]

(d) (Generalization to manifolds with cylindrical ends) Suppose that \( X^\circ \) is a symplectic manifold with cylindrical end modelled on \( \mathbb{R}_{>0} \times \mathbb{Z} \). The vertical energy is defined as before in (98) and the Hofer energy of a map \( u : C^\circ \to X^\circ \) from a surface \( C^\circ \) with cylindrical ends to \( X^\circ \) is defined by dividing \( X^\circ \) into a compact piece \( X^{\text{com}} \) and a cylindrical end \( \mathbb{R}_{\geq 0} \times \mathbb{Z} \), and defining

\[ E(u) = E(u|X^{\text{com}}) + E(u|\mathbb{R}_{\geq 0} \times \mathbb{Z}). \]

**Theorem 6.5.** Any sequence of finite energy broken pseudoholomorphic maps \( u_\nu : C_\nu \to X^\circ_\nu \) resp. \( u_\nu : C_\nu \to X[k]^\circ \) with bounded Hofer energy \( \sup \nu E(u_\nu) < \infty \) has a convergent subsequence, and any such convergent sequence has a unique limit.
Proof. We begin with some historical remarks. In the case that the almost complex structure is compatible and preserves the horizontal subspace, Theorem 6.5 is essentially a special case of the compactness result in symplectic field theory [20, Section 5.4] (with further details and corrections in Abbas [1] and alternative approach given in [22]) with the additional complication of Lagrangian boundary conditions. Since the Lagrangian is compact in $X^\circ$, these Lagrangian boundary conditions do not affect any of the arguments except that an additional argument for energy quantization for curves with cylindrical ends and boundary in $L$ is required.

For transversality reasons later we need the case of tamed almost complex structures not necessarily preserving the horizontal subspace. Our particular setup corresponds to the case of relative stable maps in Ionel-Parker [54] and Li-Ruan [72], as explained in Bourgeois et al. [20, Remark 5.9]. In particular, asymptotic convergence follows from asymptotic convergence for holomorphic maps to $Y$; energy quantization for disks in $X_C$ implies energy quantization for finite energy holomorphic maps of half-cylinders to $X_C^\circ$, where the boundary of the cylinder maps to the Lagrangian $L$. Energy quantization for holomorphic maps of spheres to $Y$ implies energy quantization for maps of holomorphic spheres to $\mathbb{P}(N \pm \oplus \mathbb{C})$: there exists a constant $\hbar > 0$ such that any holomorphic map $\mathbb{P}(N \pm \oplus \mathbb{C})$ with non-trivial projection to $Y$ has energy at least $\hbar$. Matching of intersection multiplicities is [20, Remark 5.9], Tehrani-Zinger [106, Lemma 6.6]: By removal of singularities, there is a one-to-one correspondence between finite energy holomorphic curves in $X^\circ_C$ resp. $X[k]^\circ$ and those in $X_\pm$ resp. $X[k]$ that are not contained in the divisor $Y$ resp. divisors at zero and infinity. Thus the intersection multiplicity is the degree of the cover of the Reeb orbit at infinity.

We sketch the argument. It suffices to show that on each tree or surface part of the domain, a subsequence converges to some limit in the Gromov sense. Since each domain is stable, each surface part has a unique hyperbolic metric so that the boundary is totally geodesic, see Abbas [1, I.3.3]. We denote by $r_\nu: C_\nu \to \mathbb{R}_{>0}$ the injectivity radius. The argument of Bourgeois et al. [20, Chapter 10], see also Abbas [1], shows that after adding finitely many sequences of points to the domain we may assume that the domain $C_\nu$ converges to a limit $C$ such that the first derivative sup $|du_\nu|/r_\nu$ is bounded with respect to the hyperbolic metric on the surface part, and with respect to the given metric on the tree part. This implies that there exists a limiting map $u: C^\times \to X$ on the complement $C^\times$ of the nodes so that on compact subsets of the complement of the nodes a subsequence of $u_\nu$ converges to $u$ in all derivatives. Removal of singularities and matching conditions then follows from the corresponding results for holomorphic maps: the matching condition for nodes mapping into the cylindrical end is simply the matching condition for the maps to $Y$, in addition to matching of intersection degrees which is immediate from the description as a winding number. Convergence on the tree part of the domain follows from uniqueness of solutions to ordinary differential equations. □

Remark 6.6. (Exponential decay) In fact any holomorphic map $u: C^\circ \to X^\circ$ with finite Hofer energy converges exponentially fast to the corresponding Reeb orbit: In coordinates $s,t$ on the cylindrical end diffeomorphic to $\mathbb{R}_{>0} \times Z$, there exists a
constant $C$ and constants $s_0, s_1 > 0$ such that for $s > s_1$
$$\text{dist}(u(s, t), (\mu(s - s_0), \gamma(t))) < C \exp(-s).$$
This follows from the correspondence with holomorphic maps to the compactification. This property will be used later for the gluing result.

6.3. Broken perturbations. Each stratum of the moduli space of broken maps is smooth under a certain regularity condition involving the surjectivity of a suitable linearized operator. Because the situation is Morse-Bott, there are zero modes in the tangential operator at infinity and the linearized Cauchy-Riemann operator in standard Sobolev spaces is not Fredholm. Instead, the linearized operator acts on Sobolev spaces involving a choice of Sobolev weight for the cylindrical ends which is not in the spectrum of the tangential operator at infinity.

Remark 6.7. (Linearized operators) We first introduce suitable Sobolev spaces of maps modelled on multiple covers of Reeb orbits at infinity. Let $X, Y$ be as above. Let $\lambda \in (0, 1)$ be a Sobolev weight. Choose a cutoff function

\begin{equation}
\beta \in C^\infty(\mathbb{R}), \quad \begin{cases}
\beta(s) = 0 & s \leq 0 \\
\beta(s) = 1 & s \geq 1
\end{cases}
\end{equation}

We equip the complement $X^\circ = X - Y$ with a cylindrical end modelled on $\mathbb{R}_{>0} \times Z$. Let $\pi_\mathbb{R} : \mathbb{R}_{>0} \times Z \to \mathbb{R}_{>0}$ denote projection on the first factor. We fix a multiplicity $\mu \in \mathbb{Z}$. For the moment we assume that the domain $C^\circ$ has no tree parts.

First we define a weighted Sobolev space of maps asymptotic to a Reeb orbit. For $p > 2$ and any map $u : C^\circ \to X^\circ$ asymptotic to $(\mu(s - s_0), \gamma(t - t_0))$ as $s \to \infty$ in the sense that

$$\lim_{s \to \infty} \text{dist}(u(s, t) - (\mu(s - s_0), \gamma(t))) = 0$$

define

\begin{equation}
\|u\|_{1,p,\mu,\lambda} := \int_C (\|du - \mu \beta(\alpha + d\pi_\mathbb{R})\|^p + \|\beta(\pi_\mathbb{R} \circ u - \mu(s - s_0))\|^p) e^{\beta(s)p\lambda} \text{d Vol}_C.
\end{equation}

Here $s, t$ are cylindrical coordinates on the cylindrical end of $C$, $\mu \beta \alpha$ and $\beta(s)p\lambda$ are by definition equal to 0 on the complement of the support of $\beta$, where the function $s$ and connection form $\alpha$ are not necessarily defined. Note that the form $du - \mu \beta \alpha$ is zero exactly if $u$ is an $\mu$-fold cover of a Reeb orbit $\gamma(t)$ on the cylindrical end. Let

$$\text{Map}(C^\circ, X^\circ)_{1,p,\mu,\lambda} \subset W^{k,p}_{\text{loc}}(C^\circ, X^\circ)$$
denote the space of $W^{1,p}_{\text{loc}}$ maps from $C^\circ$ to $X^\circ$ with finite $1, p, \mu, \lambda$-norm \([100]\). For any smooth map $u : C^\circ \to X^\circ$ with finite $1, p, \lambda$ norm we denote by $\Omega^0(C^\circ, u^*TX^\circ)$ the space of smooth sections of $u^*TX^\circ$. For any section $\xi : C^\circ \to u^*TX^\circ$ vanishing sufficiently rapidly at infinity we denote by

$$\|\xi\|_{1,p,\lambda} := \int_{C^\circ} (\|\nabla\xi\|^p + \|\xi\|^p) e^{p\lambda} \text{d Vol} C^\circ.$$

More generally if $\xi$ is asymptotic to $\xi_\infty \in \ker(\alpha) \subset TZ$ we denote by

\begin{equation}
\|\xi\|_{1,p,\lambda} := \left(\|\xi - \xi_\infty\|_{1,p,\lambda}^p + |\xi_\infty|_p^p\right)^{1/p}.
\end{equation}
The space of $W^{1,p}_{loc}$-sections with finite $1,p,\lambda$-norm (101) is denoted $\Omega^0(C^0, u^*TX^0)_{1,p,\lambda}$; these are sections which are equal to a constant valued in $\ker(D)$ plus an exponentially decaying term. For any smooth map $u$ with finite $(1,p,\lambda)$-norm, pointwise geodesic exponentiation defines a map (using Sobolev multiplication estimates)
\[
\exp_u : \Omega^0(C^0, u^*TX)_{1,p,\lambda} \to \text{Map}(C^0, X)_{1,p,\mu,\lambda}.
\]
The maps (102) provide charts on $\text{Map}(C, X)_{1,p,\mu,\lambda}$ making $\text{Map}(C, X)_{1,p,\mu,\lambda}$ into a smooth Banach manifold. More generally, in the case that $C = S^0 \cup T$ is the union of surface and tree parts we define similar Sobolev spaces by taking the standard $W^{1,p}$ spaces on the tree parts. Thus, for example, we denote for simplicity
\[
\Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda} := \Omega^{0,1}(S^0, u^*TX^0)_{0,p,\lambda} \oplus \Omega^1(T_0, u^*TL)_{0,p} \oplus \Omega^1(T_\bullet, u^*TY)_{0,p}.
\]

The pseudoholomorphic maps are cut out locally by a smooth map of Banach spaces: Define
\[
\|\eta\|_{0,p,\lambda} := \int_{C^0} \|\eta\|_{p,\mu,\lambda}^p d\text{Vol} C^0
\]
and let $\Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda}$ denote the space of $L^p_{loc}$ sections with finite $0,p,\lambda$-norm. Let $\mathcal{F}_u$ denote the local trivialization of the Cauchy-Riemann operator:
\[
\mathcal{F}_u^0 : \Omega^0(C^0, u^*TX^0, (\partial u)^*TL)_{1,p,\lambda} \to \Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda},
\]
\[
\xi \mapsto \left( T_u(\xi) - \frac{1}{s} \partial \exp_u(\xi|s), \left( \frac{d}{ds} - \text{grad}(F_T) \right) \left( \xi|_{T_\bullet} \right), \left( \frac{d}{ds} - \text{grad}(H_T) \right) \left( \xi|_{T_\bullet} \right) \right)
\]
denotes parallel transport using an almost complex connection. The linearization of $\mathcal{F}_u^0$ at 0 is
\[
\mathcal{D}_u^0 : \Omega^0(C^0, u^*TX^0)_{1,p,\lambda} \to \Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda},
\]
\[
\xi \mapsto \left( \nabla^0 \xi - \frac{1}{2} / (J\nabla^0 \xi) \partial J u, \left( \frac{d}{ds} \xi - \nabla^0 \xi|_{T_\circ} \text{grad}(F_T), \left( \frac{d}{ds} \xi - \nabla^0 \xi|_{T_\bullet} \text{grad}(H_T) \right) \right)
\]

For sufficiently small Sobolev weight, the linearized operator above has the same kernel as the kernel of the standard linearized operator. Consider the map on the compact curve
\[
\mathcal{D}_u : \Omega^0(C, u^*TX)_{1,p} \to \Omega^{0,1}(C, u^*TX^0)_{0,p}.
\]
We claim that for $\lambda \in (0,1)$, the kernel resp. cokernel of $\mathcal{D}_u$ is isomorphic to the kernel resp. cokernel of the linearized operator $\mathcal{D}^0_u$. Indeed, elliptic regularity implies that any element of $\ker(\mathcal{D}_u)$ is smooth and so is equal to a constant plus a term that is at most linear in the local coordinate, hence exponentially decaying in the coordinates $s,t$; while conversely any element $\xi$ of $\ker(\mathcal{D}^0_u)$ lifts to a finite energy element of $\mathcal{D}_u$ on $L^p$ sections; by removal of singularities $\xi$ extends to an element of $\mathcal{D}_u$. A similar argument gives the identification of cokernels: the adjoint $\mathcal{D}^0_u^*$ acts on the space $\Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda}$ to $\Omega^0(C^0, u^*TX^0)_{1,p,\lambda}$ with small negative exponential weight $-\lambda$, allowing elements with small exponential growth in the domain and range. However for $\lambda$ small standard analysis implies that any
element in the cokernel converges exponentially to a constant along the cylindrical end (c.f. [3, 4.7]). The map \( u \) is regular if the linearized operator is surjective.

In order to achieve transversality we introduce stabilizing divisors satisfying a compatibility condition with the degeneration and introduce domain-dependent almost complex structures. By a broken divisor we mean a divisor which arises from degeneration of a divisor in the original manifold via neck stretching.

**Definition 6.8.** (Broken divisors) A broken divisor for the broken almost complex manifold \( X := X_\C \cup_Y X_{\bar{\C}} \) consists of a pair

\[
\mathbb{D} = (D_-, D_+) \quad \text{where} \quad D_- \subset X_\C, \quad D_+ \subset X_{\bar{\C}}
\]

of codimension two almost complex submanifolds such that each intersection \( D_\C \cap Y = D_Y \) is a codimension two almost complex submanifold in \( Y \). Given a broken divisor we may view the space \( \mathbb{D} = D_- \cup_D Y \cup_D D_+ \) as a subspace of the broken manifold \( \mathbb{X} \). Given a broken divisor \( D_\C, D_{\bar{\C}} \) as above we obtain a divisor

\[
D_N := \mathbb{P}(N_\pm|D_Y \oplus \underline{\mathbb{C}}) \subset \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}).
\]

We suppose that each \([D_\C, D_{\bar{\C}}]\) is dual to a large multiple of the symplectic class on \( X_\pm \), that is, \([D_\C] = k[\omega_\C], [D_{\bar{\C}}] = k[\omega_{\bar{\C}}]\). Then \([D_N] = k\pi_Y^*[\omega_Y]\), where \( \pi_Y \) is projection onto \( Y \), and as a result does not represent a multiple of any symplectic class on \( \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \). Thus the divisor \( D_N \) can be disjoint from non-constant holomorphic spheres in \( \mathbb{P}(N_\pm \oplus \underline{\mathbb{C}}) \), namely the fibers. However, holomorphic spheres whose projections to \( Y \) are non-constant automatically intersect \( D_N \).

As in the unbroken case, to achieve transversality we use almost complex structures equal to a fixed almost complex structure on the stabilizing divisor. We introduce the following notations. For a symplectic manifold \( X^\circ \) with cylindrical end, denote by \( \mathcal{J}(X^\circ) \) the space of tamed almost complex structures on \( X^\circ \) that are of cylindrical form on the end. Define

\[
\mathcal{J}(X):=\mathcal{J}(X^\circ) \times_{\mathcal{J}([R \times Z])} \mathcal{J}(X^\circ_{\mathbb{R}})
\]

the space of almost complex structures on \( X^\circ, R \times Z, X^\circ_{\mathbb{R}} \) that are cylindrical form on the cylindrical ends and induce the same almost complex structure on \( Y \). Given \( J_{D_\pm} \in \mathcal{J}(D_\C, D_{\bar{\C}}) \), we denote by \( \mathcal{J}(X, J_D) \) the space of almost complex structures in \( \mathcal{J}(X) \) that agree with \( J_{D_\C,D_{\bar{\C}}} \) on \( D_\C, D_{\bar{\C}} \):

\[
\mathcal{J}(X, J_D) = \{(J_\C,J_\bar{\C}) \in \mathcal{J}(X), J_\C|D_\C = J_{D_\C}, \quad J_{\bar{\C}}|D_{\bar{\C}} = J_{D_{\bar{\C}}})\}
\]

Fix a tamed almost complex structure \( J_{D_\C,D_{\bar{\C}}} \) such that \( D_\C, D_{\bar{\C}} \) contains no non-constant \( J_{D_\C,D_{\bar{\C}}} \)-holomorphic spheres of any energy and any holomorphic sphere meets \( D_\C, D_{\bar{\C}} \) in at least three points, as in [26 Proposition 8.14]. By [24 Proposition 8.4], for any energy \( E > 0 \) there exists a contractible open neighborhood \( \mathcal{J}^*(X_\pm, J_{D_\C,D_{\bar{\C}}}E) \) of \( J_{D_\C,D_{\bar{\C}}} \) agreeing with \( J_{D_\C,D_{\bar{\C}}} \) on \( D_\C, D_{\bar{\C}} \) with the property that \( D_\C, D_{\bar{\C}} \) still contains no non-constant holomorphic spheres and any holomorphic sphere of energy at most \( E \) meets \( D_\C, D_{\bar{\C}} \) in at least three points. We denote by \( \mathcal{J}^*(X, J_D, E) \) the fiber product \( \mathcal{J}^*(X_\C, J_{D_-}, E) \times_{\mathcal{J}^{\text{hol}}([R \times Z])} \mathcal{J}^*(X_{\bar{\C}}, J_{D_+}, E) \).

Given a broken divisor define perturbation data for a broken symplectic manifold, as before, by allowing the almost complex structure etc. to depend on the additional
intersection points with the divisor. The new data is the choice of perturbations of the Morse function on the degenerating divisor. For base almost complex structures $J_{D, \pm}$ agreeing on $Y$, a *perturbation datum* for type $\Gamma$ of broken maps is datum

$$P_\Gamma = (J_\Gamma, F_\Gamma, G_\Gamma, H_\Gamma)$$

where

$$J_\Gamma : \mathcal{S}_\Gamma \to J(X, J_D), \quad F_\Gamma : \mathcal{T}_{\Gamma, \circ} \to C^\infty(L)$$

$$G_\Gamma : \mathcal{T}_{\Gamma, \circ} \to \mathcal{G}(L), \quad H_\Gamma : \mathcal{T}_{\Gamma, \bullet} \to C^\infty(Y)$$

and $J_\Gamma$ is equal to the given almost complex structures on $D_C, D_\gamma$.

