Abstract. We prove that small blow-ups or reverse flips (in the sense of the minimal model program) of rational symplectic manifolds with point centers create Floer-non-trivial Lagrangian tori. We give examples of explicit mmp runnings and descriptions of Floer non-trivial tori in the case of toric manifolds, polygon spaces, and moduli spaces of flat bundles on punctured two-spheres (moduli of parabolic bundles). These results are part of a conjectural description of generators for the Fukaya category of a compact symplectic manifold with an orbifold running of the minimal model program.

Contents

1. Introduction 2
2. Minimal model transitions 9
  2.1. Mmp runnings 10
  2.2. Mmp for toric varieties 18
  2.3. Mmp for moduli spaces of polygons 19
  2.4. Mmp for moduli spaces of flat bundles 24
  2.5. Mmp for flag varieties 29
3. Fukaya algebras 30
  3.1. A-infinity algebras 30
  3.2. Treed holomorphic disks 33
  3.3. Spectral sequence and divisor equation 40
4. Broken Fukaya algebras 44
  4.1. Broken curves 45
  4.2. Broken maps 46
  4.3. Broken perturbations 51
  4.4. Broken divisors 59
  4.5. The case of a blow-up or reverse flip 62
  4.6. Getting back together 67
5. The break-up process 72
  5.1. Breaking a symplectic manifold 72
  5.2. Breaking perturbation data 73
  5.3. Getting back together, redux 79
References 84

This work was partially supported by NSF grant DMS 1207194 and a Simons Fellowship.
1. Introduction

Lagrangian Floer cohomology was introduced in [36] as a version of Morse theory for the space of paths from a Lagrangian submanifold to itself. Despite the fact that the theory was introduced almost thirty years ago, it has been far from clear which Lagrangian submanifolds have well-defined or non-trivial Floer cohomology, or whether a symplectic manifold contains any Lagrangians with non-trivial Floer cohomology at all. The purpose of this paper is to describe a method for producing Floer-non-trivial Lagrangians via an analog of the minimal model program in algebraic geometry.

A reason for expecting Floer-non-trivial Lagrangians can be found in the homological mirror symmetry conjecture of Kontsevich. Recall that the Fukaya category of a symplectic manifold is a homotopy-associative category whose objects are Lagrangian submanifolds and morphisms spaces are Floer cochain groups. The homological mirror symmetry conjecture [54] relates the derived Fukaya category of a compact symplectic manifold with a derived category of matrix factorizations of a potential on a mirror complex manifold. If the derived category of matrix factorizations has many non-trivial objects, one expects the Fukaya category also to be non-empty, that is, contain Lagrangians with well-defined and non-trivial Floer cohomology. This is often the case in the birationally-Fano examples considered here.

Existence of Floer-non-trivial Lagrangians would have the following concrete consequence in symplectic topology. The Floer cohomology is invariant under diffeomorphisms \( \phi \in \text{Ham}(X) \) generated by time-dependent Hamiltonians; thus the non-triviality of the Fukaya category also implies that there should exist Lagrangian submanifolds \( L \) with non-trivial Floer cohomology \( HF(L) \) and hence non-displaceable by Hamiltonian diffeomorphisms \( \phi \in \text{Ham}(X) \):

\[
\exists L \subset X, \forall \phi \in \text{Ham}(X), \quad \phi(L) \cap L \neq \emptyset.
\]

For example, if \( X = Y^- \times Y \) is the product of a compact symplectic manifold \( Y \) with its dual \( Y^- \) then the diagonal in \( X \) is Hamiltonian non-displaceable by the Arnold conjecture. More generally if \( X \) admits an anti-symplectic involution then in many cases the fixed point set of the involution has non-trivial Floer cohomology [39].

Despite this, it has been far from clear how to actually construct Floer-non-trivial (and so Hamiltonian non-displaceable) Lagrangians in symplectic manifolds that are not equipped with anti-symplectic involutions or how to produce generators of the Fukaya category. In many situations, an argument of Moser implies that “almost all” Lagrangians are displaceable, see McDuff [4] and Abreu-Borman-McDuff [77]. Existence of Floer non-trivial Lagrangians in compact symplectic manifolds is known only in isolated cases, such as toric varieties by work of Cho-Oh [26] and Fukaya-Oh-Ohta-Ono [40] and in semi-Fano hypersurfaces by Seidel [93] and Sheridan [95]. The examples given in [26] are Lagrangian tori, which generalize the Clifford torus in complex projective space, while the examples in [93], [95] are Lagrangian spheres which can be seen as singular fibers in a Lagrangian fibration.
A natural testing ground for Floer cohomology is that of Lagrangian torus orbits in toric manifolds. In this setting Kontsevich’s conjecture is proved in forthcoming work of Abouzaid-Fukaya-Oh-Ohta-Ono, using previous separate work of the authors [2], [40]. In particular an explicit collection of Lagrangians is identified which, when equipped with suitable bounding cochains, generates the derived Fukaya category.

A connection between the generating Lagrangians and the minimal model program was noted in work with Gonzalez [44] on quantum cohomology of toric varieties; this gives a more conceptual picture for the generators. Recall that a running of the mmp for a smooth projective variety $X$ is a sequence of varieties $X = X_0, X_1, \ldots, X_k$ such that each $X_{i+1}$ is obtained from $X_i$ by an mmp transition. The simplest mmp transition is a blow-down of a divisor, or more generally a divisorial contraction the other operations are flips and the minimal model program ends (in the birationally-Fano case) with a Mori fiber space. Each blow-up or flip has a center $Z_i$ which is a subquotient of both $X_i$ and $X_{i+1}$ and, in the cases considered here, involves replacing a weighted-projective bundle $\mathbb{P}(N^+_i) \to Z_i$ over the center with another weighted-projective bundle $\mathbb{P}(N^-_{i+1}) \to Z_i$:

\[
X_i \xleftarrow{\mathbb{P}(N^+_i)} Z_i \xrightarrow{\mathbb{P}(N^-_{i+1})} X_{i+1}.
\]

Here a weighted-projective bundle is obtained by removing the zero section from a vector bundle and taking the quotient by a linear $\mathbb{C}^\times$-action with only the zero section as fixed point set. In the case of a divisorial contraction, the map $\mathbb{P}(N^+_i) \to Z_i$ would be an isomorphism, while in the case of a Mori fibration, one may think of the center as the result of the transition: $\mathbb{P}(N^+_i) \simeq X_i, Z_i \simeq X_{i+1}$. Sometimes singularities in the spaces $X_i$ are unavoidable; however, in good cases (such as toric varieties) one may assume that the $X_i$’s are smooth orbifolds. In this case we say that $X$ has a smooth running of the mmp. Existence of mmp runnings is known for varieties of low dimension and in many explicit examples. For toric varieties, mmp runnings exist by work of Reid [89]. We discuss several other examples such as moduli spaces of points on the projective line and moduli spaces of parabolic bundles.

In this paper we initiate a strategy for producing Floer-non-trivial Lagrangians (and hopefully generators for the Fukaya category) by running the mmp backwards. We prove some results in the direction of the (somewhat vague) conjecture:

**Conjecture 1.1.** Suppose $X = X_0, X_1, \ldots, X_k = X'$ is a sequence of compact symplectic manifolds (or orbifolds) such that each $X_{i+1}$ is obtained from $X_i$ by an mmp transition. Then the idempotent-closure of the derived Fukaya category $D^\pi \text{Fuk}(X_0)$ is isomorphic to the disjoint union of categories of centers of the mmp transitions:

\[
D^\pi \text{Fuk}(X) \cong D^\pi \text{Fuk}(X') \sqcup \bigcup_{i=1}^k D^\pi \text{Fuk}(Z_i)^{m_i}
\]
where

\[ m_i = \dim(QH(X_i)) - \dim(QH(X_{i+1})) \]

is the multiplicity of the \( i \)-th mmp transition given as the difference of the quantum cohomology rings \( QH(X_{i+1}), QH(X_i) \).