In order to define perturbation data we have to keep in mind that the domain of a broken curve is not necessarily stable because of Morse trajectories of infinite length. However, given a broken curve $C$ we obtain a stable broken curve $f(C)$ by collapsing unstable components, and a perturbation system $P_\Gamma$ for curves of such type by pulling back $P_{f(\Gamma)}$ under the stabilization map $C \to U_{f(C)}$.

**Definition 6.9.** (Adapted broken maps) Given a type $\Gamma$ of broken disk and a perturbation datum $P_\Gamma$, an *adapted stable broken map* is a stable map $u : C \to X[k]$ from a broken weighted treed disk $C$ to $X[k]$ for some $k$ such that

(a) (Stable domain) The surface part $S$ of the domain $C$ is stable: each disk component resp. sphere component on which $u$ is constant has at least three boundary special points or one boundary special point and one special interior point resp. three special points;

(b) (Marking property) each interior marking maps to $D$ and each component of $u^{-1}(D) \cap S$ contains an interior marking; and

(c) (Sphere property) any holomorphic sphere contained in $D_C, D_\gamma$ is constant, while any holomorphic sphere contained in $D_Y$ is contained in a fiber of $\mathbb{P}(\mathbb{N}_\pm \oplus \mathbb{C}) \to Y$.

Let $\mathcal{M}_\Gamma(X, L, \mathbb{D})$ denote the moduli space of stable adapted broken weighted treed disks with boundary in $L$ of type $\Gamma$.

We generalize the various conditions on perturbation data from the unbroken case:

(a) A perturbation system $\mathcal{P} = (P_\Gamma)$ is *coherent* if it satisfies natural conditions with respect to (Collapsing edges, Making edge lengths or weights finite and non-zero, Cutting edges of infinite length, and Forgetting edges of infinite weight) and on any disconnected type $\Gamma = \Gamma_1 \cup \Gamma_2$, $P_\Gamma$ is obtained as the product of perturbation data $P_{\Gamma_1}$ and $P_{\Gamma_2}$.

(b) A perturbation datum $P_\Gamma$ is *regular* if every broken map of uncrowded, stable type $\Gamma$ is regular. Under this assumption, $\mathcal{M}_\Gamma(X, L, \mathbb{D})$ is smooth of expected dimension. Note in particular that regularity means that, for each type of expected dimension zero, each Morse trajectory occurring in the configuration is disjoint from the stabilizing divisor, $u_T^{-1}(D) = \emptyset$, since the condition of meeting the stabilizing divisor is codimension one in the space of Morse trajectories.
(c) A perturbation system \( P = (P_\Gamma) \) is stabilizing if for each type \( \Gamma \), \( J_\Gamma \) takes values in \( J^*(X, J, n(\Gamma)/k) \) on any component \( C_i \) having \( n(\Gamma_i) \) interior markings, where \( \Gamma_i \) is the single-vertex graph corresponding to \( C_i \).

(d) A perturbation system \( P_\Gamma \) of type \( \Gamma \) with one leaf is divisorial if it is pulled back under the forgetful map \( \overline{f} \) forgetting the incoming leaf.

The following generalizes the compactness and transversality results for Fukaya algebras to the broken case:

**Theorem 6.10.** Let \( \Gamma \) be an uncrowded type of stable broken treed disk of expected dimension at most one and suppose that regular, stabilizing perturbation data \( P_\Gamma \) have been chosen for all uncrowded boundary strata \( \overline{U}_\Gamma \subset \overline{U}_\Gamma \). There exists a comeager subset of the space of regular perturbation data \( P_\Gamma \) equal to the given perturbation data on lower-dimensional strata and sufficiently close to the base almost complex structure and metric such that

(a) (Transversality) Every element of \( \mathcal{M}_\Gamma(X, L, D) \) is regular.

(b) (Compactness) The closure \( \overline{\mathcal{M}}_\Gamma(X, L, D) \) is compact and contained in the stable, uncrowded locus.

(c) (Tubular neighborhoods) Each uncrowded stratum \( \mathcal{M}_\Gamma(X, L, D) \) of dimension zero has a tubular neighborhood of dimension one in any adjoining uncrowded strata of one higher dimension.

(d) (Orientations) The uncrowded strata \( \mathcal{M}_\Gamma(X, L, D) \) of formal dimension one or two are equipped with orientations satisfying the standard gluing signs for inclusions of boundary strata as in the unbroken case; in particular we denote by \( \epsilon([u]) \in \{\pm 1\} \) the orientation sign associated to the zero-dimensional moduli spaces.

(e) (One-leaf forgetful morphisms) among the regular perturbations there exist a comeager subset that are divisorial, and for these there exist forgetful morphisms

\[
\mathcal{M}_\Gamma(X, L, D) \to \mathcal{M}_{f(\Gamma)}(X, L, D)
\]

where \( \Gamma \) is any type with a single leaf and \( f(\Gamma) \) the type with no leaves obtained by forgetting the leaf.

**Proof.** Without the domain-dependent perturbations, compactness is Theorem 6.5. The case with domain-dependent perturbations is proved similarly, using the coherence condition on the perturbations. To see that the limit is stable and uncrowded, note that the stabilization condition on the divisor implies that any sphere bubble resp. disk bubble appearing in the limit has at least three resp. one interior intersection points \( u^{-1}(D) \) with the stabilizing divisor \( D \). Furthermore, by preservation of intersection multiplicity with the divisor, each maximal ghost component \( C_1 \subset C \) mapping to the divisor \( D \) must contain at least one marking \( z_j \in C_i \). Now \( C_i \) must be adjacent either to at least two non-ghost components \( C_j, C_k \), a single non-ghost component \( C_j \), or adjacent to two tree segments \( T_j, T_j \subset C \). Since strata with a component with a point with intersection multiplicity two, or mapping the node to the divisor are codimension at least two, they do not occur in the limit. Hence the \( C_i \) contains at most one marking, so the limit is of uncrowded type.
Transversality is an application of Sard-Smale as in Charest-Woodward \[24\] on the universal space of maps, after removing maximal ghost components. We say a ghost component in \(\mathbb{P}(N_\pm \oplus \mathbb{C})\) is one that projects to a point in \(Y\). Such maps are automatically multiple covers of a fiber of \(\mathbb{P}(N_\pm \oplus \mathbb{C}) \to Y\) and so automatically regular. We begin by covering the universal treed broken disk \(\mathcal{U}_\Gamma \to \mathcal{M}_\Gamma\) by local trivializations \(\mathcal{U}_\Gamma^i \to \mathcal{M}_\Gamma^i, i = 1, \ldots, N\). For each local trivialization consider an moduli space defined as follows. Let \(\text{Map}_{\Gamma}^{k,p}(C, X, L, \mathbb{D})\) denote the space of maps of class \(k, p\) mapping the boundary of \(C\) into \(L\), the interior markings into \(\mathbb{D}\), and constant (or constant after projection to \(Y\), if the component maps to a neck piece) on each disk with no interior marking. Let \(\mathcal{U}^i_{\Gamma, \text{thin}}\) be a small neighborhood of the nodes and attaching points in the edges \(\overline{\mathcal{U}}\), so that the complement in each edge and surface component is open. Let \(\mathcal{P}_k(X, L, \mathbb{D})\) denote the space of perturbation data \(P = (J, F, G, H)\) of class \(C^1\) equal to the given pair \((J, F, G, H)\) on \(\mathcal{U}_{\Gamma, \text{thin}}\), and such that the restriction of \(P\) to \(\overline{\mathcal{U}}\) is equal to \(P_{\Gamma}\), for each boundary type \(\Gamma\). Let \(l \gg k\) be an integer and and

\[
\mathcal{B}_{k,p,l,\Gamma}^i := \mathcal{M}_{\Gamma}^{i, \text{univ}} \times \text{Map}_{\Gamma}^{k,p}(C, X, L, \mathbb{D}) \times \mathcal{P}_k(X, L, \mathbb{D}).
\]

Consider the map given by the local trivialization

\[
\mathcal{M}_{\Gamma}^{i, \text{univ}} \to \mathcal{J}(S), \ m \mapsto j(m).
\]

Let \(S^{\text{nc}} \subset S\) be the union of disk and sphere components with interior markings (on which the stability condition requires the map to be non-constant, hence the notation). Let \(m(e)\) denote the function giving the intersection multiplicities with the stabilizing divisor, defined on edges \(e\) corresponding to intersection points, and consider the fiber bundle \(\mathcal{E}^i = \mathcal{E}^i_{k,p,l,\Gamma}\) over \(\mathcal{B}_{k,p,l,\Gamma}^i\) given by

\[
(\mathcal{E}^i_{k,p,l,\Gamma})_{m,u,J} \subset \Omega^0_{j,J}(S^{\text{nc}}, (u|S)^*TX)_{k-1,p}
\]

\[
\oplus \Omega^1(T_\nu, (u|T_\nu)^*TL)_{k-1,p} \oplus \Omega^1(T_\nu, (u|T_\nu)^*TY)_{k-1,p}
\]

the space of 0,1-forms with respect to \(j(m), J\) which vanish to order \(m(e) - 1\) at the node or marking corresponding to each contact edge \(e\). The Cauchy-Riemann and shifted gradient operators applied to the restrictions \(u_S\) resp. \(u_T\) of \(u\) to the two resp. one dimensional parts of \(C = S \cup T\) define a \(C^q\) section

\[
\bar{\mathcal{J}}_{\Gamma} : \mathcal{B}_{k,p,l,\Gamma}^i \to \mathcal{E}^i_{k,p,l,\Gamma},
\]

\[
(m, u, J, F, H) \mapsto \left(\bar{\mathcal{J}}_{j(m), J} u_S, \left(\frac{d}{ds} - \text{grad}_{F} \right) u_T, \left(\frac{d}{ds} - \text{grad}_{H} \right) u_T\right)
\]

where

\[
\bar{\mathcal{J}}_{j(m), J} u := \frac{1}{2} (J du_S - du_S j(m)),
\]

and \(s\) is a local coordinate with unit speed. The local universal moduli space is

\[
\mathcal{M}_{\Gamma}^{i, \text{univ}}(X, L, \mathbb{D}) = \bar{\mathcal{J}}^{-1} \mathcal{B}_{k,p,l,\Gamma}^i.
\]
where $B^i_{k,p,l,\Gamma}$ is embedded as the zero section. This subspace is cut out transversally: by [26] Lemma 6.5, Proposition 6.10, the linearized operator is surjective on the two-dimensional part of the domain mapping to $X$ on which $u$ is non-constant, while at any point $z$ in the interior of an edge in $C$ with $du(z) \neq 0$ the linearized operator is surjective by a standard argument. Furthermore, the matching conditions at the nodes are cut out transversally, by an inductive argument given in the unbroken case.

It remains to deal with components that map to a neck piece and project to a constant map. $u : \mathbb{P}^1 \to \mathbb{P}(N_{\pm} \oplus \mathbb{C})$ such that $\pi u$ is constant. Let $\eta \in \Omega^{0,1}(u^*T(\mathbb{P}(N_{\pm} \oplus \mathbb{C})))$ be a one-form on one of the intermediate broken pieces $S_i$ such that $\eta$ lies in the cokernel of the universal linearized operator

$$D_{u,j}((\xi, K) = D_u \xi + \frac{1}{2} KDuj.$$ 

for the universal moduli space. Variations of tamed almost complex structure of cylindrical type are $J$-antilinear maps $K : T\mathbb{P}(N_{\pm} \oplus \mathbb{C}) \to T\mathbb{P}(N_{\pm} \oplus \mathbb{C})$ which vanish on the vertical subbundle. Since the horizontal part of $D_z u$ is non-zero at some $z \in C$, we may find an infinitesimal variation $K$ of almost complex structure of cylindrical type by choosing $K(z)$ so that $K(z)D_z u j(z)$ is an arbitrary $(j,z,J(z))-\text{antilinear map from } T_z C$ to $T_{u(z)}\mathbb{P}(N_{\pm} \oplus \mathbb{C})$. Choose $K(z)$ so that $K(z)D_z u j(z)$ pairs non-trivially with $\eta(u(z))$ and extend $K(z)$ to an infinitesimal almost complex structure $K$ by a cutoff function. By the implicit function theorem, $\mathcal{M}^{\text{univ},i}_{\Gamma}(X,L,\mathbb{D})$ is a Banach manifold of class $C^q$, and the forgetful morphism

$$\varphi_i : \mathcal{M}^{\text{univ},i}_{\Gamma}(X,L,\mathbb{D})_{k,p,l} \to \mathcal{P}_{\Gamma}(X,L,\mathbb{D})_l$$

is a $C^q$ Fredholm map. Let

$$\mathcal{M}^{\text{univ},i}_{\Gamma}(X,L,\mathbb{D})_d \subset \mathcal{M}^{\text{univ},i}_{\Gamma}(X,L,\mathbb{D})$$

denote the component on which $\varphi_i$ has Fredholm index $d$. By the Sard-Smale theorem, for $k,l$ sufficiently large the set of regular values $\mathcal{P}^{\text{reg}}_{\Gamma}(X,L,D)_l$ of $\varphi_i$ on $\mathcal{M}^{\text{univ},i}_{\Gamma}(X,L,\mathbb{D})_d$ in $\mathcal{P}_{\Gamma}(X,L,\mathbb{D})_l$ is comeager. Let

$$\mathcal{P}^{\text{reg}}_{\Gamma}(X,L,\mathbb{D})_l = \cap_i \mathcal{P}^{\text{reg},i,\text{reg}}_{\Gamma}(X,L,\mathbb{D})_l.$$ 

A standard argument shows that the set of smooth domain-dependent $\mathcal{P}^{\text{reg}}_{\Gamma}(X,L,\mathbb{D})$ is also comeager. Fix $(J_\Gamma, F_\Gamma) \in \mathcal{P}^{\text{reg}}_{\Gamma}(X,L,\mathbb{D})$. By elliptic regularity, every element of $\mathcal{M}_{\Gamma}(X,L,\mathbb{D})$ is smooth. The transition maps for the local trivializations of the universal bundle define smooth maps $\mathcal{M}^i_{\Gamma}(X,L,\mathbb{D})|_{\mathcal{M}^{i}_L \cap \mathcal{M}^j_{1}} \to \mathcal{M}^1_{\Gamma}(X,L,\mathbb{D})|_{\mathcal{M}^i_{1} \cap \mathcal{M}^j_{1}}$. This construction equips the space $\mathcal{M}_{\Gamma}(X,L,\mathbb{D}) = \cup_i \mathcal{M}^i_{\Gamma}(X,L,\mathbb{D})$ with a smooth atlas. Since $\mathcal{M}_{\Gamma}$ is Hausdorff and second-countable, so is $\mathcal{M}_{\Gamma}(X,L,\mathbb{D})$ and it follows that $\mathcal{M}_{\Gamma}(X,L,\mathbb{D})$ has the structure of a smooth manifold.

Existence of orientations and tubular neighborhoods for codimension one strata involving broken Morse trajectory is standard. However, for strata corresponding to a trajectory of length zero, there is a new gluing result necessary which is proved in Section 6.6. \[\square\]
Remark 6.11. (True and fake boundary components) The codimension one strata are of the following types:

(a) Strata of maps to $X[1]$ such that one component has an interior node or boundary, that is, connected by a gradient trajectory of $H$ or $F$ of length 0. See Figure 35.

(b) Strata of maps with pieces mapping to $X[2]$, that is, with three levels connected by gradient trajectories of $H$. See Figure 39.

(c) Strata of maps to $X[1]$ with a broken Morse trajectory for $F$ passing through a critical point. See Figure 36.

Of these three types, the first two are *fake* boundary types in the sense that they do not represent points in the topological boundary. In the first case one can make the length of either the first gradient trajectory or the second finite, but not both (since the total length must be infinite). In the second case, one can either make the length finite and non-zero or resolve the node; this shows that the stratum is in the closure of two strata of top dimension. The last type is a true boundary component since the only deformation is that which deforms the length of the trajectory to a finite real number.
Using the regularized moduli spaces of broken maps we define the higher composition maps of the broken Fukaya algebra
\[ \mu^n : \hat{CF}(X,L)^{\otimes n} \to \hat{CF}(X,L) \]
on generators by
\[ \mu^n(<l_1>, \ldots, <l_n>) = \sum_{[u] \in \mathcal{M}(X,L,D,l)} (-1)^{\heartsuit}(\sigma([u])!)^{-1} \text{Hol}_L([\partial u]) q^{E([u])} \epsilon([u]) <l_0> \]
where \( \heartsuit = \sum_{i=1}^n i |l_i| \).

**Theorem 6.12.** (Broken Fukaya algebra) For any regular coherent stabilizing divisorial perturbation system \( P = (P_\Gamma) \) as above sufficiently \( C^2 \) close to the base datum \((J,F,H)\), the maps \( (\mu^n)_{n \geq 0} \) satisfy the axioms of a convergent \( A_\infty \) algebra \( \hat{CF}(X,L,D) \) with strict unit and weak divisor axiom. The homotopy type of \( \hat{CF}(X,L,D) \) and non-vanishing of the broken Floer cohomology is independent of all choices up to homotopy equivalence.

**Proof.** Transversality and compactness properties of the moduli space in Theorem 6.10. Note that boundary components corresponding to broken stable maps with three levels are negative expected dimension. The estimate needed for the definition of convergence follows for a fixed \((J,G,H)\) by finiteness of the set of homotopy classes for a given energy, by compactness; then the same finiteness holds for perturbations sufficiently \( C^2 \) small. \( \square \)

6.4. Broken divisors. In the rest of the section we show that broken stabilizing divisors exist. The result is an analog of a result in algebraic geometry that follows from Kodaira vanishing: Let \( X \) be a smooth complex projective variety equipped with an ample line bundle \( E \) and \( i : Y \subset X \) a smooth subvariety of codimension one. Let \( \mathcal{E}(Y) \) denote the sheaf of sections vanishing on \( Y \). The exact sequence of sheaves
\[ 0 \to \mathcal{E}(Y) \to \mathcal{E} \to i_* i^* \mathcal{E} \to 0 \]
induces a long exact sequence of cohomology groups including the sequence
\[ 0 \to H^0(\mathcal{E}(Y)) \to H^0(\mathcal{E}) \to H^0(i_* i^* \mathcal{E}) \to H^1(\mathcal{E}(Y)) \to \ldots \]
By Kodaira vanishing \( H^1(\mathcal{E}(Y)) \) vanishes for sufficiently positive \( \mathcal{E} \) and furthermore \( \mathcal{E}(Y) \) is generated by its global sections. By the long exact sequence \( H^0(\mathcal{E}) \to H^0(i_* i^* \mathcal{E}) \) is surjective. By Sard’s theorem, the relative version of Bertini holds: For any \( s_Y \in H^0(i_* \mathcal{E}) \) there exists a section \( s \in H^0(\mathcal{E}) \) restricting to \( s_Y \) and cutting out a smooth divisor.

The symplectic version of this statement is obtained by a modification of Donaldson’s argument in [33]. Let \( \hat{X} \to X \) be a line-bundle with connection \( \alpha \) over \( X \) whose curvature two-form \( \text{curv}(\alpha) \) satisfies \( \text{curv}(\alpha) = (2\pi/i) \omega \); since our symplectic manifolds are rational we may always assume this to be the case after taking a suitable integer multiple of the symplectic form.

**Definition 6.13.** (Asymptotically holomorphic sequences of sections) Let \( (s_k)_{k \geq 0} \) be a sequence of sections of \( \hat{X}^k \to X \).
(a) The sequence \((s_k)_k \geq 0\) is asymptotically holomorphic if there exists a constant \(C\) and integer \(k_0\) such that for \(k \geq k_0\),
\[
|s_k| + |\nabla s_k| + |\nabla^2 s_k| \leq C, \quad |\bar{\partial} s_k| + |\nabla \bar{\partial} s_k| \leq Ck^{-1/2}.
\]
(b) The sequence \((s_k)_k \geq 0\) is uniformly transverse to 0 if there exists a constant \(\eta\) independent of \(k\) such that for any \(x \in X\) with \(|s_k(x)| < \eta\), the derivative of \(s_k\) is surjective and satisfies \(|\nabla s_k(x)| \geq \eta\).

In both definitions the norms of the derivatives are evaluated using the metric \(g_k = k\omega(\cdot, J \cdot)\).

**Theorem 6.14.** For \(k \gg 0\) there exist approximately holomorphic codimension two submanifolds \(D \subset D \supset \subset X\) such that \(D \subset \cap Y = D \supset \cap Y = D_Y\) is asymptotically holomorphic and codimension two, and each \(D \subset, D \supset, D_Y\) represents \(k[\omega]\) resp. \(k\[\omega\]\) resp. \(k[\omega_Y]\).

The proof will be given after two lemmas below.

**Lemma 6.15.** (Extension of asymptotically holomorphic sequences) Let \(X\) be an integral symplectic manifold equipped with a compatible almost complex structure, \(\tilde{X} \to X\) a line bundle with connection whose curvature is the symplectic form and \(Y \subset X\) an almost complex (hence symplectic) submanifold. Denote by \(\tilde{Y} \to Y\) the restriction of \(\tilde{X}\) to \(Y\). Given any asymptotically holomorphic sequence \(s_{Y,k}\) of \(\tilde{Y}^k \to Y\), there exists an asymptotically holomorphic sequence \(s_k\) of \(\tilde{X} \to X\) such that \(s_k, \pm |Y = s_{Y,k}\).

**Proof.** We may identify \(X\) near \(Y\) with the normal bundle \(N\) of \(Y\) in \(X\) on a neighborhood \(U\) of the zero section. Let \(\pi : N \to Y\) denote the projection and \(\phi : N \to \mathbb{R}_{\geq 0}\) the norm function. We may assume that the linearization \(\tilde{X}\) is given by a pull-back \(\pi^* \tilde{Y}^k\). Furthermore we may assume that the connection in the normal direction is given to leading order by the expression \(\frac{k}{2}(z_1 d\bar{z}_1 - \bar{z}_1 dz_1)\) where \(z_1\) is a coordinate on the fiber of the normal bundle. Define a Gaussian sequence
\[
s_{k,\pm} = \pi^* s_{Y,k} \exp(-k|z_1|^2/4)
\]
in a neighborhood of the divisor \(Y\), using parallel transport along the normal directions to trivialize, and then multiply by a cutoff function supported in a neighborhood of size \(k^{-1/6}\) of \(Y\) and extend by zero to all of \(X\). The bounds
\[
|s_k| + |\nabla s_k| + |\nabla^2 s_k| \leq C
\]
follows immediately from the fact that the derivatives of the Gaussian are bounded, and the derivatives are with respect to the metric \(g_k\); the bound
\[
|\bar{\partial} s_k| + |\nabla \bar{\partial} s_k| \leq Ck^{-1/2}
\]
follows from the fact that the Gaussian is holomorphic to leading order as in [33 (10)]: Consider the splitting \(TN\) into the vertical part \(T^\text{ver} N\) and its orthogonal complement \(T^\text{hor} N\), the difference between the almost complex structures \(\pi^* J_Y \oplus\)
$J^{\text{vert}}$ and $J$ is represented by a map $\mu : \Lambda^{1,0}TN \to \Lambda^{0,1}T^{\text{vert}}N$. Then as on [33, p. 10] we have
\[
\mathcal{D}_k = \sum_{i \geq 2} \mu(dz_i) \partial_i s_k + \sum_{i \geq 2} (\partial_i \pi^* s_{Y,k}) e^{-k|z_i|^2/4} + \pi^* s_{Y,k} \mu(\pi_1 dz_1) e^{-k^{1/2}|z_1|^2/4}
\]
where the last term represents the derivative in the normal direction. Since $\mu$ and all its derivatives are uniformly bounded this implies
\[
|\mathcal{D}_k s_k| \leq Ck^{-1/2}|k^{1/2}z_1|e^{-k|z_1|^2/4} \sup \mathcal{D}_k s_{Y,k} | \leq Ck^{-1/2}
\]
and
\[
|\nabla \mathcal{D}_k s_k| \leq C(|\nabla \mu||k^{1/2}z_1|e^{-k^{1/2}|z_1|^2/4} + |\mu||\nabla k^{1/2}\pi_1 dz_1 e^{-k|z_1|^2/4} |
\]
\[
\leq Ck^{-1/2}(|z_1| + |z_1|^3)e^{-k|z_1|^2/4} + \sup |\nabla \mathcal{D}_k s_{Y,k}|
\]
\[
\square
\]

Lemma 6.16. For any $p \in X - Y$ with $d_k(p,Y) \geq k^{-1/2}$, there exists an approximately section $s_{p,k}$ satisfying the estimates (109) and (110) with the property that $s_{p,k}$ vanishes on $Y$.

Proof. If $d_k(p,Y) \geq k^{-1/6}$, then the previously chosen locally Gaussian section satisfies the required properties since in this case $s_{k,p}$ vanishes on $Y$ since $Y$ is outside the support of the cutoff function. So it suffices to assume that $p$ is in the intermediate region
\[
d_k(p,Y) \in (k^{-1/2}, k^{-1/6}).
\]
So fix $p' \in Y$ and choose a local Darboux chart $(z_1, \ldots, z_n)$ near $p'$ so that $Y$ is described locally by $z_1 = 0$ and $\mathcal{D}_1(z_1) = 0$. Let $p$ lie in this Darboux chart, satisfying the estimate (111). Let $p_1 \neq 0$ denote the first coordinate of the point $p$. Given an approximately holomorphic sequence $s'_{p,k}$ with sufficiently small support (for example, a Gaussian $s(z) = \exp(-k|z - p|^2)$ the section $s_{p,k}(z) = s'_{p,k}(z)z_1/p_1$ is also approximately holomorphic, uniformly in $p$ as long as $|p_1| > k^{-1/2}$. Indeed the bound
\[
|s_{k,p}| + |\nabla s_{k,p}| + |\nabla^2 s_{k,p}| \leq C
\]
is immediate, and uniform if $|p_1| > k^{-1/2}$ since $s'_{p,k}$ is Gaussian in $k^{1/2}z_1$. The bound
\[
|\mathcal{D}_k s_{k,p}| + |\nabla \mathcal{D}_k s_{k,p}| \leq Ck^{-1/2}
\]
follows from the fact that $z_1/p_1$ is holomorphic to leading order; see Auroux [9]
Proof of Proposition 3] where similar approximately holomorphic sections were used to simplify Donaldson’s construction.

Proof of Theorem 6.14 Let $X = X_{\leq k} \cup_X X_{>}$ be a broken symplectic manifold. The sections $s_{<,k}, s_{>,k}$ from Lemma 6.15 are already asymptotically holomorphic and uniformly transverse in a neighborhood of size $\sqrt{k}$ around $Y$. Recall that in Donaldson’s construction, one has for each $k$ a collection of subsets $V_0 \subset \ldots V_N = X$, where $N$ is independent of $k$, and one shows that given a combination of the local Gaussians that is approximately holomorphic and transverse section $s_{k,i}$ on $V_i$,
that one can adjust the coefficients of the locally Gaussian functions so that approximately holomorphic and transverse over $V_{i+1}$. Here we may use sections $s_{p,k}$ in Lemma 6.16 for $p$ of distance at least $k^{-1/2}$ to achieve transversality off of $Y$. More precisely, in Donaldson’s construction \cite[p. 681]{33} taking $V_0$ to be, rather than empty, a neighborhood of size $k^{-1/2}$ around $Y$. Thus only the initial step of Donaldson’s construction is different.

\begin{Proposition} For any type $\Gamma$ with $k \geq 1$ components of the surface part joined by $e$ cylindrical ends and Morse trajectories and limits $l$ along the $n$ semi-infinite edges mapping to the Lagrangian, the expected dimension of the moduli space $\overline{\mathcal{M}}_\Gamma(X, L, D, l)$ of stable adapted broken maps of combinatorial type $\Gamma$ limits $l$ is given by

\begin{equation}
\dim T[u] \overline{\mathcal{M}}_\Gamma(X, L, D, l) = \dim(L) + 2I(u) - \dim(W^t_{l_0}) - \sum_{i=1}^n \dim(W^-_{l_i}) + n - 3 + (k - 1) \dim(X) - \sum_{i=1}^e (\dim(Y) - 4s_i)
\end{equation}

where $s_i$ are the multiplicities at the intersection points with $Y$.
\end{Proposition}

\begin{proof} We apply Riemann-Roch for Cauchy-Riemann operators on surfaces with boundary, \cite[Appendix]{77}. We obtain $\dim(L) + (k - 1) \dim(X) + 2I(u)$ for the sum of the indices on the various surface components, with corrections $\dim(W^-_{l_i})$ appearing from the constraints at the joining points between the tree parts mapping to $L$ and the disk parts, the $n - 3$ equal to the dimension of the moduli space of stable disks, and finally the contribution $\sum_{i=1}^e (\dim(Y) - 4s_i)$ from the matching and tangency conditions for the Morse trajectories in $Y$.
\end{proof}

6.5. The case of a blow-up or reverse flip. Now we specialize to the case that the symplectic manifold is obtained by a small simple reverse flip or blow-up, and the broken manifold is obtained by stretching a hypersurface separating the exceptional locus. We will show that in this case the broken Fukaya algebra of regular Lagrangian branes is unobstructed and the Floer cohomology is non-trivial.

As mentioned in the introduction we want to restrict to the case of flips occurring in smooth runnings of the mmp, where the center of the flips are trivial. These flips are called simple:

\begin{Definition} \begin{itemize}
\item[(a)] (Smooth flips) $X$ is obtained from a smooth reverse flip if the local model $\tilde{V}$ in \eqref{5.6} has positive weights $\mu_i$ all equal to 1, that is, $(\mu_i > 0) \implies (\mu_i = 1)$. It follows that $X$ is free of orbifold singularities.
\item[(b)] (Simple flips) $X$ is obtained from a small reverse simple flip or blow-up if the local model $\tilde{V}$ in \eqref{5.6} has all weights equal to $\pm 1$. In this case there exists an embedded projective space $\mathbb{P}^{n_+ - 1}$ in $X$ and a tubular neighborhood of $\mathbb{P}^{n_+ - 1}$ in $X$ symplectomorphic to $U^{\oplus m-}$ where $n_+ \oplus n_- = n + 1$ and $B$ is the tautological bundle over $\mathbb{P}^{n_+ - 1}$.
\end{itemize}\end{Definition}

\begin{Remark} (Exceptional and unexceptional pieces) Let $Z$ denote the unit sphere bundle in $B^{n_-}$. Then $Z$ is circle-fibered coisotropic fibering over the projectivized

\end{Remark}
bundle $Y = \mathbb{P}(B^{n-}) = \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. The variety $X$ degenerates to a broken manifold $(X_C, X_{\supset})$ where $X_C$ is a toric variety

$$X_C = \mathbb{P}(U^{n-} \oplus \mathbb{C})$$

and $X_{\supset}$ is the blow-up of $X$ at $\mathbb{P}^{n-1}$. We call $X_C$ resp. $X_{\supset}$ the exceptional resp. unexceptional piece of $X$.

**Remark 6.20.** (Exceptional piece as a symplectic quotient) The space $X_C$ may also be realized as a symplectic quotient

$$X_C = (\mathbb{C}^{n+1}_- \oplus \mathbb{C}^{n+1}_1 \oplus \mathbb{C})/(U(1) \times U(1))$$

where the action is by weights $\mathbb{C}^{n+1}_-$ with weight $(-1,1)$ and on $\mathbb{C}^{n+1}_1$ with weight $(+1,1)$, and on the last factor of $\mathbb{C}$ with weight $(0,-1)$. The flip is obtained by variation of git quotient in the above local model, which changes the unstable locus from $\{0\} \oplus \mathbb{C}^{n+}$ to $\mathbb{C}^{n-} \oplus \{0\}$. It follows that under the flip the exceptional locus $((0) \oplus \mathbb{C}^{n+})/\mathbb{C}^\times \cong \mathbb{P}^{n-1}$ is replaced by the exceptional locus $(\mathbb{C}^{n-} \oplus \{0\})/\mathbb{C}^\times \cong \mathbb{P}^{n-1}$.

**Remark 6.21.** (Exceptional piece as a toric manifold) The description of $X_C$ as a symplectic quotient from [5.7] implies that $X_C$ is a symplectic toric manifold obtained by symplectic quotient of $\mathbb{C}^{n+1}$ by $(\mathbb{C}^\times)^2$. We denote by $T = (S^1)^{n+1}/(S^1 \times S^1)$ the residual torus acting on $X_C$. The canonical moment map for the action of $(S^1)^{n+1}$ on $\mathbb{C}^{n+1}$ induces a moment map $\Phi : X_C \to T^\vee$. The moment polytope $\Phi(X_C)$ of $X_C$ has facets defined by normal vectors arising from projection of the standard basis vectors in $\mathbb{R}^{n+1} = \text{Lie}((S^1)^{n+1})$. Using the parametrization

$$(S^1)^{n-1} \cong T, \quad (z_1, \ldots, z_{n-1}) \mapsto [z_1, \ldots, z_{n-1}, 1, 1]$$

we have in terms of the standard basis vectors

$$(113) \quad \nu_1 := \epsilon_1, \quad \nu_2 := \epsilon_2, \ldots, \nu_{n-1} := \epsilon_{n-1} \in T^\vee.$$ 

On the other hand, from the description of the weights we have

$$(114) \quad \nu_n := \epsilon_1 + \ldots + \epsilon_{n-} - \epsilon_{n+1} - \ldots - \epsilon_{n-1}, \quad \nu_{n+1} := -\epsilon_1 - \ldots - \epsilon_{n-}.$$

Let $P_C$ denote the moment polytope of the toric piece $X_C$

$$P_C = \{\mu \mid \langle \mu, \nu_k \rangle \geq c_k, \quad k = 1, \ldots, n + 1\}$$

where $c_1 = c_2 = \ldots c_{n-1} = 0$, $c_n = \epsilon$, $c_{n+1} \gg 0$ and $\epsilon > 0$ represents the size of the exceptional divisor. Let

$$\Phi_C : X_C \to P_C$$

denote the moment map for the action of the $n$-torus $T^n$. The fiber over

$$(115) \quad \lambda = \epsilon(1, \ldots, 1)/(n_- - n_+ - 1)$$

is a Lagrangian torus $L$. For example, if $n_+ = 2, n_- = 1$ then the corresponding transition is a blow-down of curve in a surface. The moment polytope has normal vectors $\epsilon_1, \epsilon_2, \epsilon_1 - \epsilon_2, -\epsilon_1$. The moment polytope is

$$\{ (\lambda_1, \lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 - \lambda_2 \geq \epsilon, \lambda_1 \leq c_4 \}$$
Figure 37. The polytope for the blow-up of the projective plane

where $c_4 \gg 0$. Then $L$ is the fiber over $(\epsilon, \epsilon)$. See Figure 37 where the point $\lambda$ is shown as a shaded dot inside the moment polytope, which is a trapezoid.

We compute the Floer cohomology of the torus. Recall from Remark 6.19 that the pieces $X_{-1}, X_2$ are joined along the hypersurface

$$Y = \mathbb{P}(U^{n-1}) = \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}.$$ 

Let $H : Y \to \mathbb{R}$ be the standard Morse function on $Y$ obtained as the pairing of the moment map with a generic vector. For $i \leq k$ let

$$\epsilon_i := [0, \ldots, 1, 0, \ldots] \in \mathbb{P}^k$$

denote the points whose homogeneous coordinates are all zero except for the $i$-th coordinate; these are the fixed points for the standard torus action on $\mathbb{P}^k$. The critical points are the fixed points for the torus action

$$\text{crit}(H) = \left\{ ([\epsilon_{i-}], [\epsilon_{i+}]) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n+1}, \quad i_\pm \leq n_\pm \right\}.$$ 

The choice of $H$ corresponds to a choice of one-parameter subgroup $\mathbb{C}^\times$ inside the standard torus of dimension $n_- \times n_+$ acting on $\mathbb{P}^{n-1} \times \mathbb{P}^{n+1}$. The Morse cycles consist of those points that flow to $(\epsilon_{i-}, \epsilon_{i+})$ under the $\mathbb{C}^\times$-action:

$$W^-_{([\epsilon_{i-}], [\epsilon_{i+}])} = \left\{ (z_1, z_2) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n+1} \mid \lim_{z \to 0} z(z_1, z_2) = (\epsilon_{i-}, \epsilon_{i+}) \right\}.$$ 

Here it suffices to consider the one-parameter subgroup generated by the element $(1, \ldots, n_-), (1, \ldots, n_+)$; it is standard that the Morse cycles are those points $(z_-, z_+)$ such that the homogeneous coordinates of $z_\pm$ above index $i_\pm$ vanish. That is,

$$W^-_{([\epsilon_{i-}], [\epsilon_{i+}])} \cong \mathbb{P}^{i-1} \times \mathbb{P}^{i+1}, \quad \text{for } i_- \leq n_-, i_+ \leq n_+.$$ 

For example, when $n_- = 1, n_+ = 2$ so the transition corresponds to the blow-down of a curve in a surface, $Y = \mathbb{P}^1$ and the Morse cycles are either all of $\mathbb{P}^1$ or a point. This ends the Remark.