By the disjoint union of categories we mean the category whose objects are the disjoint union, and whose morphism groups between elements of different sets in the disjoint union are trivial. The conjecture 1.1 is somewhat vague because the meaning of the minimal model program for symplectic manifolds is somewhat unclear; we have in mind various examples where the symplectic manifold is a smooth birationally-Fano projective variety. Since our evidence is mostly in the birationally-Fano case, it is possible that the conjecture 1.1 needs some similar restriction. The decomposition above is compatible with the decomposition by quantum multiplication of the first Chern class in the following sense: Recall that quantum multiplication by the first Chern class \( c_1(X) \in H^2(X) \) induces an endomorphism

\[ c_1(X) \star : QH(X) \to QH(X), \quad \alpha \mapsto c_1(X) \star \alpha. \]

Here we work with the version of quantum cohomology that is a module over the universal Novikov field \( \Lambda \) of formal series in a single formal variable \( q \), so that the structure coefficients are weighted with \( q \)-exponent given by the symplectic areas of pseudoholomorphic spheres. The eigenvalues \( \lambda_1, \ldots, \lambda_k \in \Lambda \) which are non-zero have a well-defined \( q \)-valuation \( \text{val}_q(\lambda_i) \in \mathbb{R} \), given by the exponent of the leading order term. We expect that the \( q \)-valuations of the eigenvalues of quantum multiplication by the first Chern class are the transition times in the minimal model program, in the sense that the decomposition (1) is preserved by \( c_1(X) \star \) and the non-zero eigenvalues \( \lambda \) of \( c_1(X) \star \) on \( \text{Fuk}(Z_i)^{m_i} \) have the property that \( \text{val}_q(\lambda) \) is the transition time. The decomposition (1) is intended to be a geometric version of this eigenvalue decomposition, and the mmp is conjectured to (in good cases) produce generators for the summands.

The conjecture 1.1 above provides a method of attack on another conjecture of Kontsevich, that the Fukaya category \( \text{Fuk}(X) \) of a symplectic manifold \( X \) is expected to be a categorification of the quantum cohomology \( QH(X) \), at least in many cases, in the following sense [59]: there should be an isomorphism

\[ H(\text{Fuk}(X)) \cong QH(X) \]

from the Hochschild cohomology \( H(\text{Fuk}(X)) \) of the Fukaya category to the quantum cohomology \( QH(X) \). For any pair of A-infinity categories, the space of A-infinity functors between them is itself an A-infinity category; in particular, the space of endomorphisms of the identity functor is itself an A-infinity algebra and the Hochschild cohomology is the cohomology of the algebra of endomorphisms. We expect that if the centers of the mmp transitions have the property (2) then the manifold has the same property and hence also the quantum cohomology decomposes into summands corresponding to transitions.

In this paper we prove one piece of the conjecture 1.1: that mmp transitions produce Floer non-trivial Lagrangians, and so non-trivial objects in the Fukaya category. The version of Floer theory that we use requires the notion of bounding
cochain of Fukaya-Oh-Ohta-Ono [39], and we begin with a brief discussion of what we mean by Fukaya algebras. There are several foundational systems which at the moment which are not known to be equivalent (or in some cases, completely written down.) The foundational system we will use is joint with F. Charest [23], [24], and uses stabilizing divisors as in Cieliebak-Mohnke [27]. While this foundational scheme is somewhat less general than the other approaches, it requires no discussion of virtual fundamental classes and so makes the necessary foundational arguments substantially shorter. In particular for any compact oriented spin Lagrangian submanifold equipped with a rank one local system there is a strictly unital Fukaya algebra, defined by counting weighted treed holomorphic disks, independent of all choices up to homotopy equivalence. The resulting moduli space of solutions to the weak Maurer-Cartan equation is independent of all choices and for each solution there exists a well-defined Floer cohomology group. We say that the Lagrangian is weakly unobstructed if the space of solutions is non-empty and Floer non-trivial if the Floer cohomology group is non-zero for some solution to the weak Maurer-Cartan equation.

The summands in the decomposition (1) are expected to be the images of certain quilt functors developed in joint work with Ma'u and Wehrheim [70]. For any small blow-up or flip with center $Z_i \subset X$ we have a regular coisotropic $\hat{Z}_i \subset X$ with the property that $\hat{Z}_i$ is a torus fibration over $Z_i$, and the area of the Maslov index two disks in the fibers are all equal (hence the terminology regular); this means essentially that the fibers correspond to critical points of the disk potential. Via the theory of pseudoholomorphic quilts the coisotropic gives rise to a functor

$$\Phi(\hat{Z}_i) : \text{Fuk}(Z_i) \to \text{Fuk}(X)$$

(at least under suitable monotonicity assumption [70]; note that the extended Fukaya category of [70] is not needed in this case.) In this paper we discuss only the case that the centers $Z_i$ are points, in which case the quilt functor assigns to the point the Lagrangian submanifold $\hat{Z}_i$ itself. The main result of this paper is:

**Theorem 1.2.** Suppose that $X$ is a compact symplectic manifold obtained from a symplectic orbifold $X'$ by a small reverse simple flip or blow-up with trivial center. Then the Lagrangian torus $\hat{Z} \subset X$ has weakly unobstructed and non-trivial Floer cohomology, $HF(\hat{Z}) \cong H(\hat{Z}) \neq 0$.

In particular, the small blowup or flip of any such symplectic manifold with trivial center contains a Floer-non-trivial (hence Hamiltonian non-displaceable) Lagrangian torus; related discussions of blow-ups can be found in Smith [97]. In the case of toric varieties, the existence of Floer-non-trivial Lagrangians is a result of Fukaya-Oh-Ohta-Ono [39]. Overlapping results, using integrable systems, were constructed by Entov-Polterovich [35]. Theorem 1.2 implies the existence of Hamiltonian non-displaceable Lagrangians in a large class of Kähler manifolds. A more precise description of how “small” the blow-up has to be is given in Corollary 5.16 below.

We emphasize that in a number of examples, the Lagrangians appearing in the above theorem may be described explicitly. We discuss several examples in detail just to give an idea of how the scheme works (or might work) in practice:
Example 1.3. (Toric manifolds) Recall that a toric variety is a normal variety with an action of a complex torus with an open orbit, see for example Cox-Little-Schenck [29]. An equivariant projective embedding of a smooth toric variety induces a symplectic structure and a Hamiltonian action of the unitary part of the complex torus. From the symplectic point of view, Hamiltonian torus actions with Lagrangian orbits are classified by a theorem of Delzant [30]. Let $X$ be a smooth projective toric variety, $T$ the unitary torus acting on $X$, and $\Psi : X \to t^\vee$ a moment map induced by a compatible symplectic structure. The moment polytope of $X$ is the image

$$P = \Psi(X) \subset t^\vee.$$  

We write

$$P = \{ \lambda \in t^\vee \mid \langle \lambda, \nu_j \rangle \geq c_j, \quad j = 1, \ldots, k \}$$

where $\nu_j \in t_\mathbb{Z}$ are the minimal lattice vectors that are inward pointing normal vectors to the facets of $P$, and $c_j$ are constants determining their position. The condition that $X$ is smooth implies that the normal vectors $\nu_j$ meeting each vertex form a lattice basis. Delzant’s classification theorem states that [30] the moment polytope and generic stabilizer classify compact connected Hamiltonian torus actions with Lagrangian torus orbits completely. Floer non-triviality of toric moment fibers was studied in Fukaya et al. [40], [41]. As a corollary of the main result Theorem 1.2 we have the following: Suppose that a compact toric manifold $X$ is obtained by a small reverse flip or blow-up and $t$ is the parameter representing the size of the exceptional locus $E \subset X$. Let $\lambda \in P$ denote the unique point satisfying

$$\langle \lambda, \nu_j \rangle - c_j = t$$

for every normal vector $\nu_j$ of a facet whose inverse image intersects the exceptional locus $E$:

$$\Psi^{-1}(\{ \langle \lambda, \nu_j \rangle = c_j \}) \supset E.$$  

Then the fiber $L = \Psi^{-1}(\lambda)$ is a Lagrangian torus with weakly unobstructed and non-trivial Floer cohomology.

In Figure 1 we show the case of a twice blow-up of the product of projective lines. Fibers corresponding to the blowups are those over the darkly shaded points shown in the Figure. These results are substantially weaker than the results in [40], [41], which give a whole collection of Floer non-trivial tori. For example, in the example in Figure 1, the results of [26], [40] show that the lightly shaded point in the middle also gives a Floer non-trivial torus. However, once the results in the sequels to this paper are taken into account we will recover the entirety of the results in [40], [41].

Similar results for non-toric blow-ups were announced by those authors at about the same time as our results were announced. This ends the example.

Example 1.4. (Moduli of flat bundles on a compact oriented surface) The moduli space of flat bundles appears in several places in mathematical physics, for example, in Witten’s interpretation of the Jones polynomial [105]; since Fukaya categories are one method of “categorification” of quantum invariants, the Fukaya category of this moduli space is one that several mathematicians have been interested in. It
Figure 1. Floer non-trivial tori for the twice blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$

has a natural family of Lagrangian tori described by result of Goldman [42] and the Lagrangians in the theorem may be taken to be part of Goldman’s family.