**Lemma 6.22.** The exceptional piece $X_C$ is a Fano toric variety.

**Proof.** Since $X_C$ is a toric variety it suffices to check that the anticanonical degree of rational invariant holomorphic curves is positive. By Remarks \[5.7\] \[6.21\] $X_C$ is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^{n+1}$, obtained by git from quotienting $\mathbb{C}^{n-1+n+1}$ by the action of $(\mathbb{C}^\times)^2$ with weights $(-1, 1)$ with multiplicity $n_-$, and $(1, 1)$ with multiplicity $n_+$. 

Hence the anticanonical bundle of $X_\subset$ is the quotient of the trivial bundle with weight $(n_++n_-+1, n_++n_-+1)$:

$$K_{X_\subset}^{-1} \cong (\mathbb{C}^{n_++n_-+1} \times \mathbb{C}^{n_++n_-+n_++n_-+1})/(\mathbb{C}^{n_+})^2.$$ 

By standard toric arguments the irreducible invariant holomorphic curves either lie in a fiber of the projection to $\mathbb{P}^{n_+-1}$, the zero section $\mathbb{P}^{n_+-1}$ or the divisor at infinity $\mathbb{P}^{n_+-1} \times \mathbb{P}^{n_+-1}$. We denote by $O(k)$ the $k$-th tensor power of the hyperplane line bundle. The restriction of $K_{X_\subset}^{-1}$ to the zero section $\mathbb{P}^{n_+-1}$ is $O(n_++n_-+1)$, while the restriction to the divisor at infinity $\mathbb{P}^{n_+-1} \times \mathbb{P}^{n_+-1}$ is

$$K_{X_\subset}^{-1}|_{\mathbb{P}^{n_+-1} \times \mathbb{P}^{n_+-1}} \cong \pi_+O(n_++1) \otimes \pi_-O(n_-+1)$$

where $\pi_\pm$ are the projections on the factors. Finally the anticanonical bundle on the fibers $\mathbb{P}^{n_-}$ is positive. The claim follows from positivity on the zero section, divisor at infinity, and fibers. 

Next we describe the broken holomorphic disks corresponding to the decomposition into exceptional and unexceptional pieces.

**Corollary 6.23.** There exist regular perturbation data for the broken manifold $X_\subset$ with the property that the almost complex structure on $X_\subset \subset X$ is the standard one.

**Proof.** The Fano condition implies that any constant holomorphic spheres have positive Chern number. Hence the only configurations of broken weighted treed disks with index zero or one consist of broken curves whose piece mapping to $X_\subset$ is a single disk. As in Cho-Oh [29], this implies that all holomorphic disks (necessarily given by the Blaschke products in (91)) are regular.

The following is the main result of this section; it states that the broken Floer theory of the Lagrangian torus is unobstructed and non-trivial.

**Theorem 6.24.** Let $L \subset X_\subset$ be a regular Lagrangian brane. If $(P_?)$ is a collection of (regular, coherent, stabilizing, divisorial) perturbations such that each almost complex structure is constant equal to the standard complex structure on $X_\subset$ and the Morse function on $Y$ for the nodes attaching to the disk components are the standard one on $Y \cong \mathbb{P}^{n_+-1} \times \mathbb{P}^{n_+-1}$ then

(a) the broken Fukaya algebra $CF(X,L)$ is a projectively flat for any local system $\rho$, that is, $\mu_\rho(1)$ is a multiple of the identity and so $(\mu_\rho)^2 = 0$; and

(b) there exists some local system $\rho \in \mathcal{R}(L)$ such that the Floer cohomology is non-vanishing: $H(\mu_\rho^1) = H(L) \neq 0$.

**Proof.** As mentioned in the introduction, moduli spaces of disks in toric varieties with boundary on Lagrangian torus orbits generally have “excess dimension”; for example, the smallest dimension moduli space has dimension that of the Lagrangian. More generally, let $l \in \text{crit}(F)$ and $(x_1, \ldots, x_m) \in \text{crit}(H).$ Denote by $\mathcal{M}(X_\subset, L, x, l)$ the moduli space of disks with a single level and no leaves, limiting to critical points $x \in \text{crit}(H)$ along the semi-infinite edges attached to interior points and $l$ along the semi-infinite edge attached to the boundary point. (That is, $\mathcal{M}(X_\subset, L, x, l)$ is the “disk part” of a component of the moduli space of broken disks.) In order
that the moduli space $\mathcal{M}(X, L, x, l)$ is non-empty and of expected dimension zero for perturbations with almost complex structures close to the standard complex structure, we must have

\begin{equation}
I(u) - \sum \deg(x_i) + 2 + \deg(l) - 2 = 0
\end{equation}

for any element $[u]$. Indeed, the moduli space of disks with boundary in $L$ is the moduli space of Blaschke products (91) which is regular and has evaluation map at any interior marking transverse to the Morse cycles $P_i \times P_j$. Thus we may in fact assume that the almost complex structure on $X$ is the standard torus-invariant complex structure. The requirement that $u(z)$ meet $P_i \times P_j$ at a marking $z_k$ implies that $u$ has a zero at $z_k$ for the codim($P_i \times P_j$) coordinates defining $P_i \times P_j$. Each such zero contributes two to the Maslov index $I(u)$ of $u$, hence for the moduli space to contain some element $[u]$ we must have

\begin{equation}
I(u) - 2 \geq \sum \deg(x_i) + 2.
\end{equation}

Combining (117) with (116) It follows that the moduli space can be non-empty only when $\deg(l) = 0$, in which case $\mu^0(1)$ is a count of Maslov index two disks and $CF(X, L)$ is projectively flat, that is, $\mu^0(1)$ is a multiple of the identity.

The leading order terms in the potential $W(y)$ are as in the toric case and can be read off from (113), (114):

\begin{equation*}
W(y_1, \ldots, y_n) = q^\epsilon \left( y_1 + \ldots + \frac{y_1 \ldots y_{n_+}}{y_{n+1} \ldots y_n} \right) + \text{higher order}
\end{equation*}

where the higher order terms have $q$-exponent at least $\epsilon$. The leading order potential

\begin{equation*}
W_0(y_1, \ldots, y_{n-1}) = y_1 + \ldots + \frac{y_1 \ldots y_{n_+}}{y_{n+1} \ldots y_{n-1}}.
\end{equation*}

Its partial derivatives are for $i \leq n_+$

\begin{equation*}
y_i \partial_{y_i} W_0(y_1, \ldots, y_{n-1}) = y_i + \frac{y_1 \ldots y_{n_+}}{y_{n+1} \ldots y_{n-1}}.
\end{equation*}

For $i > n_+$ the partial derivatives are

\begin{equation*}
y_i \partial_{y_i} W_0(y_1, \ldots, y_{n-1}) = y_i - \frac{y_1 \ldots y_{n_+}}{y_{n+1} \ldots y_{n-1}}.
\end{equation*}

Setting all partial derivatives equal to zero we obtain

\begin{equation*}
y_1 = \ldots = y_{n_+} = -y_{n+1} = \ldots = -y_{n-1}
\end{equation*}

and $y_1^{n_+ - n_-} = (-1)^{n_-}$. Hence $W_0(y)$ has a non-degenerate critical point at certain roots of unity. Now one can solve for a critical point of $W = W_0 + \text{higher order}$ using the implicit function theorem in [42, Theorem 10.4] by varying the local system. This gives a local system for the broken Fukaya algebra $CF(X, L)$ for which the Floer cohomology is non-vanishing. \qed
6.6. **Getting back together.** In this section we prove a gluing result which constructs from a broken configuration a family of unbroken maps. The result is similar but slightly different from that in Bourgeois-Oancea [19]. Much more complicated gluing theorems in symplectic field theory have been proved in Hutchings-Taubes [53] and Hofer-Wysocki-Zehnder (see e.g. [52]) both of which involve obstructions arising from multiple branched covers of Reeb orbits. Here any such cover corresponds to a fiber of a bundle with projective line fibers, and so one has transversality automatically (although one also has to achieve transversality with the diagonal at the nodes).

**Theorem 6.25.** Suppose that \( u : C \to X \) is a regular adapted broken map with either a broken Morse trajectory or a Morse trajectory of length zero. Then there exists an \( \delta > 0 \) and a family of adapted broken maps \( u_\delta : C_\delta \to X \) such that \( \lim_{\delta \to 0} [u_\delta] = [u] \).

**Proof.** Given a broken curve \( C \) with two sublevels \( C_+, C_- \), a standard gluing procedure creates, for any small gluing parameter \( \delta \in \mathbb{C} \), a curve \( C_\delta \) obtained by removing small disks around the node and gluing in using a map given in local coordinates by \( z \mapsto \delta/z \). Similarly, given a broken Morse trajectory a similar gluing procedure replaces the trajectory with one of finite length by removing a small ball around the breaking and gluing together the pieces on either side. The gluing procedure for the second kind of gluing (replacing a broken gradient trajectory with one of finite length) is rather standard so we focus on the first kind. For simplicity, we focus on the case that \( C \) has a single level and two sublevels \( C_+, C_- \); the general case is similar.

The proof is an application of a quantitative version of the implicit function theorem. The steps are: construction of an approximation solution; construction of an approximate inverse to the linearized operator; quadratic estimates; application of the contraction mapping principle. We assume for simplicity that \( X[k_{\pm}] = \mathbb{P}(N_{\pm} \oplus \mathbb{C}) \). We think of any curve in \( X[k_{\pm}] \) as a curve with cylindrical ends in \( \mathbb{R} \times Z \) where \( Z \) is the circle-fibered coisotropic with base \( Y \).

**Step 0:** Fix local trivializations of the universal treed disk and the associated families of complex structures and metrics on the domains. If \( \Gamma_{\pm} \) denote the combinatorial types of \( u_{\pm} \) let

\[
\mathcal{U}_{\Gamma_{\pm}}^i \to \mathcal{M}_{\Gamma_{\pm}}^i \times S_{\pm}, \quad i = 1, \ldots, l
\]

be a local trivializations of the universal treed disk, identifying each nearby fiber with \( (C_{\pm}^i, \mathcal{Z}, \mathcal{W}) \) such that each point in the universal treed disk is contained in one of these local trivializations. We may assume that \( \mathcal{M}_{\Gamma_{\pm}}^i \) is identified with an open ball in Euclidean space so that the the fiber \( C_{\pm}^o \) correspond to 0. Similarly, we assume we have a local trivialization of the universal bundle near the glued curve giving rise to a family of complex structures

\[
\mathcal{M}_{\Gamma}^i \to \mathcal{J}(S_{\delta})
\]

of complex structures on the two-dimensional locus \( S_{\delta}^o \subset C_{\delta}^o \), which are constant on the neck region.
**Step 1:** Define an approximate solution is given by gluing together the two solutions using a cutoff function. With \( \beta \) as in \([99]\) let \( y \in Y \) be the evaluation of \( u_\pm \) at the node and \( \gamma(t) \) a Reeb orbit in the fiber over \( y \) in \( Z \) with multiplicity \( \mu \) so that \( u_\pm \) considered locally as maps to \( X^\circ \) are asymptotic to \( \gamma(t) \):

\[
\lim_{s \to \infty} d(u_+(s, t), (\mu s, \gamma(t))) = \lim_{s \to -\infty} d(u_-(s, t), (\mu s, \gamma(t))) = 0.
\]

Any any point \( x \in X \) let \( \exp_x : T_x X \to X \) denote the map given by geodesic exponentiation with respect to the metric defined by the given compatible almost complex structure. We write in cylindrical coordinates near the divisor at infinity \( Y \),

\[
u_\pm(s, t) = \exp_{(\mp \mu s, \gamma(t))}(\zeta_\pm(s, t)).
\]

Define \( u'^{\text{pre}}_\delta \) to be equal to \( u_\pm \) away from the neck region, while on the neck region of \( C_\delta \) with coordinates \( s, t \) define

\[
u'^{\text{pre}}_\delta(s, t) = \exp_{(\mu s, \gamma(t))}(\beta(-s)\zeta_\pm(-s + |\ln(\delta)|/2, z) + \beta(s)\zeta_\pm(s - |\ln(\delta)|/2, t)).
\]

In other words, one translates \( u_+, u_- \) by some amount \(|\ln(\delta)|\), and then patches them together using the cutoff function and geodesic exponentiation.

**Step 2:** Define a map between suitable Banach spaces whose zeroes describe pseudoholomorphic curves near to the approximate solution. Since \( C_\delta \) satisfies a uniform cone condition, one has uniform Sobolev embedding estimates and multiplication estimates for the spaces. We will need Sobolev spaces on the curves \( C_\delta \) which approximate the weighted Sobolev spaces appearing in Section 6.2. We denote by \((s, t) \in [-|\ln(\delta)|/2, |\ln(\delta)|/2] \times S^1 \) the coordinates on the neck region created by the gluing. Define a Sobolev weight function

\[
\kappa_\lambda : C_\delta \to [0, \infty), \quad (s, t) \mapsto \beta(|\ln(\delta)|/2 - |s|)p\lambda(|\ln(\delta)|/2 - |s|)
\]

where \( \beta(|s| - |\ln(\delta)|/2)p\lambda(|s| - |\ln(\delta)|/2) \) is by definition zero on the complement of the neck region.

Pseudoholomorphic maps near the pre-glued solution are cut out locally by a smooth map of Banach spaces. Given a smooth map \( u : C_\delta \to X \), element \( m \in \mathcal{M}^1_T \) and a section \( \xi : C_\delta \to u^*TX \) define

\[
\|(m, \xi)\|_{1, p, \lambda} := \left( \|m\|^p + \int_{C_\delta} (\|\nabla \xi\|^p + \|\xi\|^p) \exp(\kappa_\lambda) d\text{Vol}_{C_\delta} \right)^{1/p}. \tag{119}
\]

Let \( \Omega^0(C_\delta, u^*TX)_{1, p, \delta} \) be the space of \( W^{1,p}_{\text{loc}} \) sections with finite norm \([119]\). Pointwise geodesic exponentiation defines a map (using Sobolev multiplication estimates)

\[
\exp_{u'^{\text{pre}}_\delta} : \Omega^0(C_\delta, (u'^{\text{pre}}_\delta)^*TX)_{1, p, \lambda} \to W^{1,p}(\text{Map}(C_\delta, X)). \tag{120}
\]

The space of pseudoholomorphic maps is cut out locally by a smooth map of Banach spaces. For a 0, 1-form \( \eta \in \Omega^{0,1}(C_\delta, u^*TX) \) define

\[
\|\eta\|_{0, p, \lambda} = \left( \int_{C_\delta} \|\eta\|^p \exp(\kappa_\lambda) d\text{Vol}_{C_\delta} \right)^{1/p}.
\]
Parallel transport using an almost-complex connection defines a map
\[ T_u^{\text{pre}}(\xi) : \Omega^{0,1}(C, (u_\delta^{\text{pre}})^*TX)_{0,p,\lambda} \rightarrow \Omega^{0,1}(C, (\exp_{u_\delta^{\text{pre}}}(\xi))^*TX)_{0,p,\lambda}. \]
We write \( C = S \cup T_0 \cup T_* \) where \( T_0 \) are the tree parts mapping to \( L \) and \( T_* \) are the tree parts mapping to \( Y \).

\[ (121) \quad F_\delta : M_1^L \times \Omega^0(C_\delta, (u_\delta^{\text{pre}})^*TX)_{1,p} \rightarrow \Omega^{0,1}(C_\delta, (u_\delta^{\text{pre}})^*TX)_{0,p}, \]

\[ \xi \mapsto \left( T_{u_\delta^{\text{pre}}}^{\text{pre}}(\xi|S)^{-1}\partial_j(m) \exp_{u_\delta^{\text{pre}}}(\xi|S), \right. \]

\[ \left. \left( \frac{d}{dt} - \text{grad}(H_\Gamma)(\xi_T), \frac{d}{dt} - \text{grad}(F_\Gamma)(\xi_T) \right) \right). \]

Treed pseudoholomorphic maps close to \( u_\delta^{\text{pre}} \) correspond to zeroes of \( F_\delta \). In addition, because we are working in the adapted setting, our curves \( C_\delta \) have a collection of markings \( z_1, \ldots, z_n \). We require
\[ (\exp_{u_\delta^{\text{pre}}}(\xi))(z_i) \in D, \quad i = 1, \ldots, n. \]
By choosing local coordinates near \( u(z_i) \), these constraints may be incorporated into the map \( F_\delta \) to produce a map
\[ (122) \quad F_\delta : M_1^L \times \Omega^0(C_\delta, (u_\delta^{\text{pre}})^*TX)_{1,p} \]

\[ \rightarrow \Omega^{0,1}(C_\delta, (u_\delta^{\text{pre}})^*TX)_{0,p} \oplus \bigoplus_{i=1}^n T_{u(z_i)}X/T_{u(z_i)}D \]
whose zeroes correspond to adapted pseudoholomorphic maps near the preglued map \( u_\delta^{\text{pre}} \).