Let $\Sigma$ be a compact, oriented genus zero Riemann surface with $n$ boundary components. For any collection $\mu_1, \ldots, \mu_n \in [0, 1/2]$ of holonomy parameters let

$$ R(\mu_1, \ldots, \mu_n) \subset \text{Hom}(\pi_1(\Sigma), SU(2))/SU(2) $$

denote the moduli space of $SU(2)$-representations (isomorphism classes of flat $SU(2)$-bundles) on $\Sigma$ with holonomy around the $j$-th boundary component conjugate to $\text{diag}(\exp(\pm 2\pi i \mu_j))$ for $j = 1, \ldots, n$. For generic parameters $\mu_1, \ldots, \mu_n$, the moduli space $R(\mu_1, \ldots, \mu_n)$ is a smooth symplectic manifold. Given a pants decomposition $\Sigma = \bigcup_{P \in \mathcal{P}} P$ define a label map

$$ \Psi_P : R(\mu_1, \ldots, \mu_n) \rightarrow [0, 1/2]^{n-3} $$

by taking the logarithms of eigenvalues of the holonomies around the interior circles. If $Z_j$ is the $j$-th circle in the pants decomposition then (similar to the boundary condition in (8))

$$ (\Psi_P)_j([\varphi]) = \mu_j, \quad [\text{diag}(\exp(\pm 2\pi i \mu_j))] = [\varphi([Z_j])]. $$

Goldman [42] shows that the generic fibers of $\Psi$ are Lagrangian tori. We say that a labelling $\lambda \in \mathbb{R}^{n-3}$ is uniformly loose if for pair of pants $P \in \mathcal{P}$ with boundary circles $C_i, C_j, C_k$ with corresponding components $\lambda_i, \lambda_j, \lambda_k$ of $\lambda$, the quantity (which we call the looseness of the pair of pants)

$$ l(P) := \min(\lambda_i + \lambda_j - \lambda_k, \lambda_i + \lambda_k - \lambda_j, \lambda_j + \lambda_k - \lambda_i, 1 - \lambda_i - \lambda_j - \lambda_k) $$

is independent of the choice of $P$.

Theorem 1.2 implies the existence of a Floer non-trivial torus described as follows. Suppose that $\mu_1, \ldots, \mu_n$ are generic labels and $R(\mu_1, \ldots, \mu_n)$ is the corresponding moduli space of flat $SU(2)$-bundles on the $n$-holed two-sphere. Let $\mathcal{P} = \{P\}$ be a pants decomposition with ordered boundary circles $C_1, \ldots, C_{n-3}$ and suppose that $\lambda = (\lambda_1, \ldots, \lambda_{n-3})$ is a uniformly loose labelling with looseness given by the first transition time in the mmp. Then the Goldman Lagrangian $\Psi^{-1}(\lambda)$ has non-trivial Floer cohomology:

$$ HF(\Psi^{-1}(\lambda)) \cong H(\Psi^{-1}(\lambda)) \neq 0. $$

Figure 2 gives an example of a labelled pants decomposition giving rise to a Floer non-trivial Lagrangian, corresponding to an mmp transition at time $t = .06$. The
Figure 2. A labelling giving a Floer-non-trivial Goldman torus

results are similar to those of Ueda and collaborators [84], [85] via toric degenerations. The viewpoint in those papers is that one should look for generators in the Fukaya category among fibers in the resulting Lagrangian fibrations. The viewpoint here is slightly different: it suggests that in order to obtain generators for the Fukaya category, one should look for a Lagrangian torus for each mmp transition. However, the results in this paper only go a small part of the way towards establishing this program, and only establish Floer non-triviality of one of these tori. We remark that from the point of view of low-dimensional topology, the case that all boundary markings are equal is especially interesting because of the existence of braid group actions; essentially in this case one expects a connection between Lagrangian Floer homology and the instanton knot homology of Kronheimer-Mrowka [62]. We hope to explain elsewhere how the Lagrangians resulting from mmp’s for generic markings give rise to Lagrangians in the monotone case of equal markings, by taking a limit. One then obtains a collection of Lagrangians parametrized by triangulations, remarkably close to the “crossingless matchings” considered in Khovanov-Rozansky [61]. This ends the example.

We remark that the idea that the quantum cohomology of a symplectic manifold should behave well under minimal model transitions is not new, in for example Ruan [91], Lee-Lin-Wang [64], Bayer [12], Acosta-Shoemaker [5]; we also heard related results in talks of H. Iritani. The present paper is the natural generalization to Fukaya categories, which in many respects is easier than quantum cohomology since one can often explicitly identify the objects created by the transitions.

The minimal model program has also proved useful for understanding the derived category of matrix factorizations: A method for producing generators for the derived category of matrix factorizations was studied by Bondal-Orlov [16], Kawamata [53] and others. The minimal model program on the symplectic side corresponds under mirror symmetry to a deformation of the mirror potential by a change of variables in the potential

$$W \mapsto \phi_t^* W, \quad \phi_t(y) := yq^{t c_1(X)}$$
(the discussion here is on a heuristic level) because the areas of the holomorphic disks of index two change at the same rate. Because the mirror should be understood as a formal completion at \( q = 0 \), such a deformation changes the mirror by eliminating some critical loci which are killed by the formal completion.

We comment briefly on the proof of Theorem 1.2. Locally near the exceptional locus the reverse flip is toric, and the computations in the toric case imply the existence of a Floer-non-trivial torus in the corresponding toric variety, which is the unique Lagrangian torus which collapses at the singularity of a running of the minimal model program; in this sense the Lagrangian is a “vanishing cycle”. The exceptional locus of the flip is separated from the rest of the symplectic manifold by a coisotropic submanifold fibered over a toric variety. Stretching the neck, as in symplectic field theory, produces a homotopy-equivalent broken Fukaya algebra associated to the Lagrangian, which counts maps to the pieces combined with Morse trajectories on the toric variety. (Actually, the homotopy equivalence is much stronger than the result that we need.) Similar arguments are common in the literature, for example, in the work of Iwao-Lee-Lin-Wang [64], [63]. One computes explicitly, using a Morse function arising as component of a moment map, that the resulting broken Fukaya algebra is weakly unobstructed and the broken Floer cohomology of the Lagrangian is non-vanishing. Moduli spaces of disks in toric varieties are somewhat larger in dimension than they have any right to be; for example, in order to hit one of the boundary divisors the dimension of the moduli space must be at least that of the ambient manifold, whereas ordinarily the condition the moduli space should only be at least two. This “excess dimension” is the reason for the unobstructedness, and then the classification of disks of small area implies the existence of a critical point of the potential.

2. Minimal model transitions

In this section we review some aspects of the minimal model program for birationally-Fano varieties introduced by Mori and others, see [58]. The goal of the minimal model program is to classify algebraic varieties by finding a minimal model in each birational equivalence class. Ideally, the minimal model should have canonical class with a definite sign, depending on the Kodaira dimension of the original variety. We are mainly thinking of the case of Kodaira dimension \(-\infty\) in which case the hoped-for minimal model is Mori fibration; a fibration with Fano fiber. In general, singularities play an important role in the minimal model program but here we are interested in the case that the variety admits a smooth (or at least orbifold) running of the minimal model program. While this case is considered somewhat trivial by algebraic geometers, it includes a number of smooth projective varieties whose symplectic geometry is poorly understood. We introduce a symplectic version of the minimal model program which is a path (with singularities corresponding to surgeries) of symplectic manifolds. We recall that by a suggestion of Song-Tian and others [96], an mmp running is conjecturally equivalent to running Kähler-Ricci flow with surgery; the paths in our symplectic mmp are essentially a weak version of the Kähler-Ricci flow.
2.1. Mmp runnings. We begin with a description of the transitions in the minimal model program. Each transition is a special kind of birational transformation, for which we review the definition: Let $X_±$ be normal projective varieties. A rational map from $X_+$ to $X_-$ is a Zariski open subset $U \subset X_+$ and a morphism $U \rightarrow X_-$. A rational map is a birational equivalence if it has a rational inverse, that is, a rational map from $X_-$ to $X_+$ such that the composition is the identity on the domain of definition.

Definition 2.1. A minimal model transition of $X_+$ is a birational transformation of one of the following types, see for example Hacon-McKernan [47]:

(a) (Divisorial contractions) A morphism $\tau : X_+ \rightarrow X_-$ that is the contraction of a Cartier divisor (codimension one subvariety); a typical example is a blow-down.

(b) (Flips) Let $\tau_+ : X_+ \rightarrow X_0$ be a birational morphism. The morphism $\tau_+$ is small iff $\tau_+$ does not contract a divisor. The morphism $\tau_+$ is a flip contraction iff the relative Picard number of $\tau_+$ is one. The flip of $X_+$ is another small birational morphism $\tau_- : X_- \rightarrow X_0$ of relative Picard number one.

(c) (Mori fibrations) A fibration $\tau : X_+ \rightarrow X_-$ with Fano fiber, that is, a fiber whose anticanonical bundle is well-defined and very ample.