**Step 3:** Estimate the failure of the approximate solution to be an exact solution. The one-form \( F_\delta(0) \) has contributions created by the cutoff function as well as the difference between \( J_{u^\pm} \) and \( J_{u_\delta^{\text{pre}}} \):
\[ \|F_\delta(0)\|_{0,p,\lambda} = \|\partial \exp_{(\mu_\delta, \gamma(t))}(\beta(-s)\zeta_-(-s + |\ln(\delta)|/2, t) + \beta(s)\zeta_+(s - |\ln(\delta)|/2, t))\|_{0,p,\lambda} \]
\[ = \|D \exp_{(\mu_\delta, \gamma(t))}(d\beta(-s)\zeta_-(-s + |\ln(\delta)|/2, z) + d\beta(s)\zeta_+(s - |\ln(\delta)|/2, t)) + \beta(-s)d\zeta_-(-s + |\ln(\delta)|/2, z) + \beta(s)d\zeta_+(s - |\ln(\delta)|/2, t))\|_{0,p,\lambda}. \]

The terms involving \( d\zeta_{\pm} \) are approximately \( J \)-holomorphic: The almost complex structures \( J_{\exp(\beta \zeta_{\pm})} \), transported to \( T_{(\mu_\delta, \gamma(t))}X \) via \( D \exp_{(\mu_\delta, \gamma(t))}^{-1} \), are exponentially close as \( \beta \) ranges from 0 to 1, and the fact that \( u_{\pm} \) are holomorphic implies an estimate
\[ \|((\beta(-s)d\zeta_-(-s + |\ln(\delta)|/2, z) + \beta(s)d\zeta_+(s - |\ln(\delta)|/2, t))\|_{0,p,\lambda} \leq Ce^{\ln(\delta)(1-\lambda)} = C\delta^{\lambda-1}, \]
c.f. Abouzaid [5, 10]. Similarly from the terms involving the derivatives of the cutoff function and exponential convergence of \( \zeta_{\pm} \) to 0 we obtain an estimate
\[ (123) \quad \|F_\delta(0)\|_{0,p,\lambda} < C \exp(-|\ln(\delta)|(1 - \lambda)) = C\delta^{\lambda-1}. \]
Step 4: Construct a uniformly bounded right inverse for the linearized operator of the approximate solution from the given right inverses of the pieces. Given an element \( \eta \in \Omega^{0,1}(u^{\text{pre}})^*T(\mathbb{R} \times \mathbb{Z})_{0,p} \), one obtains elements \( \bar{\eta} = (\eta_-, \eta_+) \) by multiplication with the cutoff function:

\[
\eta_+ = \beta \eta, \quad \eta_- = (1 - \beta) \eta.
\]

By Remark 6.7, there exist right inverse \( Q \) to the linearized operators \( D_u \) for the broken map \( u^\circ \), that is, for any \( \eta = (\eta_+, \eta_-) \in \Omega^{0,1}(C^0, u^*TX^\circ)_{0,p} \) there exists

\[
\bar{\xi} = (\xi_+, \xi_-) \in \Omega^{0,1}(C^0, u^*TX^\circ)_{1,p}, \quad D_u \xi^\pm = \eta^\pm, \quad \xi^\pm(w_{-+} - w_{+-}) = \xi_- - \xi_-(w_{-+})
\]

where \( w_{-+} \in C_\pm \) are the nodal points, and furthermore,

\[
\bar{\xi}(z_i) \in T_{u(z_i)}D_i, \quad i = 1, \ldots, n.
\]

We write

\[
u_\pm = \exp_{\gamma_\pm}^\text{pre}(\zeta_\pm), \quad \bar{\xi} = (Q_\pm^\delta(\eta), Q_\pm^\delta(\eta))\]

for some \( \zeta_\pm, \eta \). The limits of the elements \( \xi^\pm \) are by assumption equal to some element \( \xi_{\infty} \in \Omega^{0,1}(S^1, \gamma^*TX) \). Therefore we may define \( Q^\delta \eta \) equal to \((Q_{-}^\delta(\eta_-, \eta_+), Q_{+}^\delta(\eta_-, \eta_+))\) away from \( Z \) and near \( Z \) by patching these solutions together using a cutoff function

\[
Q^\delta \eta := \beta(-s + 1/4|\ln(\delta)|)Q_-(\eta) - \xi_{\infty} + \beta(s + 1/4|\ln(\delta)|)Q_+(\eta) - \xi_{\infty} + \xi_{\infty} \in \Omega^{0,1}(C, (u^{\text{pre}})^*TX)_{1,p,\lambda}.
\]

Then for any \( \rho \) sufficiently small, there exists \( \delta_0 \) such that for \( \delta > \delta_0 \)

\[
\|D_{u^\text{pre}}^\delta Q^\delta \eta - \eta\|_{1,p,\lambda} = \|D_{u^\text{pre}}^\delta Q_+^\delta \eta - \mathcal{T}_+D_u^\delta Q_+^\delta \eta\|_{1,p,\lambda} \leq \rho\|\beta(s + 1/4|\ln(\delta)|)Q_+^\delta \eta\|_{0,p,\lambda} + \|\beta(-s + 1/4|\ln(\delta)|/4)Q_+^\delta \eta\|_{0,p,\lambda} + \|d\beta(s - 1/4|\ln(\delta)|/4)Q^\delta \eta\|_{0,p,\lambda}
\]

where \( \mathcal{T}_\pm \) denote parallel transport from \( u^\circ \) to \( u \). The difference in the exponential factors in the definition of the Sobolev norms implies that possibly after changing the constant \( C \), we have

\[
\|d\beta(s - 1/4|\ln(\delta)|/4)Q^\delta \eta\|_{1,p,\lambda} = Ce^{-\lambda\delta}.
\]

Hence one obtains an estimate as in Fukaya-Oh-Ohta-Ono [11 7.1.32], Abouzaid [3, Lemma 5.13]: for some constant \( C > 0 \), for any \( \delta > 0 \)

\[
\|D_{u^\text{pre}}^\delta Q_+^\delta \eta - \mathcal{T}_+D_u^\delta Q_+^\delta \eta\|_{1,p,\lambda} \leq Ce^{-\lambda\delta}.
\]

(125) It follows that for \( \delta \) sufficiently large an actual inverse may be obtained from the Taylor series formula

\[
D_{u_\delta^\text{pre}}^{-1} = (Q^\delta D_{u_\delta^\text{pre}})^{-1}Q = \sum_{k \geq 0}(I - Q^\delta D_{u_\delta^\text{pre}})^kQ.
\]
Step 5: Obtain a uniform quadratic estimate for the non-linear map. After redefining $C > 0$ we have for all $\xi_1, \xi_2$
\begin{equation}
(126) \quad \|D_{\xi_1}F^D(\xi_1) - D_{u^{\text{pre}}_\delta \xi_1}\| \leq C\|\xi_1\|_{1,p,\lambda}\|\xi_2\|_{1,p,\lambda}.
\end{equation}
To prove this we require some estimates on parallel transport. Let
\[T^\delta_x(m, \xi) : \Lambda^{0,1}T^*\xi C_\delta \otimes TX \rightarrow \Lambda^{0,1}_{j^\delta(m)}T^*\xi C_\delta \otimes T_{\text{exp}_\delta(\xi)}X\]
denote pointwise parallel transport. Consider its derivative
\[DT^\delta_x(m, \xi, m_1, \xi_1; \eta) = \nabla_{|t=0}T^\delta_{u^{\text{pre}}_\delta(m + tm_1, \xi + t\xi_1)\eta}.
\]
For a map $u : C \rightarrow x$ we denote by $DT^\delta_u$ the corresponding map on sections. By Sobolev multiplication, there exists a constant $c$ such that
\begin{equation}
(127) \quad \|DT^\delta_u(m, \xi, m_1, \xi_1; \eta)\|_{0,\lambda} \leq c\|\xi\|_{1,\lambda}\|m_1\|_{1,\lambda}\|\xi_1\|_{1,\lambda}\|\eta\|_{0,\lambda}.
\end{equation}
Differentiate the equation
\[T^\delta_u(m, \xi)F^D(m, \xi) = \overline{D}_j^\delta(m)(\text{exp}_{u^{\text{pre}}_\delta(\xi))}
\]
with respect to $(m_1, \xi_1)$ to obtain
\begin{equation}
(128) \quad DT_{u^{\text{pre}}_\delta}(m, \xi, m_1, \xi_1, F^D(m, \xi)) + T^\delta_u(m, \xi)(DF^D(m, \xi, m_1, \xi_1)) = (\overline{D}_j^\delta(m, \exp_{\xi}(Dj^\delta(m_1, 1, \exp_{\xi}(\xi, \xi_1))).
\end{equation}
Using the pointwise inequality
\[|F^D(m, \xi)| < c|\text{exp}_{u^{\text{pre}}_\delta(z)}(\xi)| < c(|du^{\text{pre}}_\delta| + |\nabla\xi|)
\]
for $m, \xi$ sufficiently small, the estimate (127) yields a pointwise estimate
\[|T^\delta_{u^{\text{pre}}_\delta}(\xi)^{-1}DT_{u^{\text{pre}}_\delta}(m, \xi, m_1, \xi_1, F^D(m, \xi))| \leq c(|du^{\text{pre}}_\delta| + |\nabla\xi|)(m, \xi)(\xi, m_1)|.
\]
Hence
\begin{equation}
(129) \quad \|T^\delta_{u^{\text{pre}}_\delta}(\xi)^{-1}DT_{u^{\text{pre}}_\delta}(m, \xi, m_1, \xi_1, F^D(m, \xi))\|_{0,\lambda} \leq c(1 + \|du^{\text{pre}}\|_{0,\lambda} + \|\nabla\xi\|_{0,\lambda})\|\xi\|_{L^\infty}(\xi, m_1)\|_{L^\infty}.
\end{equation}
It follows that
\begin{equation}
(130) \quad \|T^\delta_{u^{\text{pre}}_\delta}(\xi)^{-1}DT_{u^{\text{pre}}_\delta}(m, \xi, m_1, \xi_1, F^D(m, \xi))\|_{0,\lambda} \leq c\|\xi\|_{1,\lambda}\|m_1\|_{1,\lambda}\|\xi_1\|_{1,\lambda}
\end{equation}
since the $W^{1,p}$ norm controls the $L^\infty$ norm by the uniform Sobolev estimates. Then, as in McDuff-Salamon [17] Chapter 10, Abouzaid [13] there exists a constant $c > 0$ such that for all $\delta$ sufficiently small, after another redefinition of $C$ we have
\begin{equation}
(131) \quad \|T^\delta_{u^{\text{pre}}_\delta}(\xi)^{-1}D_{\text{exp}_{u^{\text{pre}}_\delta}(\xi)}(Dm_j^\delta(m_1), D_{\text{exp}_{u^{\text{pre}}_\delta}(\xi)}(\xi_1)) - D_{u^{\text{pre}}_\delta}(m_1, \xi_1)\|_{0,\lambda} \leq C\|m, \xi\|_{1,\lambda}\|m_1, \xi_1\|_{1,\lambda}.
\end{equation}
Combining these estimates and integrating completes the proof of claim (126).

Step 6: Apply the implicit function theorem to obtain an exact solution. Applying the estimates (123), (125), (126) produces a unique solution $m(\delta), \xi(\delta)$ to the equation
FLOER COHOMOLOGY AND FLIPS

7. The break-up process

In this section we show that the Fukaya algebra is homotopy equivalent to the broken Fukaya algebra. The proof is a degeneration which produces holomorphic curves which “match”, in a certain sense, at the resulting hypersurface. A second degeneration, discussed in Bourgeois [21] and Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [20], involves choosing a Hamiltonian perturbation on the neck and letting the perturbation tend to zero as the neck lengthens. This degeneration produces holomorphic curves together with Morse flow lines and the result is the broken Fukaya category of the previous section.

7.1. Breaking a symplectic manifold. First we recall some terminology from Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [20].

Definition 7.1. (a) (Neck-stretched manifold) Let \( X \) be a closed almost complex manifold, and \( Z \subset X \) a separating real hypersurface. Let \( X^\circ \) denote the manifold with boundary obtained by cutting open \( X \) along \( Z \). Let \( Z', Z'' \) denote the resulting copies of \( Z \) obtained by gluing together the ends of \( X^\circ \) using a neck of length \([−τ, τ]\).

(b) (Neck-stretched form with perturbation) For \( ϵ \) small we consider the following perturbations corresponding to a Morse function \( H : Y \to \mathbb{R} \). We denote by the same notation the pull-back of \( H \) to \( Z \). Choose a cutoff function \( β \in \pi^* \mathbb{Y} \) equal to 1 on \([−τ + 1, τ − 1]\) \times \( Z \), and with support in \([−τ, τ] \times Z \). Let \( α \in Ω^1(Z) \) be a connection one-form, and let

\[
\omega_ϵ = π_Y^* \omega_Y + ϵ d(βHα) \in Ω^2(\mathbb{X})^\tau. 
\]

Define the perturbed Reeb vector field by

\[
v_ϵ \in \text{Vect}(\mathbb{X}^\tau), \quad \iota(v_ϵ)ω_ϵ = 0, \quad α(v_ϵ) = 1. 
\]

Differentiating \( v_ϵ \) with respect to \( ϵ \) at \( ϵ = 0 \) we obtain

\[
\iota \left( \frac{d}{d ϵ} v_ϵ \big|_{ϵ=0} \right) π_Y^* ω_Y = −\iota(v) d(βHα), \quad α \left( \frac{d}{d ϵ} v_ϵ \big|_{ϵ=0} \right) = 0. 
\]

This implies that on the neck region, \( \frac{d}{d ϵ} v_ϵ \big|_{ϵ=0} \) is the horizontal lift of minus the Hamiltonian vector field of \( H \).

Definition 7.2. (Almost complex structures on neck-stretched manifolds) Let \( π_Y : Z \to Y \) be a circle bundle over a symplectic manifold \( Y \) and \( X = \mathbb{R} \times Z \). Let

\[
π_X : X \to \mathbb{R}, \quad π_Z : X \to Z 
\]
the projections onto factors and \( \ker(\pi_Z) \subset X \) the vertical subspace. Let \( \omega_Z = \pi_\gamma^* \omega \subset \Omega^2(Z) \) denote the pullback of the symplectic form \( \omega \) to \( Z \). Then

\[
V = \ker(D\pi_Y) \oplus \ker(D\pi_Z) \subset TX
\]

is a rank two subbundle complementary to

\[
H = \ker(\alpha) \subset \pi_\gamma^* TZ \subset TX.
\]

The \( \mathbb{R} \) action by translation and \( U(1) \) action on \( Z \) combine to a smooth \( \mathbb{C}^\times \cong \mathbb{R} \times U(1) \) action on \( X \). We say that an almost complex structure on \( X \) is cylindrical iff \( J_\frac{\partial}{\partial t} = v \) in (135) and is invariant under the \( \mathbb{C}^\times \)-action. Any cylindrical almost complex structure induces an almost complex structure \( J_Y \) on \( Y \) by projection.

There are isomorphisms of complex vector bundles

\[
TX \cong H \oplus V, \quad H \cong \pi_\gamma^* TY, \quad V \cong X \times \mathbb{C}.
\]

Given a cylindrical almost complex structure and a Hamiltonian perturbation as in (133) define the perturbed almost complex structure by requiring that \( J_\epsilon \in \mathcal{J}(X^\tau) \)

\[
J_\epsilon \frac{\partial}{\partial t} = v_\epsilon.
\]

7.2. Breaking perturbation data. The next lemmas allow us to develop a perturbation scheme for broken pseudoholomorphic maps which will be compatible with degeneration. Suppose that \( J_t \) is a family of almost complex structures on \( X \) obtained by stretching the neck at \( Z \), using the Morse function \( H \).

Lemma 7.3. (Symplectic sums for pairs) Suppose that \( X_{\subset}, X_{\supset} \) are symplectic manifolds both containing a symplectic hypersurface \( Y \subset X_{\subset}, X_{\supset} \) such that the normal bundles \( N^\pm_Y \) of \( Y \) in \( X_{\subset}, X_{\supset} \) are inverse. Suppose furthermore that \( D_{\subset}, D_{\supset} \subset X_{\subset}, X_{\supset} \) are codimension two symplectic submanifolds such that \( D_{\subset}, D_{\supset} \cap Y = D_Y \) for some \( D_Y \subset Y \). Then the symplectic sum

\[
D = D_{\subset} \# D_{\supset}
\]

is naturally a symplectic submanifold of \( X = X_{\subset} \#_Y X_{\supset} \) preserved by the family of almost complex structures \( J_t \).

Proof. This is an application of symplectic local models as in Lerman [66] and Gompf [46]. Choose a metric on \( X_{\subset}, X_{\supset} \) near \( Y \) so that \( D_{\subset}, D_{\supset} \) is totally geodesic in \( X_{\subset}, X_{\supset} \), as in [71] Lemma 6.8; this can be done in stages so that the metrics on \( Y \) (considered as submanifolds of \( X_{\subset}, X_{\supset} \)) agree. The geodesic exponential map \( N_{\subset} \to X_{\subset}, N_{\supset} \to X_{\supset} \) identifies a neighborhood of the zero section in \( N_{\subset}, N_{\supset} \) restricted to \( D_Y \) with a neighborhood of \( D_Y \) in \( D_{\subset}, D_{\supset} \). Choose a unitary connection \( \alpha_+ \) on \( N_{\subset} \), let \( \alpha_- \) denote the dual connection on \( N_{\supset} \) and let \( \rho : N \to \mathbb{R}_{\geq 0} \) denote the norm function. Let \( \pi : N \to Y \) denote the projection. The two forms

\[
\pi^* \omega_Y + d(\alpha_+, \rho) \in \Omega^2(N_{\subset}), \quad \pi^* \omega_Y - d(\alpha_-, \rho) \in \Omega^2(N_{\supset})
\]

are symplectic in a neighborhood of the zero section and the punctured normal bundles \( N^\circ_{\subset}, N^\circ_{\supset} \) may be glued together to form the symplectic sum.
In order to globalize this procedure we have to check that the symplectic normal forms may be chosen so that the identification of the divisors is preserved. The constant rank embedding theorem in Marle [68] identifies a neighborhood of \( D_Y \) in \( D \) with \( N|D_Y \) symplectically. Now consider the identification of \( N \) with a neighborhood of \( Y \) on \( X_C, X_D \) which maps \( N|D_Y \) to \( D \). Let \( \omega_1 \) denote the pull-back of the symplectic form on \( X_C, X_D \) and \( \omega_0 \) the symplectic form induced from a connection on \( N \). We have \( \omega_1 - \omega_0 |(D \cup Y) = 0 \). Now \( D \cap Y \) is by assumption transverse, hence there exists a deformation retract of a neighborhood \( U \) of \( Y \) to \( (D \cup Y) \cap U \). The standard homotopy formula produces a one-form \( \beta \in \Omega^1(U) \) such that

\[
d\beta = \omega_1 - \omega_0, \quad \beta|(D \cup Y) \cap U = 0.
\]

Then \( \omega_t = t\omega_1 + (1-t)\omega_0 \) satisfies \( \frac{d}{dt}\omega_t = d\beta \). Define a vector field

\[
v_t = \frac{d}{dt}\omega_t, \quad \nu(v_t)\omega_t = \beta.
\]

Then \( v_t \) vanishes on \((D \cup Y) \cap U \) and defines a symplectomorphism from \( \omega_0 \) to \( \omega_1 \) in a neighborhood of \( Y \) equal to the identity on \( D \). Thus the gluing of local models induces an identification of the divisors as well.