Here the relative Picard number is the difference in Picard numbers, that is, the difference in dimensions in the moduli spaces of line bundles. The relative Picard number is one iff every two curves contracted by $\tau_-$ are numerical multiples of each other, that is, define proportional linear functions on the space of degree two cohomology classes.

Definition 2.2. (a) A running of the minimal model program (mmp) is a sequence of smooth projective varieties $X = X_0, \ldots, X_{k+1}$ such that for $i = 0, \ldots, k$, $X_{i+1}$ is obtained from $X_i$ by a divisorial contraction or a flip, and $X_{k+1}$ is obtained from $X_k$ by a Fano fibration. The variety $X_k$ may be called the minimal model of $X_0$. The different minimal models which occur are related by the Sarkisov program [48].

(b) An extended running of the mmp is a sequence of smooth projective varieties $X_0, \ldots, X_{k_1}, X_{k_1+1}, \ldots, X_{k_l+1}$, where $k_1, \ldots, k_l$ is some increasing sequence of integers, such that $X_{i+1}$ is obtained from $X_i$ by a divisorial contraction or flip for $k \neq k_i$, while $X_{k_i+1}$ is obtained from $X_{k_i}$ by a Mori fibration.

It is expected that runnings of the mmp exist for all smooth projective varieties. This is proved up to dimension three [58].

Example 2.3. (Toric surfaces) The simplest example of the minimal model program occurs for compact toric surfaces. Recall that any compact toric surface $X$ is projective and corresponds to a convex polygon $P \subset \mathbb{R}^2$ whose edge vectors at any vertex give a lattice basis. Each edge vector corresponds to an invariant prime divisor $D_i$ of $P$. Elementary combinatorics shows that if $P$ has more than four edges then there always exists some invariant prime divisor $D_i$ with $D_i^2 = -1$; blowing down $D_i$ one eventually reaches a Hirzebruch surface or a projective plane, which is the minimal
model, see Audin [10, Theorem VIII.2.9]. A different way of obtaining a running, which allows orbifold singularities, is discussed below in Section 2.2. Note that toric surfaces often admit several runnings of the toric mmp, depending on the order in which the $-1$-curves are blown down.

Example 2.4. (del Pezzo surface) Let $dP_n$ denote the blow up of $\mathbb{P}^2$ at $n-1$ generic points. For $1 \leq n \leq 9$, $dP_n$ is Fano and admits a running of the mmp with (in our notation) one transition, since $dP_n$ is itself a Mori fibration over a point. However, clearly $dP_n$ for $n < 8$ admits multiple mmp runnings. For example, for $dP_4$ one can blow down the exceptional curves in any order; on the other hand, one may view $dP_4$ as the thrice-blow-up of the dual projective plane via the Cremona transformation, and blowing down these curves in any order gives another six runnings of the mmp. Of course, $dP_4$ is toric with moment polytope a hexagon, and the 12 runnings above correspond to the ways of “restoring the corners” to make a triangle. See Figure 3; the points giving Floer non-trivial tori in Theorem 1.2 are darkly shaded in the figure. The fiber over the medium-shaded point is also Floer-non-trivial, but not covered by the results of this paper.

\begin{center}
Figure 3. Blowing down $dP_4$ to $\mathbb{P}^2$
\end{center}

Runnings of the minimal model program can often be obtained from geometric invariant theory as follows, see Reid [90] and Thaddeus [102].

Remark 2.5. (Mmp runnings via git) Let $G$ be a complex reductive group and $X$ a smooth projective $G$-variety. Recall that a linearization of $X$ is an ample $G$-line bundle $\hat{X}$. Given such a linearization the geometric invariant theory quotient $X//G$ is the quotient of the semistable locus, defined as the set of points where an invariant section is non-vanishing,

$$X^{ss} := \{ x \in X \mid \exists k > 0, \ s \in H^0(\hat{X}^k)^G, s(x) \neq 0 \},$$

by the orbit equivalence relation

$$x_1 \sim x_2 \iff \overline{Gx_1} \cap \overline{Gx_2} \neq 0.$$

The git quotient depends only on the ray generated by $\hat{X}$ in the equivariant Picard group, that is, tensor powers of $\hat{X}$ define the same git quotient. In particular, git quotients are defined for ample elements of the rational Picard group $\text{Pic}_Q(X) = $
Pic(X) ⊗_{\mathbb{Z}} \mathbb{Q}. Varying the linearization \hat{X} by rational powers of the anti-canonical bundle \( K^{-t} \) produces a family of rational linearizations
\[
\hat{X}_t = \hat{X} \otimes K^{-t} \in \text{Pic}_Q(X), \ t \in \mathbb{Q}.
\]
Let
\[
X//_t G := X^s_t / \sim
\]
be the corresponding family of geometric invariant theory quotients. By results of Thaddeus [102] and others, \( X//_t G \) undergo a sequence of divisorial contractions, flips, and fibrations each obtained as follows: The transition times are the set of values of \( t \) where the quotient \( X//_t G \) has stabilizer groups of positive dimension:
\[
T := \{ t \in \mathbb{R}, \ \exists x \in X^s_t, \ #G_x = \infty \}.
\]
Thaddeus [102] reduces the study of the wall-crossings to the case that \( G \sim C \times \) is a one-parameter subgroup by replacing \( X \) with the master space
\[
X_{t_1, t_2} := \mathbb{P}(K^{-t_1} \oplus K^{-t_2})//G
\]
for some \( t_1, t_2 \): the residual \( \mathbb{C}^\times \)-action has git quotients
\[
X_{t_1, t_2}/\mathbb{C}^\times \cong X//_t G
\]
for \( t \in (t_1, t_2). \)

Having reduced to the case of a circle action, one can now check that variation of quotient produces a flip. In the circle group case \( G = \mathbb{C}^\times \) let \( F \subset X^s_t \) be a component of the fixed point set which is stable at time \( t \). Let \( \mu_i, i = 1, \ldots, k \in \mathbb{Z} \) denote the weights of \( \mathbb{C}^\times \) on the normal bundle \( N \) to \( F \), and \( N_i \) the weight space for weight \( \mu_i, i = 1, \ldots, k \). Let
\[
N_\pm := \bigoplus_{\pm \mu_i > 0} N_i
\]
denote the positive resp. negative weight subbundle. For \( t_- < t < t_+ \) with \( t_\pm \) close to \( t \) the Hilbert-Mumford criterion and Luna slice theorem imply that the semistable locus changes by replacing a variety isomorphic to \( N_- \) with one isomorphic to \( N_+ \), see [102]. Hence \( X//_{t_+} G \) is obtained from \( X//_{t_-} G \) by replacing the (weighted)-projectivized bundle \( N_-^\times /G \) of with \( N_-^\times /G \):
\[
(X//_{t_-} G)\backslash (N_-^\times /G) \cong (X//_{t_+} G)\backslash (N_+^\times /G).
\]
The definition of the anticanonical bundle implies that the sum of the weights on the anticanonical bundle at any fixed point are positive, and the morphisms are relatively \( K \)-ample resp. \( -K \)-ample over the center. Thus, in the absence of singularities, the spaces \( X//_t G \) yield a smooth running of the minimal model program.

The symplectic story can be described as follows in terms of Morse theory of the moment map. We suppose that we have a Hamiltonian \( U(1) \)-action on a symplectic manifold with proper moment map \( \Phi : X \to \mathbb{R} \). Given a critical value \( c \) of \( \Phi \), we denote by \( c_\pm \in \mathbb{R} \) regular values on either side of \( c \), so that \( c \) is the unique critical value in \( (c_-, c_+) \). We suppose for simplicity that \( \Phi^{-1}(c) \) contains a unique
critical point $x_0 \in X$. By the equivariant Darboux theorem, there exist Darboux coordinates $z_1, \ldots, z_n$ near $x_0$ and weights $\mu_1, \ldots, \mu_n \in \mathbb{Z}$ so that
\[
\Phi(z_1, \ldots, z_n) = c - \sum_{j=1}^{n} \mu_j |z_j|^2 / 2.
\]
In particular, $\Phi$ is Morse and we denote by $S^\pm_{x_0}$ the stable and unstable manifolds of $-\text{grad}(\Phi)$. The gradient flow of $-\text{grad}(\Phi)$ induces a diffeomorphism between level sets on the complement of the stable unstable manifolds:
\[
\Phi^{-1}(c_+)|S^+_{x_0} \to \Phi^{-1}(c_-)|S^-_{x_0}.
\]
Assuming the gradient vector field is defined using an invariant metric one obtains an identification of symplectic quotients except on the symplectic quotients of the stable and unstable manifolds:
\[
(X//_{c_+} U(1))\setminus(S^+_{x_0}//_{c_+} U(1)) \to (X//_{c_-} U(1))\setminus(S^+_{x_0}//_{c_-} U(1)).
\]
By the description from equivariant Darboux, one sees that the symplectic quotients of the stable and unstable manifolds are weighted projective spaces with weights given by the positive resp. negative weights:
\[
(S^+_{x_0}//_{c_+} U(1)) \cong \mathbb{P}[\pm \mu_i, \pm \mu_i > 0].
\]
Thus $X//_{c_+} U(1)$ is obtained from $X//_{c_-} U(1)$ by replacing one weighted projective space with another.

The change in the symplectic class under variation of symplectic quotient is described by Duistermaat-Heckman theory [33]. Let $c_0 \in \mathbb{R}$ be a regular value of $\Phi$. Consider the product $\Phi^{-1}(c_0) \times [c_-, c_+]$ for $c_+ \to c_0$. Let $\pi_C, \pi_R$ be the projections on the factors of $\Phi^{-1}(c_0) \times [c_-, c_+]$. Choose a connection one-form $\alpha \in \Omega^1(\Phi^{-1}(c_0))^{U(1)}$ and let $\text{curv}(\alpha) \in \Omega^2(X//_{c_0} U(1))$ denote its curvature two-form. Define a closed two-form on $\Phi^{-1}(c_0) \times [c_-, c_+]$ by
\[
\omega_0 = \pi_C^* \omega_c + d(\alpha, \pi_R - c_0).
\]
For $c_-, c_+$ sufficiently close to $c_0$, $\omega_0$ is symplectic and has moment map given by $\pi_R$. By the coisotropic embedding theorem, for $c_-, c_+$ sufficiently small there exists an equivariant symplectomorphism of a neighborhood of $\Phi^{-1}(c_0)$ in $X$ with $\Phi^{-1}(c_0) \times [c_-, c_+]$. Hence the symplectic quotients $X//_{c} U(1)$ for $c$ close to $c_0$ are diffeomorphic to $X//_{c_0} C$, with symplectic form $\omega_c + (c - c_0) \text{curv}(\alpha)$. This ends the remark.

We now explain the symplectic setting for our results which requires in a path of symplectic forms and local models arising from variation of symplectic quotient in the canonical direction as above.

**Definition 2.6.** (a) (Symplectic flips) Let $X\pm$ be non-empty symplectic manifolds. We say that $X_+$ is obtained from $X_-$ by a symplectic flip if there exist symplectic manifolds $U, V_-, V_+$, open embeddings $U \to X_-, U \to X_+, V_- \to X_-, V \to X_+$ such that
(i) Each $X\pm$ is covered by $U, V\pm$, that is, $X\pm = U \cup V\pm$. 


(ii) The manifold $U$ admits a family of symplectic forms $\omega_{t,t} \in \Omega^2(U)$, $t \in [-\epsilon, \epsilon]$ and embeddings $i_\pm : U \to X_\pm$ such that $i_\pm^* \omega_{t,t} = \omega_{t,\pm \epsilon}$.

(iii) The manifolds $V_\pm$ are obtained by symplectic reduction from a Hamiltonian $U(1)$-space with moment map $\Psi : \tilde{V} \to \mathbb{R}$ and a unique critical component mapping to 0 such that the sum of the weights is positive, at least two weights are positive, at least two weights are negative, and $V_\pm := \tilde{V} // U(1)$ are the symplectic quotients at $\pm 1$; and

(iv) Under the canonical identification $H^2(X_-) \to H^2(X_\pm)$ induced by the diffeomorphism in codimension at least four we have for some $\epsilon > 0$

$$[\omega_+] - [\omega_-] = \epsilon c_1(X_\pm).$$

(b) (Symplectic divisorial contraction) Symplectic divisorial contractions are defined in the same way as symplectic flips, but in this case all but one weight is positive. So that there exists a projection $\pi : X_+ \to X_-$ with exceptional locus $E \subset X_+$ a (weighted)-projective bundle. We require

$$[\omega_+] = \pi_+^*[\omega_-] + \epsilon(\pi_+^* c_1(X_-) + [E]^\vee)$$

where $[E]^\vee \in H^2(X_+)$ is the Poincaré dual of the exceptional divisor. Note that this differs from the usual definition of symplectic blow-down because of the presence of the additional change in symplectic class $c_1(X_-)$.

(c) (Symplectic Mori fibration) By a symplectic Mori fibration we mean a symplectic fibration $X_+ \to X_-$ such that the fiber is a monotone symplectic manifold. In all our examples flips $X_-$ will be obtained from $X_+$ by a variation of symplectic quotient using a global Hamiltonian circle action, and symplectic Mori fibrations will be Mori fibrations in the usual sense.

(d) (Symplectic mmp running) A symplectic mmp running we mean a sequence $X = X_0, X_1, \ldots, X_k$ together with, for each $i = 0, \ldots, k$, a path of symplectic forms $\omega_{i,t} \in \Omega^2(X_i)$, $t \in [t_i^-, t_i^+]$ such that

$$\frac{d}{dt} [\omega_{i,t}] = c_1(X_i)$$

and each $(X_i, \omega_{i-})$ is obtained from $(X_{i-1}, \omega_{i}^{-})$ by a symplectic mmp transition for $i = 1, \ldots, k$.

Remark 2.7. (Exceptional piece) Suppose that $X_+$ is obtained from $X_-$ by a reverse minimal model transition of flip or blow-up type, replacing a projective bundle $\mathbb{P}(N_-) \to Z$ with a projective bundle $\mathbb{P}(N_+) \to Z$. By the constant rank embedding theorem [69], a neighborhood of $X_+$ is symplectomorphic to a neighborhood of the zero section in a symplectic vector bundle $E_+$ over $\mathbb{P}(N_+)$. Any fiber of $\mathbb{P}(N_+)$ has a canonical family of Lagrangian tori, given by orbits of a maximal torus of the unitary group. Similarly, the fibers of $E_+$ have canonical family of Lagrangian tori given by the orbits of the maximal torus of the structure group.

By construction $X_+$ contains a neighborhood $U$ of the exceptional locus which is a toric fibration over the $\mathbb{P}(N_-)$. The boundary of the neighborhood $U$ may be taken to be fibered coisotropic by, for example, taking the neighborhood to be a ball.
in each fiber in the symplectic local model. Collapsing the coisotropic boundary of $U$ yields as in Lerman [66] a fibration with compact toric fibers denoted $\overline{U}$.

We introduce a class of Lagrangians associated to minimal model transitions which we call regular; these will later be shown to be Floer non-trivial.

**Definition 2.8.** Let $X$ be obtained by a reverse flip or blow-up, $\mathbb{P}(N_+)$ the projective bundle produced by the reverse flip or blow-up, and $E_+$ the normal bundle as above.

(a) A Lagrangian in $\mathbb{P}(N_+)$ is toric if it fibers over a Lagrangian submanifold of $Z$ with fiber a standard Lagrangian (in the fiber) torus in a fiber of $\mathbb{P}(N_+)$. 

(b) A Lagrangian in $E_+$ is toric if it fibers over a toric Lagrangian in $\mathbb{P}(N_+)$ with fiber a toric torus in the fiber of $E_+$.

To specify which Lagrangians are regular we need to recall a few facts about disks in toric varieties.

**Remark 2.9.** (Toric manifolds as symplectic/git quotients) Any smooth toric Deligne-Mumford stack with projective coarse moduli space has a presentation as a geometric invariant theory quotient [25, Section 3.1]. Take $V$ be a Hermitian vector space of dimension $k$ with an action of a torus $G$ with weights $\mu_1, \ldots, \mu_k$ in the weight lattice $\mathfrak{g}_\mathbb{C} \subseteq \mathfrak{g}_{\mathbb{Z}}$. A linearization of $V$ is determined by an equivariant Kähler class $\omega_{V,G} \in H^2_G(V) \cong \mathfrak{g}_{\mathbb{Z}}$. The geometric invariant theory $V//G$ is, if locally free, the quotient of the semistable locus $V^{ss} = \{ (v_1, \ldots, v_k) \in V, \text{span}\{\mu_k|v_k \neq 0}\} \ni \omega_{V,G}$ by the action of $G$. Suppose $G$ is contained in a maximal torus $H$ of the unitary group of $V$; then the residual torus $T = H/G$ acts on $X = V//G$ making $X$ into a toric variety.