**Lemma 7.4.** Suppose that \( L \subset X_C \) is a rational Lagrangian. There exists a family of divisors \( D_t \) on \( X \) so that the image of \( L \) in \( X \) is exact in the complement of \( D_t \), and \( D_t \) degenerates as \( t \to 0 \) to a broken symplectic manifold \( \mathbb{D} \) where \( D_C, D_D \) are divisors in \( X_C, X_D \) so that \( L \) is exact in the complement of \( D_C \).

**Proof.** Let \( \mathbb{D} = (D_C, D_D) \) be a broken divisor. Since the normal bundles \( N_Y^\pm \) of \( Y \) in \( X_C, X_D \) are inverses, the restrictions \( N_Y^\pm |D_Y \) are also inverses. Then \((Y, D_Y)\) has a symplectic tubular neighborhood of the form \((N_Y^+, N_Y^- |D_Y)\), where the latter has symplectic structure induced by the choice of connection as in the Lemma 7.3. Furthermore, \( D \) is almost complex with respect to every element of the family \( J_t \) by construction.

We now construct a system of perturbations for the breaking process. We say that a **breaking disk** is a disk with a **breaking parameter** \( \rho \in [0, \infty) \) such that if \( \rho < \infty \) resp. \( \rho = \infty \) then the disk is unbroken resp. broken. The compactified moduli space of breaking disks of type \( \Gamma \) is denoted \( \overline{M}_\Gamma \) with a universal curve \( \overline{U}_\Gamma \) that has surface part \( \overline{S}_\Gamma \) and tree part \( \overline{T}_\Gamma \).

**Definition 7.5.** (Perturbation data) A **perturbation datum** for breaking curves of type \( \Gamma \) is a datum

\[
P_\Gamma = (J_\Gamma, F_\Gamma, G_\Gamma, H_\Gamma),
\]

\[
J_\Gamma : \overline{S}_\Gamma \to \mathcal{J}(X), \quad F_\Gamma : \overline{T}_{\Gamma, \rho} \to C^\infty(L), \quad G_\Gamma : \overline{T}_{\Gamma, \rho} \to \mathcal{G}(L), \quad H_\Gamma : \overline{T}_\Gamma \to C^\infty(Y)
\]

such that \( J_\Gamma \) takes values in the space of almost complex structures adapted to the neck in a neighborhood of the curves with parameter \( \epsilon = 0 \), that is, are obtained by gluing from a map to \( \mathcal{J}(X^0_\circ) \times \mathcal{J}(R \times Z) \mathcal{J}(X^0_\circ) \) via the map \( \Omega^1 \).

Associated to any morphism of combinatorial types is a morphism of perturbation data, as before. However in this case there is a new kind of morphism of combinatorial types corresponding to **gluing of cylindrical end symplectic manifolds**. The
corresponding morphism of perturbation data takes the almost complex structure on the cylindrical parts of the corresponding curve by pull-back as in [132], [136].

We say that a collection of perturbations are coherent if, near the boundary of the moduli space of domains, the perturbations are given by pulling back under the morphisms induced by morphisms of graphs of type (Breaking an edge of infinite length), (Product axiom), (Collapsing edges), (Making edges or weights finite or non-zero).

Given a domain $C$, we obtain a perturbation datum on $C$ by pulling back the perturbation data on $\mathcal{U}_{f(\Gamma)}$, where $f(\Gamma)$ is the type of the stabilization of $C$.

**Definition 7.6.** Given a perturbation datum an adapted breaking stable map $u$ of type $\Gamma$ is a breaking disk $C$ with breaking parameter $\rho \in [0, \infty)$ together with a map $C \to X$ that is $J_{\Gamma}$-holomorphic for $\rho > 0$, or a map $C \to \bar{X}$ for $\rho = 0$ that is $J_{\Gamma,-}$ resp. $J_{\Gamma,0}$ resp. $J_{\Gamma,+}$-holomorphic on the pieces $C_0$ resp. $C_1, \ldots, C_k$ resp. $C_{k+1}$.

The following compactness result is a Lagrangian version of a result of Bourgeois [21], which technically seems to be new but is a combination of already established cases. (Bourgeois et al. [21] does not treat curves with boundary or treed disks, while Abbas [1] treats, to some extent, curves with boundary but not the Morse-Bott limit.)

**Theorem 7.7.** Let $P = (P_{\Gamma})$ be a coherent collection of perturbation data for breaking disks. Let $\tau_\nu \to \infty$ be a sequence of neck lengths with $0 < \epsilon_\nu \leq \epsilon_0$ and $\epsilon_\nu \to 0$ a sequence of perturbation parameters such that $\tau_\nu \epsilon_\nu \to \infty$. Any sequence of adapted stable maps $u_\nu : C_\nu \to X^\tau_\nu$ holomorphic with respect to $J_{\Gamma}$ with bounded energy converges, after passing to a subsequence, to a stable adapted broken map $u : C \to X$ with the same index.

The proof of the compactness result Theorem 7.7 is similar to standard Gromov compactness, with the exception that because the Hamiltonian perturbation is vanishing in the limit, the annulus lemma fails so that the bubbles appearing in the limit need not connect. Instead, the bubbles are connected by Morse gradient segments. We first prove the following lemma.

**Lemma 7.8.** (Convergence on cylinders to Morse trajectories) Let $J$ be an almost complex structure on $X$, $H$ a smooth function on $X$, $C_\nu := [-s_\nu, s_\nu] \times S^1$ a sequence of cylinders with $s_\nu \to \infty$, $\rho_\nu \to 0$ a sequence of positive real numbers, and $u_\nu : C_\nu \to Z \times \mathbb{R}$ a sequence of $J_{\rho_\nu, \pi_{\nu}^* H}$-holomorphic maps (see (133)) such that $\rho_\nu \to 0$ has sup $|\pi_Y du_\nu|/\rho_\nu$ bounded with respect to the standard metric on the domain $C_\nu$. Then there exists a cylinder $C := [-s_0, s_0] \times S^1$ and a smooth map $u_Y : C \to Y$ such $\partial_Y u = 0$, $\partial_s u = -\text{grad}_H(u)$ and $u_\nu$ converges to a lift $u$ of $u_Y$ uniformly in all derivatives on compact subsets of the interior.

Proof. We write $u_\nu = (\phi_\nu, v_\nu)$ in terms of the components $X = \mathbb{R} \times Z$. Because the domain is a sequence of annuli converging to a trivial cylinder, the map $\phi_\nu$ converges to $\phi(s) = \mu s$ and $\phi(t)$ converges to the Reeb orbit $\gamma(t)$ with multiplicity $\mu$. In cylindrical coordinates the map $u_\nu$ is given on $A_{w, v}$ satisfies

\begin{equation}
\partial_s u_\nu + J_{\rho_\nu, H} \partial_t u_\nu = 0, \text{ hence } \partial_s \pi_Y u_\nu + J_Y (\partial_t \pi_Y u_\nu - \alpha(\partial_t u) \rho_\nu \dot{H}(\pi_Y u)) = 0
\end{equation}
Sketch of Proof of Theorem. In the case of separating hypersurfaces of contact type, almost complex structures preserving the horizontal and vertical subspaces, and no Lagrangian boundary conditions, a proof is sketched in Bourgeois [21]. Recall that the sft compactness result (again for separating hypersurfaces of contact type) is extended to the case of Lagrangian boundary conditions in Abbas [1].

We first construct the domain of the limiting map. As in the references above and Theorem 6.5 we equip the surface parts $S_\nu$ with hyperbolic metrics with injectivity radii $r_\nu : S_\nu \to \mathbb{R}_{>0}$, and add marked points until there exist bounds on the first derivatives $\sup |du_\nu|/r_\nu$. The argument of [6.5], [1] shows that there exists a stable limit $S$ of the curve parts $S_\nu$ such that $u_\nu|S_\nu$ converges uniformly in all derivatives on compact subsets of complement of the the nodes to some smooth map $u : S^X \to X$, where $S^X$ is the complement of the nodes. However, the bubbles may not “connect” to a continuous map because of the failure of the annulus lemma. In particular, the domain captures all bubbling sequences $z_\nu \in S_\nu$ such that for some sequence local coordinate near $z_\nu$ with radius of definition bounded from below, $|d\pi_Y u_\nu(z_\nu)|$ is bounded below.

We claim that the various surface components of the limit are connected by gradient trajectories, as in Bourgeois [21]. Choose identifications of $S^X$ with the complements $S^X_\nu$ of a finite set of circles in $S_\nu$ corresponding to the nodes; such an identification may be defined by considering $S_\nu$ as fibers of a Lefschetz fibration with central fiber $S$ and choosing a connection on the complement of the singular set. Choose small disks $B^\pm_w$ on either side of the node and consider the cylinder $S^X_\nu \setminus B_w$ between $B^\pm_w$. The union of $B^\pm_w$ with the circle corresponding to the node is a finite cylinder (annulus) denoted $A_{w,\nu} \subset S_\nu$. By the collar lemma (see e.g. [1 Theorem 1.78]) the annulus may be realized as a hyperbolic surface as a subset of the quotient of the upper half plane $\mathbb{H}/(z \sim \rho_\nu z)$ satisfying an inequality of the form $\theta^\nu_+ - \arg(z)/\pi \leq \theta^\nu_+$ where $\theta^\nu_+ \to 0, \theta^\nu_+ \to 1$, and the convergence rate is sufficiently slow compared to the convergence of $\rho_\nu$ to zero so that the lengths of the paths $r \exp(i\theta^\nu_+), r \in [1,\rho_\nu]$ converges to zero. In the cylindrical metric on the finite cylinders, the bound on $c_\nu = \sup |du_\nu|/r_\nu$ still holds, since the conformal rescaling factor $\|f\|_{cy}/\|f\|_{hyp}$ between the two metrics (subscripts denoting cylindrical and hyperbolic metrics respectively) lies in $[r_\nu/2\rho_\nu, 2r_\nu/\rho_\nu]$, see the discussion in [1 p. 230] for comparison of the metric and [1 Lemma 1.83] for a formula for the injectivity radius. In the cylindrical metric the injectivity radius $r_\nu = \rho_\nu$ is a constant.

We distinguish the following cases depending on how the blow-up rate of the first derivative compares to the rate at which the radii of the cylinders converges to zero. If $c_\nu, \epsilon_\nu$ converges to infinity (after passing to a subsequence) then since
$\text{d}u_\nu$ is bounded, the cylinders must converge to a critical point of $H$, and so the maps $u_\nu$ necessarily converge to a constant. Choose a $\zeta$-neighborhood $\text{crit}(H)_\zeta$ of $\text{crit}(H)$ with $\zeta$ sufficiently small so that the balls around each critical point are disjoint, let $A_{w,\nu,\zeta}' = S^1 u_\nu^{-1}(\text{crit}(H)_\zeta)$ denote the $t$-flow-out of the inverse image of the complement, and $A_{w,\nu,\zeta}'' = A_{w,\nu} - A_{w,\nu,\zeta}'$. Hence the decomposition $A_{w,\nu} = A_{w,\nu,\zeta}' \cup A_{w,\nu,\zeta}''$ is a decomposition of the neck region $A_{w,\nu}$ corresponding to the node $w$ into further finite cylinders $A_{w,\nu,\zeta,k}', A_{w,\nu,\zeta,l}''$ (possibly infinite in number) so that $A_{w,\nu,\zeta,k}'$ map to neighborhoods of the critical locus $\text{crit}(H)$ and on $A_{w,\nu,\zeta,l}''$ the derivative $\text{d}H$ is bounded from below.

\[\text{Figure 38. Limit of cylinders with small derivative}\]

Hence if $c_\nu$ denotes the supremum of the first derivative on $A_{w,\nu,\zeta,l}''$, then $c_\nu \epsilon_\nu$ is bounded. We rescale by $\epsilon_\nu$ so that the first derivative of the rescaled map $u_\nu(\epsilon_\nu s, \epsilon_\nu t)$ is bounded (possibly converging to zero). By Lemma 7.8 in the limit $\nu \to \infty$ one obtains for each sequence $A_{w,\nu,\zeta,l}'$ a subsequence that converges to a piece $u : [-s_l, s_l] \times S^1 \to Y$ constant in $t$ and a gradient trajectory of $H$, possibly of finite or infinite length. Any trajectory connecting distinct components of $\text{crit}(H)_\zeta$ must decrease the value of $H$, and it follows that the numbers of pieces $A_{w,\nu,\zeta,l}''$ with non-trivial limits (and so connecting different components of $\text{crit}(H)_\zeta$) must be finite. The same argument with $\zeta$ replaced with $\zeta/2^m$ further gives limiting Morse trajectories which connect components of $\text{crit}(H)_{\zeta/2^m}$. A standard diagonal argument gives a sequence which connects $\text{crit}(H)_{\zeta/2^m}$ for any $m$, hence a broken Morse trajectory. See Figure 38 where the regions $\text{crit}(H)_\zeta$ are the shaded balls and $A_{w,\nu,\zeta,l}'$ are the shaded regions on the cylinder.

We claim that the limit is adapted and has uncrowded type. By assumption, the Morse trajectories are disjoint from the stabilizing divisor, so there are no ghost components containing interior markings attached to Morse trajectories. Thus any maximal ghost component with at least two interior markings is connected to at least one non-constant component. Either it is connected to a single component, then the adjacent non-constant component has intersection multiplicity two at the node, by conservation of intersection multiplicity under degeneration. Since this is a codimension two condition, it cannot occur. Similarly if the ghost bubble is adjacent to two non-constant components then by removing the ghost bubble we obtain a configuration with a node mapping to the divisor, which is again a codimension two condition and so does not occur.

$\square$
Theorem 7.9. Suppose that \( u : C \to X \) is a regular broken map with \( k \) levels. Then there exists \( \delta_0 > 0 \) such that for each gluing parameter \( \delta \in (0, \delta_0) \) there exists an unbroken map \( u_\delta : C_\delta \to X \), with the property that \( u_\delta \) depends smoothly on \( \delta \) and \( \lim_{\delta \to 0} [u_\delta] = [u] \).

We prove Theorem 7.9 in Section 7.3. In fact, we only use the following weaker version for Theorem 7.9 whose proof is easier:

Theorem 7.10. Suppose that a sequence of holomorphic weighted treed disks \( u_\nu : C_\nu \to X \) converges to a regular broken weighted treed disk \( u : C \to X \) as \( \nu \to \infty \). Then

(a) the indices of \( u_\nu \) and \( u \) matching for \( \nu \) sufficiently large;
(b) any sequence of elements in the kernel resp. cokernel of \( D_{u_\nu} \) converges with norm one (after passing to a subsequence) to a non-zero element in the kernel resp. cokernel of \( D_u \).

Proof. Equality of indices follows from the index formula in Proposition 6.17. Identification of the kernels and cokernels is a standard argument using elliptic regularity and a gluing theorem for elements of the kernel and cokernel, see for example [109, Theorem 2.4.5]. □

Corollary 7.11. Suppose that a collection \( P \) of regular perturbations for broken maps have been chosen. For each \( E > 0 \), there exists \( \rho_0 \) such that if \( \rho > \rho_0 \) then \( \mathcal{M}^{<E}(X, L, P_\rho) \) is independent of \( \rho \) and every element is regular.

Proof. By Theorem 7.7, any sequence in \( \mathcal{M}(X_{\rho_\nu}, L, x) \) with bounded energy and \( \rho_\nu \to \infty \) converges to an element of \( \mathcal{M}(X, L, x) \). If \( \xi_\nu \) is a sequence of elements in the cokernel of \( D_{u_\nu} \) with norm one then after passing to a subsequence one obtains an element in the cokernel of the limiting linearized operator \( D_u \) by the second part of Theorem 7.10. Since \( D_u \) is surjective by assumption, \( D_{u_\nu} \) is surjective for sufficiently large \( \nu \) as well. □

Corollary 7.12. The higher composition maps \( \mu^{n,\rho} \) of the \( A_\infty \) algebra \( CF(L, P_\rho) \) have a limit as \( \rho \to \infty \):

\[
\mu^{n,\infty} := \lim_{\rho \to \infty} \mu^{n,\rho}.
\]

The limit \( CF(L, P_\infty) \) is convergent-\( A_\infty \)-homotopy equivalent to \( CF(L, P_\rho) \) for any finite \( t \).

Proof. Since for any energy bound \( E \), the terms in \( \mu^{n,\rho} \) of order at most \( q^E \) are independent of \( t \) for \( \rho > \rho_0 \) by Lemma 7.11. The \( A_\infty \) axiom follows from the \( A_\infty \) axiom for \( \mu^{n,\rho} \) for each bound \( E \). Convergent homotopy equivalence follows from the fact that for any energy \( E \) and any path \( J^t \) there are at most finitely many homotopy classes of holomorphic disks and spheres with energy at most \( E \); the same holds for sufficiently small \( C^2 \) perturbations. □

Corollary 7.13. For any energy bound \( E > 0 \), there exists \( \rho_0 > 0 \) such that if \( \rho > \rho_0 \) then there is bijection between \( \mathcal{M}^{<E}(X, L, D, P_\rho, l, x) \) and \( \mathcal{M}^{<E}(X, L, D, l, x) \).
Proof. By Theorem 7.9 and the compactness result Theorem 7.7.

\textbf{Theorem 7.14.} Suppose that \( P_{\rho} \) converges to a regular coherent stabilizing perturbation system \( P \) for broken treed disks as above. Then the limit \( \lim_{\rho \to \infty} CF(L, P_{\rho}) \) is equal to the broken Fukaya algebra \( CF(X, L) \), and in particular \( CF(L, P_{\rho}) \) is homotopy equivalent to \( CF(X, L) \).