The moment polytope for the action of the residual torus on the quotient can be computed from the moment polytope for the original action. We assume that $\omega_{V,G}$ has an equivariant extension to $H = (\mathbb{C}^*)^k$, written in terms of the standard basis vectors $\epsilon_i^\vee \in \mathfrak{h}^\vee$ as

$$\omega_{V,H} = \sum c_i \epsilon_i^\vee \in \mathfrak{h}^\vee \cong H^2_H(V).$$

After choosing Darboux coordinates $z_1, \ldots, z_k$ on $V$ the moment map for the $H$-action is

$$(z_i)_{i=1}^k \mapsto (c_i - |z_i|^2/2)_{i=1}^k.$$ 

Let $\nu_i$ denote the image of the $i$-standard basis vector $-\epsilon_i \in \mathfrak{h}$ under the projection $\mathfrak{h} \to \mathfrak{g}$. The residual action of the torus $T = H/G$ on $X = V//G$ has moment image

$$P = \{ \lambda \in \mathfrak{t}^\vee \mid \langle \lambda, \nu_j \rangle \geq c_j, \quad j = 1, \ldots, k \}.$$ 

**Remark 2.10.** (Primitive disks) The description of a toric manifold as a git quotient in the previous remark leads to the following description of disks with boundary in a torus orbit. Let $X$ be a compact symplectic toric manifold equipped with the action of a torus $T$, realized as a symplectic quotient of a vector space $V \cong \mathbb{C}^k$ by the
action of a torus $G$, and let $L \subset V$ be a Lagrangian orbit of $T$. Let $\tilde{L} \cong (S^1)^k \subset V$
 denote the lift of $L$ to $V$,

$$
\tilde{L} = \{ (e^{i\theta_1}\tilde{\mu}_1, \ldots, e^{i\theta_k}\tilde{\mu}_k) \mid \theta_1, \ldots, \theta_k \in \mathbb{R} \}.
$$

(5) For each $i = 1, \ldots, k$ there is a family of disks $\tilde{u}_i : (C, \partial C) \to (\tilde{V}, \tilde{L})$ of the form

$$
\tilde{u}_i(z) = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{i-1}, \tilde{\mu}_iz, \tilde{\mu}_{i+1}, \ldots, \tilde{\mu}_k).
$$

By passing to the quotient we obtain a collection of disks $u_i : (C, \partial C) \to (X, L)$ called the primitive disks.

More generally, disks in toric varieties are classified by a result of Cho-Oh [26]: a Blaschke product of degree $(d_1, \ldots, d_n)$ is a map from the disk $C := \{|z| \leq 1\} \subset \mathbb{C}$ to $\mathbb{C}^n$ with boundary in a toric Lagrangian prescribed by coefficients $a_{i,j} \in \mathbb{C}$ with $|a_{i,j}| < 1$ for $i \leq n$ and $j \leq d_i$.

$$
u : C \to \mathbb{C}^n, \quad z \mapsto \left( \prod_{j=1}^{d_i} \frac{z - a_{i,j}}{1 - z\overline{a_{i,j}}} \right)_{i=1, \ldots, n}.
$$

(6) As in Cho-Oh [26], the products (6) are a complete description of holomorphic disks with boundary in $L$. Since the image of $\tilde{u}(z)$ is disjoint from the semistable locus, the Blaschke products descend to disks $u : (C, \partial C) \to (X, L)$. We compute their Maslov index using the splitting $\tilde{u}^*TV \cong u^*TX \oplus g$.

Since the Maslov index is additive, and the second factor has Maslov index zero, we obtain Maslov index

$$I(u) = \sum_{i=1}^k 2d_j$$

which is twice the sum of the intersection number with the canonical divisor (the disjoint union of prime invariant divisors.)

Returning to the case of primitive disks, each primitive disk $u_j$ intersects the $j$-th prime boundary divisor once, and is disjoint from the remaining divisors. Furthermore, the area of this disk is given by

$$A(u_j) = \langle \lambda, \nu_j \rangle - c_j.$$

Indeed, $A(u_j) = A(\tilde{u}_j)$ which may be computed using the standard formula $\omega_C = r dr d\theta$ for the area form on $C$ and the fact that

$$\langle \Psi([z_1, \ldots, z_n]), \nu_j \rangle = -|z_j|^2/2 + c_j$$

so

$$\langle \lambda, \nu_j \rangle - c_j = \tilde{\mu}_j^2/2$$

where the constants on the right-hand-side are from (5). In other words, the computation reduces to the two-dimensional case, see Cho-Oh [26].
Remark 2.11. (Primitive disks in the toric piece) Let us return to the case of a symplectic manifold $X_+$ obtained by a reverse blow-up or flip. Let $U$ be the corresponding toric piece. After compactification a Lagrangian torus in $U$ is the boundary of $\dim(X) + 1$ primitive disks of Maslov index two whose projection to $Z$ is homotopically trivial; we call these the primitive disks in $X_+$. Because of the boundary at infinity, there is an additional primitive disk in $\overline{U}$ that does not lift to $X_+$. Rather, this disk lifts to an annulus in $U$ with one component mapping to $L$ and the other mapping to the boundary $\partial U$.

Example 2.12. (Blow-up of a product of projective lines) The disks in the case of blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ are shown in Figure 2.1. The image of each disk is one-dimensional, since the angular direction in each disk is tangent to the level sets of the moment map. On the other hand, the image of each disk (shown roughly as dotted lines in the figure) is non-linear since the map from $V$ to $X = V//G$ is non-linear. In the Figure, the areas of the three primitive disks of smallest area are all equal.

These disks play a role in the following definition:

Definition 2.13. Let $X$ be a symplectic manifold obtained by a symplectic reverse flip or blow-up. A Lagrangian $L \subset X$ is regular if

(a) $L$ is a toric Lagrangian in the toric piece $U \subset X$ of 2.7;
(b) the primitive disks $u : (D, \partial D) \to (X, L)$ contained in $U$ all have equal area $A_0$ and
(c) the primitive disk $u : (D, \partial D) \to (\overline{U}, L)$ meeting the complement $\overline{U}\setminus U$ has area greater than $A_0$.

The third condition says that the “rest” of the symplectic manifold is further away than the exceptional locus of the transition. More generally, in the case that the center $Z$ is non-trivial, the same definitions give rise, via the associated symplectic fiber bundle construction, to a fibered coisotropic $\mathcal{Z} \subset X$ whose fibers are isotropic tori as above. Since we only discuss the case of trivial centers in this paper, we leave the details to the reader.
In this next subsections we give a sequence of examples of mmp runnings. Each of these examples is in some sense obtained by variation of quotient as in Remark 2.5.

2.2. **Mmp for toric varieties.** The minimal model program for toric varieties was established by Reid [89]. Here we give a version involving shrinking the moment polytope, see Gonzalez-Woodward [44] and Pasquier [86]. Let $X$ be given as the symplectic quotient of a symplectic vector space by $V$ the action of a torus $G \subset H \cong U(1)^k$. The equivariant first Chern class of $V$ is represented by the element \((1, 1, \ldots, 1)\) in $H^2_{\text{eq}}(V) \cong \mathfrak{h}^* \cong \mathbb{R}^k$. Variation of symplectic quotient in this directions produces a running of the mmp, by Remark 2.5. Hence if the constants $c_j$ from (3) are generic, a running of the mmp is given by the sequence of toric varieties $X_t$ corresponding to the sequence of polytopes

$$P_t = \{ \mu \in \mathfrak{t}^* | \langle \mu, \nu_j \rangle \geq c_j + t \quad j = 1, \ldots, k\}.$$ 

Here we assume that each $P_t$ defines a smooth toric variety or toric Deligne-Mumford stack. The transition times are the set of times

$$T := \{ t \mid \exists \mu \in \mathfrak{t}^*, \{ \nu_j \mid \langle \mu, \nu_j \rangle = c_j + t \} \text{ is linearly dependent} \}.$$ 

For generic choices of constants, one obtains an mmp running in which every stage is an orbifold. If the facets stay the same, the transition is a flip; if a facet disappears then the transition is a divisorial contraction (where the divisor is the preimage of the disappearing face). For example, in Figure 1 there are two divisorial contractions, occurring at the dots shaded in the diagram. Finally if a generic point $\mu$ in $P_{t_0}$ satisfies $\langle \mu, \nu_j \rangle = c_j + t_0$ then $X_t$ undergoes a Mori fibration at $t_0$ with base given by the toric variety $X_{t_0}$ with polytope $P_{t_0}$; one can then continue the running with $X_{t_0}$ to obtain an extended running.

The regular Lagrangians are described as follows. Suppose that $X_t$ is an mmp running of the toric minimal model program with polytopes

$$P_t = \{ \mu \in \mathfrak{t}^* | \langle \mu, \nu_j \rangle \geq c_j + t \quad j = 1, \ldots, k\}$$

Let $\mu \in P$ and $t(\mu) = \min_j \langle \mu, \nu_j \rangle - c_j$; this is the time at which $\mu$ “disappears” under the mmp. Suppose the set

$$N(\mu) := \{ \mu_j \mid \langle \mu, \nu_j \rangle - c_j = t \}$$

is linearly dependent. Then $L_\mu := \Phi^{-1}(\mu)$ satisfies the first two parts of the definition of regularity in Definition 2.13. To see this we first compute the areas of disks with boundary on the Lagrangian. Suppose that $X$ is realized as the git quotient of a vector space $V \cong \mathbb{C}^k$ by a torus $G$. We may assume that $\dim(X) > 1$ so that the real codimension of the unstable locus is at least four. Let $\tilde{L}_\mu$ denote the preimage of $L_\mu$ in $\mathbb{C}^k$; this is a Lagrangian torus orbit of the group $U(1)^k$ acting on $\mathbb{C}^k$,

$$\tilde{L}_\mu = U(1)^k(\tilde{\mu}_1, \ldots, \tilde{\mu}_k)$$

for some constants $(\tilde{\mu}_1, \ldots, \tilde{\mu}_k)$. Then each disk corresponds to a collection of disks in $\mathbb{C}$ with boundary on $U(1)\tilde{\mu}_j$ as in Remark 2.10. It follows that if $L_\mu$ is a regular Lagrangian then the minimal areas of the holomorphic disks are $t$, these correspond
to disks in the components corresponding to facets distance $t$ from $\mu$, and each of these disks has Maslov index two. The last assumption in Definition 2.13 holds if the other facets are sufficiently far away. For example, in Figure 1 we have two regular Lagrangians, given as the inverse images of the shaded dots in the Figure under the moment map. By the forthcoming result of Abouzaid-Fukaya-Oh-Ohta-Ono, these generate the Fukaya category. We remark that the shaded dots may also be thought of as the $q$-valuations of the critical points of the Givental potential, see [40], [44].

2.3. MMP for moduli spaces of polygons. The moduli space of polygons is the quotient of a product of two-spheres by the diagonal action of the group of Euclidean rotations in three-space. This moduli space is often used as one of the primary examples of geometric invariant theory/ symplectic quotients, see for example Kirwan [56], and is famous as an “almost toric” example in a sense we will describe. It is also a special case of the moduli space of flat bundles on a Riemann surface, which appears in a number of constructions in mathematical physics.

First recall the Hamiltonian structure of the two-sphere via its realization as a coadjoint orbit. Let $S^2 \subset \mathbb{R}^3$ the unit two-sphere equipped with the $SO(3)$-invariant symplectic form with area one. The action of $SO(3)$ on $S^2$ is naturally Hamiltonian. Viewing $SO(3)$ as a coadjoint orbit in $\mathfrak{so}(3) \cong \mathbb{R}^3$, the moment map is the inclusion of $S^2$ in $\mathbb{R}^3$, as a special case of the Kirillov-Kostant-Souriau construction [98].

Starting with scaled two-forms on a collection of spheres we form a symplectic manifold of higher dimension by taking products. Let $n \geq 1$ be an integer, and $\lambda_1, \ldots, \lambda_n > 0$ a sequence of positive real numbers. The product

$$\tilde{X} = (S^2 \times \ldots \times S^2, \lambda_1 \pi_1^* \omega + \ldots + \lambda_n \pi_n^* \omega)$$

is naturally a symplectic manifold of dimension $2n$. The group $SO(3)$ acts diagonally on $\tilde{X}$ with moment map

$$\Psi : \tilde{X} \to \mathbb{R}^3, \quad (v_1, \ldots, v_n) \mapsto v_1 + \ldots + v_n.$$  

The symplectic quotient $X = \tilde{X} // SO(3)$ is the moduli space of $n$-gons

$$P(\lambda_1, \ldots, \lambda_n) = \left\{ (v_1, \ldots, v_n) \in (\mathbb{R}^3)^n \mid \|v_i\| = \lambda_i, \forall i, \sum_{i=1}^n v_i = 0 \right\}.$$  

The moduli space may be alternatively realized from the geometric invariant theory perspective as a quotient by the complexified group. We view each $v_i$ as a point in the projective line $\mathbb{P}^1$. A tuple $(v_1, \ldots, v_n) \in \mathbb{P}^1$ is semistable iff for each $w \in \mathbb{P}^1$, the slope inequality

$$\sum_{v_i = w} \lambda_i \leq \sum_{v_i \neq w} \lambda_i$$

holds [83]. Then $P(\lambda_1, \ldots, \lambda_n)$ is the quotient of the semistable locus by the action of $SL(2, \mathbb{C})$ by a special case of the Kempf-Ness theorem [55].

A running of the MMP for the moduli space of polygons is given by varying the lengths in a uniform way. The first Chern class of the product of spheres $\tilde{X}$ is the class of the form $\sum_{i=1}^n \pi_j^* \omega_j$ where $\pi_j$ is projection onto the $j$-th factor. It follows from the previous discussion that the sequence of moduli spaces $P(\lambda_1 - t, \ldots, \lambda_n - t)$
is a minimal model program for \( P(\lambda_1, \ldots, \lambda_n) \). Transitions occur whenever there are one-dimensional polygons. That is,

\[
T := \left\{ t \mid \exists \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}, \sum_{j=1}^{n} \epsilon_j (\lambda_j - t) = 0 \right\}.
\]

We may assume that the number of positive signs is greater at least the number of minus signs, by symmetry. Equivalently, \( T \) is the set of times for which there exist \textit{polystable configurations} where for some \( w_{\pm} \in \mathbb{P}^1 \), each \( v_i \in \{w_+, w_-\} \) and

\[
\sum_{v_i = w_+} (\lambda_i + t) = \sum_{v_i = w_-} (\lambda_i + t).
\]

For example, if the initial configuration is \( \lambda = (10, 10, 12, 13, 14) \) then there are three transitions, at \( t = 5, 7, 9 \), corresponding to the equalities

\[
5 + 5 + 7 = 8 + 9, \quad 3 + 3 + 6 = 5 + 7, \quad 1 + 1 + 5 = 3 + 4.
\]

There is a final transition when the smallest edge acquires zero length. The nature of the transitions follows from the discussion in the following paragraphs.

Each flip or blow-down replaces a projective space of dimension equal to the number of plus signs, minus one, with a projective space of dimension equal to the number of minus signs, minus one. One can explicitly describe the projective spaces involved in the flips as follows. For \( \pm \in \{+, -\} \) let

\[
I_{\pm} = \{i, \epsilon_i = \pm 1\}
\]

denote the set of indices with positive resp. negative signs. Let \( t_+ > t \) resp. \( t_- < t \) and let

\[
S_{\pm} = \{(v_1, \ldots, v_n) \in P(\lambda_1 - t_{\pm}, \ldots, \lambda_n - t_{\pm}) \mid \mathbb{R}_{>0} v_i = \mathbb{R}_{>0} v_j, \forall i, j \in I_{\pm}\}
\]

denote the locus where the vectors with indices in \( I_{\pm} \) point in the same direction. (Since \( \pm \) is a variable, this is a requirement for \( I_+ \) or \( I_- \) but not both.) Thus \( S_{\pm} \) is the symplectic quotient of a submanifold \( \tilde{S}_{\pm} \) of the product of \( S^2 \)'s with only positive resp. negative weights for the circle action. Thus \( S_+ \) is a projective space and the flip replaces \( S_+ \) with \( S_- \):

\[
\begin{array}{ccc}
S_- & \rightarrow & \text{pt} \\
\downarrow & & \downarrow \\
S_+ & \rightarrow & S_+
\end{array}
\]

For example, in the case of lengths 10, 10, 12, 13, 14 for the transition at \( t = 5 \), the configurations with edge lengths 5, 5, 7, 8, 9 with 9, 8 edges colinear are replaced with configurations with edges with lengths 5, 5, 7 colinear. The first set of configurations is a two-sphere, corresponding to a moduli space of quadrilaterals, while the latter set of configurations is a point, corresponding to a moduli space of triangles. It follows that the corresponding transition is a blow-down, which replaces the former set with the latter set.
There are two situations in which one obtains a Mori fibration. First, in the case that one of the $\lambda_i$’s becomes zero, say $\lambda_i - t$ is very small in relation to the other weights, there is a fibration

$$P(\lambda_1 - t, \ldots, \lambda_n - t) \rightarrow P(\lambda_1 - t, \ldots, \lambda_{i-1} - t, \lambda_{i+1} - t, \ldots, \lambda_n - t).$$

Symplectically, this is a special case of the results of Guillemin-Lerman-Sternberg [45, Section 4] while from the algebraic point of view, in this case the value of $v_i$ does not affect the semistability condition, and forgetting $v_i$ defines the fibration. In the case that the moduli space becomes empty before one of the $\lambda_i$’s reaches zero, the moduli space is a projective space at the last stage, by the same discussion as in the case of flips.