Proof. By Theorem 7.13, there is a bijection between the moduli spaces defining the structure coefficients of the Fukaya algebras \( CF(L, P_{\rho}) \) for \( \rho \) sufficiently large and \( CF(X, L) \). The bijection preserves the area \( A([u]) \) of the map as well as the homology class \( [\partial u] \in H_1(L) \) of the restriction of the map to the boundary, hence preserves the holonomies of the flat connection on the brane around the boundary of the disk. It follows from the gluing results in e.g. [109] that the bijection is orientation preserving. (One may treat the Morse trajectory as a cylinder of Chern number zero as in [140].) The last statement follows from Corollary 7.12.

We now specialize to the case that \( X_\subset \) is the toric piece near the exceptional locus of a reverse flip as in Theorem 6.24.

\textbf{Proposition 7.15.} Let \( X \) be obtained by a small reverse flip or blow-up, \( L \subset X \) a regular Lagrangian near the exceptional locus. Let \( P_{\rho} \) be as in Theorem 7.14. The limit \( \lim_{\rho \to \infty} CF(L, P_{\rho}) \), hence \( CF(L, P_{\rho}) \) for any finite \( \rho \), is projectively flat.


\textbf{Corollary 7.16.} Let \( X \) be obtained by a small reverse flip or blow-up, \( L \subset X \) a regular Lagrangian near the exceptional locus. For any perturbation system \( P \), there exists a weakly bounding cochain \( b \in MC(L) \) so that \( CF(L, b, P) \) is projectively flat and has non-trivial cohomology.

Proof. Let \( X_\subset \) be the toric piece from Theorem 6.24. By Theorem 6.24, \( L \) considered as a Lagrangian in the broken symplectic manifold \( (X_\subset, X_\subset) \) is weakly unobstructed with non-trivial Floer cohomology. By Proposition 7.15, the same holds for \( L \) considered as a Lagrangian submanifold of \( X \).

\textbf{Remark 7.17.} (Standard Lagrangians in Darboux charts are weakly unobstructed) If \( x \) is an arbitrary rational compact symplectic manifold, and \( (q_1, p_1, \ldots, q_n, p_n) \) are Darboux coordinates near a point \( x \in X \), the same argument shows that torus orbits \( L = \{ (p_j^2 + q_j^3) = c_j, j = 1, \ldots, n \} \) for some small constants \( c_1, \ldots, c_n \), have unobstructed but trivial Floer cohomology.

### 7.3. Getting back together, redux.

The existence of a tubular neighborhood for broken strata requires a gluing theorem generalizing that in Theorem 6.25, which is similar but slightly different from the gluing theorem for Morse trajectories and holomorphic curves in Bourgeois-Oancea [19]. Given a broken curve \( C \) with, for simplicity, levels \( C_+, C_- \) and no intermediate Morse trajectory, a standard gluing procedure creates, for any small \textit{gluing parameter} \( \delta \in \mathbb{C} \), an unbroken curve \( C_\delta \) obtained by removing small disks around the attaching points of the trajectory and gluing in using a map given in local coordinates by \( z \mapsto \delta/z \).
Theorem 7.18. Suppose that \( u : C \to X \) is a regular adapted broken map of type \( \Gamma \) with a single gradient trajectory for \( F_\Gamma \). Let \( \tilde{\Gamma} \) denote the combinatorial type of unbroken map obtained by replacing the gradient segment with a cylinder. For any function \( \epsilon(\tau) \) with \( \tau \epsilon(\tau) \to 0 \) as \( \tau \to \infty \), let \( J_\tau \) denote a Hamiltonian perturbation on the neck as in (136). There exists an \( \tau_0 > 0 \) and for \( \tau > \tau_0 \) a family of unbroken curves \( C_\tau \) and unbroken maps \( u_\tau : C_\tau \to X, \tau > \tau_0 \) of type \( \tilde{\Gamma} \) that are \((J_\tau, F_{\tilde{\Gamma}})\)-holomorphic and satisfying \( \lim_{\delta \to 0}[u_\delta] = [u] \).

Proof. The proof of this gluing result is similar to the previous result Theorem 6.25; however, now the equation to be solved has a Hamiltonian perturbation on the neck region. The fundamental new issue is gluing a Morse trajectory to a pseudoholomorphic map. To illustrate the new issue, we assume that \( X = \mathbb{R} \times Z \) is cylindrical, the broken map has a single level and two sublevels joined by a Morse trajectory of length one.

Step 1: Preliminaries with Morse trajectories We suppose the broken Morse trajectory is \( u^T : [-1, 1] \to Y \), for simplicity. Choose a map \( \phi_\epsilon : \mathbb{R} \to [-1, 1] \) satisfying

\[
(138) \quad \phi_\epsilon(s) = \begin{cases} 
1 & s \geq |1/\epsilon| \\
\epsilon s & -|1/\epsilon| + 1 \leq s \leq |1/\epsilon| - 1 \\
-1 & s \leq -|1/\epsilon|
\end{cases}
\]

Given a time-dependent Morse function on \([-1, 1] \times X\), the pull-back of the gradient flow equation to \( \mathbb{R} \) is

\[
(139) \quad \frac{d}{ds} u(s) = (\phi_\epsilon') \text{grad} H(\phi_\epsilon(s), u(s)).
\]

Define the Morse cylinder

\[
(140) \quad \tilde{u}^T(s, t) = u^T(\phi_\epsilon(s)), \quad (s, t) \in \mathbb{R} \times S^1.
\]

The Morse cylinder satisfies the equation

\[
\partial_s \tilde{u}^T(s, t) + J\partial_t(\tilde{u}^T(s, t) - (\phi_\epsilon') \phi_\epsilon^* F_{\tilde{\Gamma}}) = 0.
\]

The map \( \tilde{u}^T \) is \((J, (\phi_\epsilon') \phi_\epsilon^* F_{\tilde{\Gamma}})\)-holomorphic and equal to \( u^T(\epsilon s) \) for \( s \in (-|1/\epsilon| + 1, |1/\epsilon| - 1) \).

Replace the Morse segment with the Morse cylinder produces a map whose domain is the disjoint union of three curves (the first with boundary and segments attached.) Let \( \tilde{u} \) denote the map with disjoint domain

\[
(141) \quad \tilde{C} = C_- \sqcup (\mathbb{R} \times S^1) \sqcup C_+.
\]

We wish to relate the linearized operators of the Morse cylinder and Morse trajectory. Let \( \Omega(\tilde{C}, \tilde{u}^*TX)_{1,p,\lambda} \) denote the space of maps of Sobolev class \( 1, p, \lambda \) in the sense of (119), satisfying matching condition at the ends. There is a space of
By surjectivity of \( \eta \), then pushing-forward \( \eta \) we may assume that

\[ \text{By integrating we may find } \xi \text{ with the property that} \]

\[ J_\xi \text{ regularity, a subsequence of } (u_\xi)_{S^1} \text{ injects to consider a triple} \]

\( \text{where } \text{cokernel break into weight spaces for the } \pi \text{ where in the last factor } \pi_D \text{ is projection onto } D \) using a local coordinate chart.

Suppose that \( D_u \) is surjective. We claim that there exists \( \delta_0 > 0 \) such that for \( \delta < \delta_0 \), the linearization \( D_{\tilde{u}} \) of (142) is has a uniformly bounded right inverse.

First we apply a Fourier analysis on the neck piece. On the neck piece \( \mathbb{R} \times S^1 \in \tilde{C} \) from (141), the operator \( D_{\tilde{u}} \) is \( S^1 \)-invariant and the kernel and cokernel break into weight spaces for the \( S^1 \)-action:

\[ \text{coker}(D_{\tilde{u}}) = \bigoplus_{m \in \mathbb{Z}} \text{coker}(D_{\tilde{u}})_m \]

where \( \text{coker}(D_{\tilde{u}})_m \) is spanned by elements of the form \( \eta(s) \exp(2\pi i mt)(ds + idt) \). If \( \xi(\delta_r) \) is a sequence of sections in coker \( (D_{\tilde{u}})_m = 0 \) with \( \delta_r \to 0 \) then, via elliptic regularity, a subsequence of \( \xi(\delta_r) \) converges to a non-zero element of coker \( \nabla_s + J \nabla_t \). The latter is empty, hence no such sequence exists. It follows that for \( \delta \) sufficiently small, coker \( (D_{\tilde{u}})_m = 0 \) unless \( m = 0 \).

By the previous paragraph, to show surjectivity of the linearized operator it suffices to consider a triple \( \eta = (\eta_-, \eta_0, \eta_+) \) of one-forms on \( C_-, \mathbb{R} \times S^1, C_+ \) respectively with the property that \( \eta_0 \) is \( S^1 \)-invariant. We wish to construct \( \xi \) such that \( D_{\tilde{u}} \xi = \eta \).

By integrating we may find \( \xi_0^\pm \in \Omega^0(\mathbb{R} \times S^1, \tilde{u}^*TX) \) such that

\[ \lim_{s \to \pm \infty} \xi_0^\pm(s, t) = 0, \quad D_{\tilde{u}} \xi_0^\pm|_U = \eta_0|_U \]

for some open neighborhood \( U \) of \( [\pm \infty, \pm (|\ln(\delta)| - 1)] \). Subtracting \( D_{\tilde{u}} \xi_0^\pm \) from \( \eta_0 \) we may assume that \( \eta_0 \) is supported in the region on which \( \phi_\epsilon \) is a diffeomorphism. Then pushing-forward \( \eta \) to \( (-1, 1) \) we obtain a one-form on \( C \) with values in \( u^*TX \).

By surjectivity of \( D_u \), there exists a triple \( \xi = (\xi_-, \xi_0, \xi_+) \) such that \( D_u \xi = \eta \). Pulling back the one-dimensional piece to \( \mathbb{R} \times S^1 \) under \( \phi_\epsilon \) we obtain a triple \( \tilde{\xi} = (\xi_-, \xi_0, \xi_+) \) with \( D_{\tilde{u}} \tilde{\xi} = \eta \), as desired.

**Step 2:** Construct an approximate solution. An approximate solution is given by gluing together \( u_\pm \) and \( \tilde{u}^T \) using a cutoff function. We assume that \( \beta \in C^\infty(\mathbb{R}) \) has the property that \( \beta(s) = 0, s \leq 0 \) and \( \beta(s) = 1, s \geq 1 \) as in (99). Let \( y \in Y \) be the evaluation of \( u_\pm \) at the node and \( \gamma(t) \) the Reeb orbit in the fiber over \( y \) in \( Z \). We form a domain curve \( C_\gamma \) by gluing together \( C_-^\circ, C_+^\circ, \mathbb{R} \times S^1 \) so that the cylindrical coordinates \( s, t \) on the gluing region are such that so that \( u_\pm \) considered locally as maps to \( X^0 \) are asymptotic to \( \gamma(t) \). We write

\[ u_\pm(s, t) = \exp(\mu_s, \gamma_\pm(t)(\xi_\pm(s, t)), \quad \tilde{u}^T = \exp(\mu_s, \gamma_\pm(t)(\tilde{\xi}^T(s, t)). \]
Recall that \( \epsilon(\delta) \) is a number much smaller than \( \delta \). We assume that
\[
\lim_{\delta \to 0} \epsilon(\delta)|\ln(\delta)| = 0
\]
so that the intervals \([1/\epsilon - |\ln(\delta)|/2, 1/\epsilon + |\ln(\delta)|/2]\) are disjoint and far away from 0.

\[ u^{\text{pre}}_\delta(s, t) = \begin{cases} 
  u_-(s + |\ln(\delta)|/2 + 1/\epsilon, t) & s \leq -|\ln(\delta)|/2 - 1/\epsilon \\
  \tilde{u}^T(s, t) & |\ln(\delta)|/2 - 1/\epsilon \leq s \leq -|\ln(\delta)|/2 + 1/\epsilon \\
  u_+(s - |\ln(\delta)|/2 - 1/\epsilon, t) & s \geq |\ln(\delta)|/2 + 1/\epsilon 
\end{cases} \]

There are two gluing regions missing from the description above, corresponding to the values \(|s| \in [1/\epsilon - |\ln(\delta)|/2, 1/\epsilon + |\ln(\delta)|/2]\). In these regions, one interpolates between the two maps on either side using the cutoff function and geodesic exponentiation. On the two regions corresponding to gluing a holomorphic map to a Morse trajectory define

\[ u^{\text{pre}}_\delta(s, t) = \exp(\mu_\delta(s)(t)) (\beta(-s + |\ln(\delta)|/2 + 1/\epsilon)\zeta_-(s + |\ln(\delta)|/2 + 1/\epsilon, z) + \beta(-s - |\ln(\delta)|/2 - 1/\epsilon)\zeta^T(s, t)). \]

on the first of gluing region while on the second

\[ u^{\text{pre}}_\delta(s, t) = \exp(\mu_\delta(s)(t)) (\beta(-s - |\ln(\delta)|/2 - 1/\epsilon)\zeta_+(s - |\ln(\delta)|/2 - 1/\epsilon, z) + \beta(s - |\ln(\delta)|/2 - 1/\epsilon)\zeta^T(s, t)). \]

Thus on each gluing region, \( u^{\text{pre}}_\delta \) is approximately equal to a \( \mu \)-fold cover of the orbit corresponding to the matching condition \( u^T(\pm 1) \).

**Step 3: Define a map cutting out the moduli space locally.** Define Sobolev spaces with exponential weight on the gluing regions as follows. Define a Sobolev weight function

\[ \kappa_\lambda : C_\delta \to [0, \infty), \quad (s, t) \mapsto \beta(|\ln(\delta)|/2 + 1/\epsilon - |s|) \beta(|\ln(\delta)|/2 - 1/\epsilon + |s|) \exp(\mu_\lambda(\ln(\delta))/2 - |s - 1/\epsilon|). \]

The function \( \kappa_\lambda \) takes values \( p\lambda |\ln(\delta)|/\epsilon \) in the middle of the neck region \( s = 1/\epsilon \), and 0 away from the neck regions. Given a smooth map \( u : C_\delta \to X \), element \( m \in M^1_\lambda \) and a section \( \xi : C_\delta \to u^*TX \) define

\[ \left\| (m, \xi) \right\|_{1, p, \lambda} := \left( \left\| m \right\|^p + \int_{C_\delta} (\|\nabla \xi\|_p^p + \|\xi\|_p^p) \exp(\kappa_\lambda d \text{Vol}_{C_\delta}) \right)^{1/p}. \]

Let \( \Omega^p(C_\delta, u^*TX)_{1, p, \lambda} \) be the space of \( W^{1, p}_{\text{loc}} \) sections with finite norm (147). Similarly, for a 0, 1-form \( \eta \in \Omega^{0, 1}(C_\delta, u^*TX) \) define

\[ \left\| \eta \right\|_{0, p, \lambda} := \left( \int_{C_\delta} \left\| \eta \right\|^p \exp(\kappa_\lambda d \text{Vol}_{C_\delta}) \right)^{1/p}. \]
As before let \( \exp_{u^0_\delta} \) denote pointwise geodesic exponentiation and \( T_{u^0_\delta}(\xi) \) parallel transport along \( \exp(t\xi) \). Define

\[
(148) \quad \mathcal{F}^D : \mathcal{M}_1 \times \Omega^0(C_\delta, (u^0_\delta)^*TX)_{1,p} \to \Omega^{0,1}(C_\delta, (u^0_\delta)^*TX)_{0,p} \oplus \bigoplus_{i=1}^n T_{u(z_i)}X/T_{u(z_i)}D
\]

\[
\xi \mapsto (T_{u^0_\delta}(\xi|S)^{-1}\overline{\partial}_{i,j,m} \exp_{u^0_\delta}(\xi|S),
\]

\[
\left( \frac{d}{dt} - \text{grad}(F_T)(\xi_U), (\pi_D(\exp(\xi(z_i))))_{i=1}^n \right).
\]

Treed pseudoholomorphic maps close to \( u^0_\delta \) correspond to zeroes of \( \mathcal{F}_\delta \). In addition, because we are working in the adapted setting, our curves \( C_\delta \) have a collection of markings \( z_1, \ldots, z_n \) and we work subject to the constraint

\[
(\exp_{u^0_\delta}(\xi))(z_i) \in D, \quad i = 1, \ldots, n.
\]

First, as in the case of gluing without a Morse trajectory one has the failure of the approximate solution to be an exact solution:

\[
(149) \quad \|\mathcal{F}^D(0)\|_{0,p,\lambda} < C(\exp(-|\ln(\delta)|)(1 - \lambda)/2) = C\delta^{1-\lambda}
\]

which is proved in the same way as \([123]\), using that the cylinder \( \hat{u}^T \) is an exact solution to the perturbed Cauchy-Riemann equation. We remark that in the more complicated case of a surface component mapping to a neck piece \( X|l| = \mathbb{R} \times Z \), there is an additional contribution to \((149)\) arising from the Hamiltonian perturbation away from the gluing regions. However, the assumed estimate \((143)\) implies that this error estimate is also exponentially small in \(|\ln(\delta)|\).

**Step 4:** Prove a uniform error estimates, as well as uniform bounds on the right inverse and uniform quadratic estimates. The analogs of estimates \([123], [125]\), and \([126]\) are proved as in the case of gluing holomorphic curves. Given a one-form \( \eta \in \Omega^{0,1}(C_\delta, (u^0_\delta)^*TX) \), we obtain one-forms \( \eta_-, \eta_0, \eta_+ \) using the cutoff function:

\[
\eta_\pm = \beta(\pm(s - |\ln(\delta)|/2 - 1/\epsilon))\eta, \quad \eta(s, t) = \eta_+ = \eta_+. \quad \eta_0(s, t) = \eta - \eta_- - \eta_+.
\]

By assumption the linearized operator \( D_u \) for the limit \( u \) is surjective; this means that there exists a tuple \( \xi = (\xi_-, \xi_0, \xi_+) \) with

\[
D_u(\xi_-, \xi_0, \xi_+) = (\eta_-, \eta_0, \eta_+)
\]

and \( \xi \) satisfying matching conditions at the joining points of the Morse trajectory with the surface part of the domain. Now we define an element \( \xi \in \Omega^0(C_\delta, (u^0_\delta)^*TX) \) by interpolating between the maps \((\xi_-, \xi_0, \xi_+)\):

\[
(150) \quad \xi(s, t) = \beta(s - |\ln(\delta)|/2 + 1/\epsilon)\xi_-(s + |\ln(\delta)|/2 + 1/\epsilon, t) + (1 - \beta(s - |\ln(\delta)|/2 - 1/\epsilon))(1 - \beta(-s + |\ln(\delta)|/2 + 1/\epsilon))\xi_0(s) + \beta(s - 1/\epsilon + |\ln(\delta)|/2)\xi_+(s - |\ln(\delta)|/2 - 1/\epsilon, t)
\]
Then $D_{u^\text{pre}} \xi - \eta$ is the sum of terms arising from differentiating the cutoff functions, each of which contributes a term of size bounded by $C\delta^{p-\lambda}$. The uniform quadratic estimate is similar to that in (126).

**Step 5:** *Apply the implicit function theorem.* This produces a unique solution $\xi(\delta)$ to the equation $\mathcal{F}_{\delta}^{\text{pre}}(m, \xi(\delta)) = 0$ for each $\delta$ with $\xi(\delta)$ in the image of the uniformly bounded right inverse, giving rise to a smooth family $\exp_{u^\text{pre}}(\xi(\delta))$.

The case that $u$ is a map with two levels (rather than two sublevels) is similar, but now involves three gluing regions: two where the (necessarily broken) Morse trajectory meets the surface part of the domain, and one gluing region where the Morse trajectories are glued together. The latter is standard while in the former regions the gluing estimates are identical to those above. □

### 7.4. Relative maps

In the remainder of this section we introduce a version of the Fukaya algebra for Lagrangians in symplectic manifolds equipped with divisors, slightly different from the standard version of relative Fukaya algebras in the sense that we allow a “bulk insertion” at the divisor. We express the broken Fukaya algebra as a relative Fukaya algebra with a bulk deformation coming from the “other piece”, similar to the expression of Gromov-Witten invariants in terms of relative Gromov-Witten invariants. Then we apply the relation between the relative and broken Fukaya algebras to show Theorem 1.2 of the introduction.

The domain of a relative map is the same as a broken curve in Definition 6.1 except that the last piece $C_s$ is allowed to have outgoing edges with infinite length. Rather than repeat the lengthy definition, we refer the reader to Figure 39.

**Figure 39.** A relative disk

A *relative map* is a map from a broken curve into a relative symplectic manifold, defined as follows. Let $X$ be a compact rational symplectic manifold and $Y \subset X$ a symplectic submanifold with normal bundle $N \to Y$. Let $Z \subset N$ denote the unit circle bundle. We view $X - Y$ as a manifold with cylindrical end modelled on $Z$, that is, with an end of the form $Z \times (0, \infty)$. Denote $\mathbb{P}(N \oplus \mathbb{C})$ and

$$X[m] = X \cup_Y \mathbb{P}(N \oplus \mathbb{C}) \cup_Y \mathbb{P}(N \oplus \mathbb{C}) \cup_Y \ldots \mathbb{P}(N \oplus \mathbb{C})$$
where there are $m - 2$ copies of $\mathbb{P}(N \oplus \mathbb{C})$ called relative levels. Define
\begin{equation}
X[m]_0 = X, \quad X[m]_1 = \mathbb{P}(N \oplus \mathbb{C}), \quad \ldots, \quad X[m]_m = \mathbb{P}(N \oplus \mathbb{C}).
\end{equation}
There is a natural action of the complex torus $(\mathbb{C}^\times)^{l-2}$ on $\mathbb{P}(N \oplus \mathbb{C})$ given by scalar multiplication on each projectivized normal bundle:
\[
\mathbb{C}^\times \times \mathbb{P}(N \oplus \mathbb{C}) \to \mathbb{P}(N \oplus \mathbb{C}), \quad (z, [n, w]) \mapsto [zn, w].
\]

Relative maps with Lagrangian boundary conditions are defined as follows. Let $L \subset X$ a Lagrangian disjoint from $Y$. Let $J \in \mathcal{J}(X)$ be an almost complex structure on $X$ of cylindrical form, $H$ be a Morse function on $Y$ and $(F, G)$ a Morse-Smale pair on $L$. A relative map to $(X, Y)$ with boundary values in $L$ consists of a relative curve $C = (C_0, \ldots, C_l)$; a collection of maps $C_k$ to $X[p]_k$ (notation from \cite{151}) for $k = 0, \ldots, p$ satisfying the same conditions for broken maps, except that now the final piece $C_l$ maps to either $X$ itself or $\mathbb{P}(N \oplus \mathbb{C})$.

In order to achieve transversality we introduce relative divisors. A relative divisor for a relative symplectic manifold $(X, Y)$ equipped with an almost complex structure preserving $Y$ consists of a codimension two almost complex submanifold $D$ such that each intersection $D \cap Y = D_Y$ is a codimension two almost complex submanifold in $Y$. Given a relative divisor $D$ as above we obtain a divisor
\[
D_N := \mathbb{P}(N|D_Y \oplus \mathbb{C}) \subset \mathbb{P}(N \oplus \mathbb{C}).
\]
We suppose that each $[D]$ is dual to a large multiple of the symplectic class on $X$, that is, $[D] = k[\omega]$. Then $[D_N] = k\pi_Y^*[\omega_Y]$, where $\pi_Y$ is projection onto $Y$, and as a result does not represent a multiple of any symplectic class on $\mathbb{P}(N \oplus \mathbb{C})$. Thus the divisor $D_N$ can be disjoint from non-constant holomorphic spheres in $\mathbb{P}(N \oplus \mathbb{C})$, namely the fibers. However, holomorphic spheres whose projections to $Y$ are non-constant automatically intersect $D_N$.

To achieve transversality we use almost complex structures equal to a fixed almost complex structure on the stabilizing divisor. We introduce the following notations. For a symplectic manifold $X^\circ$ with cylindrical end, denote by $\mathcal{J}(X^\circ)$ the space of tamed almost complex structures on $X^\circ$ that are of cylindrical form on the end. Define $\mathcal{J}(X, Y)$ the space of almost complex structures on $X^\circ$ and induce the same almost complex structure on $Y$. Given $J_D \in \mathcal{J}(X, Y)$, we denote by $\mathcal{J}(X, J_D)$ the space of almost complex structures in $\mathcal{J}(X)$ that agree with $J_D$ on $D$. Fix a tamed almost complex structure $J_D$ such that $D$ contains no non-constant $J_D$-holomorphic spheres of any energy and any holomorphic sphere meets $D$ in at least three points, as in \cite{20} Proposition 8.14. By \cite{24} Proposition 8.4, for any energy $E > 0$ there exists a contractible open neighborhood $\mathcal{J}^*(X, J_D, E)$ of $J_D$ agreeing with $J_D$ on $D_{\pm}$ with the property that $D$ still contains no non-constant holomorphic spheres and any holomorphic sphere of energy at most $E$ meets $D$ in at least three points.

For a base almost complex structures $J_D$ a perturbation datum for type $\Gamma$ of relative maps is datum $P_{\Gamma} = (J_{\Gamma}, F_{\Gamma}, G_{\Gamma}, H_{\Gamma})$ as before with
\[
J_{\Gamma} : \overline{\mathcal{S}}_{\Gamma} \rightarrow \mathcal{J}(X, J_D), \quad F_{\Gamma} : \overline{T}_{\Gamma, \phi} \rightarrow C^\infty(L), \quad G_{\Gamma} : \overline{T}_{\Gamma, \phi} \rightarrow \mathcal{G}(L), \quad H_{\Gamma} : \overline{T}_{\Gamma, \phi} \rightarrow C^\infty(Y)
\]
and $J_{\Gamma}$ is equal to the fixed almost complex structures $J_{D, \pm}$ on $D$. With these definitions, transversality and compactness results for relative stable maps are almost
identical to those for broken maps. We denote by $\mathcal{M}_\Gamma(X, L, D)$ the moduli space of relative maps of type $\Gamma$.

Using the moduli spaces of relative maps of dimension zero and one we define relative Fukaya algebras with insertions along the divisor. Let $CM(Y)$ be the Morse complex for $Y$,

$$CM(Y) = \bigoplus_{y \in \mathcal{I}(H)} \Lambda <y>.$$ 

Given $y_0, y_1 \in \mathcal{I}(H)$ let $\mathcal{M}(Y, y_0, y_1)$ denote the moduli space of Morse trajectories of $-\text{grad}(H)$ connecting $y_1$ to $y_0$. Given a choice of orientations on the stable manifolds of $\text{grad}(H)$, the moduli spaces $\mathcal{M}(Y, y_0, y_1)$ naturally become oriented. In particular if $y_0, y_1$ differ by index one so that $\mathcal{M}(Y, y_0, y_1)$ is zero dimensional then we have an orientation function

$$\epsilon : \mathcal{M}(Y, y_0, y_1) \to \{\pm 1\}.$$ 

The Morse differential on $CM(Y)$ is defined on generators by

$$\partial_H : CM(Y) \to CM(Y), \quad <y_1> \mapsto \sum_{[u] \in \mathcal{M}(Y, y_0, y_1)_0} \epsilon(u) <y_0>$$

and extended by linearity. Given a Morse cocycle $c \in \ker(\partial_H)$ on $Y$, define the higher composition maps of the relative Fukaya algebra

$$\mu^{n,c} : \widehat{CF}(X, L, c)^{\otimes n} \to \widehat{CF}(X, L, c)$$

on generators as follows. We write the expansion in terms of generators

$$c = \sum_{y \in \mathcal{I}(H)} c(y)y.$$ 

Define for $y \in \mathcal{I}(H)^e$

$$c(y) = \prod_{i=1}^e c(y_i).$$

Define the composition maps

$$\mu^{n,c}(<l_1>, \ldots, <l_n>) = \sum_{[u] \in \overline{\mathcal{M}}(X, L, D, L^0, y_0)} (-1)^{\bigtriangledown} c(y)(\sigma([u]))^{-1} \text{Hol}_L([\partial u]) q^{E([u])} \epsilon([u]) <l_0>$$

where $\bigtriangledown = \sum_{i=1}^n i|l_i|$.

**Theorem 7.19.** (Relative Fukaya algebra) For any regular coherent stabilizing divisorial perturbation system $\mathcal{P} = (\mathcal{P}_\Gamma) = (J_\Gamma, F_\Gamma, H_\Gamma)$ as above sufficiently $C^2$ close to the base datum $(J, F, H)$, the maps $(\mu^{n,c})_{n \geq 0}$ satisfy the axioms of a convergent $A_{\infty}$ algebra $\widehat{CF}(X, L, c)$ with strict unit and weak divisor axiom. The homotopy type of $\widehat{CF}(X, L, c)$ and non-vanishing of the relative Floer cohomology is independent of all choices up to homotopy equivalence.
Proof. The proof is similar to that for the relative Fukaya algebra, there are additional contributions to the boundary of the one-dimensional moduli spaces of relative maps arising from breaking off Morse trajectories on the divisor. We must check that these additional contributions cancel. The contribution from breaking trajectories on $Y$ to the $A_\infty$ axiom is

$$
\sum_{[u] \in \mathcal{M}_\Gamma(X,L,D,Y)_0} (-1)^\sigma c(y)n(y_j,y_j')|\sigma([u])!|^{-1} \text{Hol}_L(\partial u)q^E([u])\epsilon([u]) <l_0>.
$$

Now using that the coefficient of $y_j$ in $\partial c$ is $\sum y_j n(y_j,y_j')c(y_j) = 0$, the additional contributions cancel. The proof of homotopy invariance of the choice of perturbations is similar to that in the non-relative case.

The broken Fukaya algebra is related to the relative Fukaya algebra in the following way. Define a relative invariant associated to a pair $(X,Y)$:

$$
c(X,Y) = \sum_{[u] \in \mathcal{M}_\Gamma(X,Y,D,Y)_0} (-1)^\sigma c(y)|\sigma([u])!|^{-1} \text{Hol}_L(\partial u)q^E([u])\epsilon([u]) <l_0>.
$$

Proposition 7.20.

(a) $c(X,Y)$ is closed in the Morse complex: $\partial c(X,Y) = 0 \in CM(Y)$.

(b) The cohomology class $[c(X,Y)] \in HM(Y)$ is independent of all choices.

Sketch of proof. The first assertion follows from the study of the ends of the one-dimensional component $\mathcal{M}_\Gamma(X,Y,D,Y)_1$, which consist of pairs of Morse trajectories in $Y$ and treed spheres in $X$. The second assertion follows from a standard study of parametrized moduli spaces associated to variation of almost complex structure and Morse function. \qed

Theorem 7.21. Let $(X_\subset,X_\supset) = X$ be a broken symplectic manifold. The broken Fukaya algebra $CF(X,L)$ for $X = (X_\subset,X_\supset)$ with $L \subset X_\supset$ is the relative Fukaya algebra $CF(X_\subset,L,c(X_\supset,Y))$ with insertion $c(X_\supset,Y) \in CM(Y)$.

Proof. The broken moduli space of dimension zero may be written as a fiber product of the pieces.

$$
\mathcal{M}(X,L,Y,D,Y)_0 \cong \mathcal{M}(X_\subset,L,Y,D,Y)_0 \times_{\mathcal{I}(H)_0} \mathcal{M}(X_\supset,Y,D,Y)_0
$$

so that each element may be written as a pair $u = (u_\supset,u_\subset)$. The construction of orientations arises from the construction of orientations on each piece, while the areas and number of interior markings are the sums of those on the pieces:

$$
\epsilon([u]) = \epsilon([u_\supset])\epsilon([u_\subset]), \quad \sigma([u]) = \sigma([u_\supset]) + \sigma([u_\subset]), \quad A([u]) = A([u_\supset]) + A([u_\subset])
$$

from which it follows

$$
\sum_{[u] \in \mathcal{M}_\Gamma(X,L,D,Y)_0} (-1)^\sigma |\sigma([u])!|^{-1} \text{Hol}_L(\partial u)q^E([u])\epsilon([u]) <l_0>
$$

$$
= \sum_{[u] \in \mathcal{M}_\Gamma(X_\subset,L,D,Y)_0} (-1)^\sigma c(y)|\sigma([u])!|^{-1} \text{Hol}_L(\partial u)q^E([u])\epsilon([u]) <l_0> c(X_\supset,Y,y)
$$

as claimed. \qed
7.5. Combining the relative and broken theories. The relative and broken theories for Fukaya algebras may be combined as follows. Suppose that a symplectic manifold $X$ is equipped with a fibered coisotropic $Z \subset X$ as well as a codimension two symplectic submanifold $Y$ disjoint from $Z$. Collapsing $Z$ results in pieces $X_{<}, X_{>}$ with $L$ contained in $X_{<}$ and $Y$ contained in $X_{>}$. A relative broken map is a broken curve $(C_{0}, \ldots, C_{k})$, possibly with outgoing edges on the last level, together with a map to

$$X_{<}, \mathbb{P}(N_{\pm} \oplus \mathbb{C}), \ldots, \mathbb{P}(N_{\pm} \oplus \mathbb{C}), X_{>}, P(N \oplus \mathbb{C}), \ldots, P(N \oplus \mathbb{C})$$

where $N_{\pm}$ is the normal bundle to $Z/S^{1}$ in $X_{<}, X_{>}$ and $N$ is the normal bundle to $Y$ in $X_{>}$, and satisfying matching conditions at the nodes as before. As before, there is a moduli space of stable relative broken maps $\mathcal{M}(X, Y)$. The transversality and compactness properties of the moduli space of stable relative broken maps are obvious generalizations of the broken case, and omitted. if $X$ is the broken manifold obtained by collapsing $Z$ then one obtains a Fukaya algebra $\widehat{CF}(X, Y, L, c)$ dependent on the choice of Morse cocycle $c \in CM(Y)$, independent of all choices up to $A_{\infty}$ homotopy. Theorem 7.14 generalizes to the relative case by the same proof:

**Theorem 7.22.** The relative Fukaya algebra $\widehat{CF}(X, Y, L, c)$ is homotopy equivalent to the relative broken Fukaya algebra $CF(X, Y, c)$.

**Corollary 7.23.** Suppose that $L$ arises from a blow-up or reverse flip with exceptional locus $E_{i}$ in $X_{i+1}$, and $Y \subset X_{i}$ is an almost complex submanifold disjoint from $E$. Then for any insertion $c \in CM(Y)$, $CF(X_{i}, L, Y, c)$ is projectively flat and Floer non-trivial.

**Proof.** The proof is essentially the same as that of Proposition 7.15. The Fukaya algebra $CF(X_{i}, L, c)$ is equivalent to that in the broken Fukaya algebra $CF(X, L, Y, c)$ where $L \subset X_{i-}$ is the toric piece. The algebra $CF(X, L, Y, c)$ is projectively flat by the same arguments in 7.15. \qed

We now prove Theorem 1.2 from the introduction.

**Proof.** We write $X_{i+1}$ as a symplectic sum of pieces

$$X_{i+1} = X_{i+1,<} \cup_{Z/S^{1}} X_{i+1,>}$$

where $X_{i+1,<}$ is a toric piece near $E_{i}$ and $X_{i+1,\subset}$ is a non-toric piece containing the Lagrangian $L$. Then $X_{i,\subset} \cong X_{i+1,\subset}$ are diffeomorphic and $X_{i,\supset}, X_{i+1,\supset}$ are related by a flip. By assumption $\omega_{i,\subset}, \omega_{i+1,\subset}$ are related by a family of symplectic forms with anticanonical variation. On the other hand, the toric pieces $X_{i,\supset}, X_{i+1,\supset}$ are related by a blow-up or flip. Now $CF(X_{i+1}, L)$ is homotopy equivalent to a broken Fukaya algebra $CF(X_{i+1}, L)$ which in turn is equal to the Fukaya algebra $CF(X_{i,\subset}, L, Y_{i+1}, c(X_{i+1,\supset}, Y_{i+1}))$ with insertion $C(X_{i+1,\supset}, Y_{i+1})$:

$$CF(X_{i+1}, L) \cong CF(X_{i+1}, L) \cong CF(X_{i,\subset}, L, Y_{i+1}, c(X_{i+1,\supset}, Y_{i+1})).$$

The latter is weakly unobstructed, and the potential in $CF(X_{i,\subset}, L, Y_{i+1}, c(X_{i+1,\supset}, Y_{i+1}))$ is equal to the potential for $CF(X_{i}, L)$ to leading order. It follows that $CF(X_{i}, L)$ is Floer-non-trivial. \qed
References


K. Fukaya. Floer homology for 3-manifolds with boundary I, 1999. unpublished manuscript.


Department of Mathematics, Barnard College - Columbia University, MC 4433, 2990 Broadway, New York, NY 10027

E-mail address: charest@math.columbia.edu

Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A.

E-mail address: ctw@math.rutgers.edu