For example, in the case of lengths $10, 10, 12, 13, 14$, one obtains a fibration when $t = 10$ over the moduli space $P(2, 3, 4)$ which is a point. Therefore, the moduli space at $t = 9.5$ is a product of two-spheres. The conclusion is that $P(10, 10, 12, 13, 14)$ is a thrice-blow-up of $S^2 \times S^2$, that is, a del Pezzo surface.

The second way that one may obtain a Mori fibration is that one can reach a chamber in which the moduli space is empty, because one of the lengths is so long compared to the others that the sum of the remaining lengths is smaller than the first length. In this case, the last non-empty moduli space is a fiber bundle over the space of reducible polygons corresponding to the transition, which form a point. The fiber is a projective space, by a repeat of the arguments above. See Moon-Yoo [82] for more details.

A natural family of Lagrangian tori is generated by the bending flows studied in Klyachko [57] and Kapovich-Millson [81]. Fix a subset $I \subset \{1, \ldots, n\}$ of the edges of the polygon and define a diagonal length function

$$\tilde{\Psi}_I : S^2 \times \cdots \times S^2 \rightarrow \mathbb{R}_{\geq 0}, \quad (v_1, \ldots, v_n) \mapsto \|v_I\|, \quad v_I := \sum_{i \in I} v_i.$$

The diagonal length is smooth on the locus where it is positive.

The Hamiltonian flow of the diagonal length function is given by rotating part of the polygon around the diagonal. More precisely, the function $\tilde{\Psi}_I$ on the locus where it is positive generates a Hamiltonian circle action given by rotating the vectors $v_i, i \in I$ around the axis spanned by $\sum_{i \in I} v_i$. To prove this one applies the symplectic cross-section theorem [46, Theorem 26.7]: the inverse of the interior of the positive Weyl chamber under the moment map is a symplectic submanifold. In our situation, the symplectic cross-section is the locus of vectors whose sum of $I$-components lies on the positive part of the first coordinate axis:

$$\tilde{X}_I = \left\{ (v_1, \ldots, v_n) \in \tilde{X} \mid v_I \in \mathbb{R}_{>0} \times \{0\} \times \{0\} \subset \mathbb{R}^3 \right\}.$$

Thus $\tilde{X}_I$ is a symplectic submanifold of $\tilde{X}$. Since the flow-out of $\tilde{X}_I$ is

$$SO(3)\tilde{X}_I = SO(3) \times_{SO(2)} \tilde{X}_I,$$

any Hamiltonian diffeomorphism of $\tilde{X}_I$ extends uniquely to a Hamiltonian diffeomorphism of $SO(3)\tilde{X}_I$ which is equivariant with respect to the $SO(3)$-action. Now on $\tilde{X}_I$, the function $\tilde{\Psi}_I$ is the first component of the moment map and so the flow of
\(\tilde{\Psi}_I\) is rotation around the first axis. It follows that the flow of \(\tilde{\Psi}_I\) is rotation around the line spanned by \(\sum_{i \in I} v_i\), as long as this vector is non-zero. In particular the flow of \(\tilde{\Psi}_I\) is \(SO(3)\)-equivariant and so descends to a function \(\Psi_I\) generating a circle action on a dense subset of \(P(\lambda_1, \ldots, \lambda_n) = \tilde{X}/SO(3)\).

Let us explain why these circle actions combine to a torus action. We assume that the ordering of vectors \(v_1, \ldots, v_n\) is such that \(I, J\) consist of adjacent indices. If \(I \subset J\) then the vectors \(v_I\) and \(v_J\) break the polygon into three pieces, and the flows of \(\Psi_I\) and \(\Psi_J\) are rotation of the first and third pieces around the diagonals \(v_I, v_J\) respectively. In particular, these flows commute. Fix a triangulation of the abstract \(n\)-gon with edges \(v_1, \ldots, v_n\) corresponding to subsets \(I_1, \ldots, I_n \subset \{1, \ldots n\}\), such that each \(I_j \subset I_{j+1}\). We obtain a map

\[
\Psi : P(\lambda_1, \ldots, \lambda_n) \to \mathbb{R}_{\geq 0}^{n-3}, \quad (v_1, \ldots, v_n) \to (\|v_{I_j}\|)_{j=1}^{n-3}
\]

given by taking the edge lengths of the diagonals. The discussion above shows that

![Triangulated polygon](image)

**Figure 4. Triangulated polygon**

the map \(\Psi\) is, where smooth, a moment map for the action of an \(n-3\)-dimensional bending torus \(T \cong U(1)^{n-3}\) which acts as follows: Let \((\exp(i\theta_1), \ldots, \exp(i\theta_{n-3})) \in T\) and \([v_1, \ldots, v_n]\) be an equivalence class of polygons. For each diagonal \(v_I\), divide the polygon into two pieces along \(v_I\), and rotate one of those pieces, say \((v_i)_{i \in I}\) around the diagonal by the given angle \(\theta_I\). The resulting polygon is independent of the choice of piece, since polygons related by an overall rotation define the same point in the moduli space.

The regular Lagrangian tori are described as follows as fibers of the Goldman map for which all triangles have the same “looseness”. Suppose that \(P(\lambda_1 - t, \ldots, \lambda_n - t)\) is an mmp running for \(P(\lambda_1, \ldots, \lambda_n)\). As noted in Example 2.2, mmp transitions correspond to partitions

\[
\{1, \ldots, n\} = I_+ \cup I_-, \quad \sum_{i \in I_+} \lambda_i - t = \sum_{i \in I_-} \lambda_i - t.
\]
For each triangle $T$ in the triangulation with labels $\mu_1, \mu_2, \mu_3$ we denote by $l(T)$ the *looseness* of the triangle

$$l(T) := \min_{i \neq j \neq k} (\mu_i + \mu_j - \mu_k)/2.$$ 

In other words, the looseness measures the failure of the triangle to be degenerate. The looseness is a tropical version of the area of the triangle given by Heron’s formula:

$$A(T) = (1/4) \sqrt{(\mu_1 + \mu_2 + \mu_3) \prod_{i \neq j \neq k} (\mu_i + \mu_j - \mu_k)}.$$ 

A labelling $\mu \in \mathbb{R}^{n-3}_{\geq 0}$ is called *uniformly loose* if $l(T)$ is independent of $T$ and greater than $\min_i \lambda_i$. For example, if the edge lengths are 2, 3, 4, 7, and the triangulation separates the first two edges from the last two, then a uniformly loose triangulation is obtained by assigning 4 to the middle edge, so that the looseness of each triangle is $1 = 2 + 3 - 4 = 4 + 4 - 7$.

![Figure 5. A uniformly loose quadrilateral](image)

We claim that if the labelling is uniformly loose with small looseness then the corresponding Lagrangian is regular. First consider the case that the looseness of each triangle is small. In this case one obtains a local toric structure given by choosing a triangulation compatible with the partition into positive and negative edges, and the action of the bending torus $T$ above. Let $v \in P(\lambda_1 - t, \ldots, \lambda_n - t)$ denote the one-dimensional polygon corresponding to the transition time. Each triangle in the triangulation is degenerate for $v$ and so for each $T$, $\mu_i + \mu_j = \mu_k$ for some edges $i, j, k$ of $T$. The inequalities defining $\Psi_t(T)$ near $v$ are of the form

$$\mu_i + \mu_j \geq \mu_k$$

as $i, j, k$ range over all possible indices. It follows that the polytope defining the image of the map $\Psi$ is given locally by the triangle inequalities, $l(T) \geq 0$ for each of the $n - 2$ triangles in the triangulation:

$$\Psi_T(P(\lambda_1, \ldots, \lambda_n)) = \left\{(\mu_1, \ldots, \mu_{n-3}) \in \mathbb{R}^{n-3}_{\geq 0} \mid \forall T \in T, (T = \{v_i, v_j, v_k\}) \implies \mu_i + \mu_j \geq \mu_k \right\}.$$ 

The uniformly loose condition is then the same as the condition for the Lagrangian to be regular in the toric case.
2.4. **Mmp for moduli spaces of flat bundles.** The moduli space of flat bundles on a Riemann surface is an example of an infinite-dimensional symplectic quotient, and studied in for example Atiyah-Bott \[8\]. Let \( \Sigma \) be a compact Riemann surface and \( G \) a compact Lie group. The trivial \( G \)-bundle \( P = \Sigma \times G \) has space of connections \( \mathcal{A}(P) \) canonically identified with the space of \( g \)-valued one-forms which are \( G \)-invariant and induce the identity on the vertical directions:

\[
\mathcal{A}(P) := \{ \alpha \in \Omega^1(\Sigma \times G), \quad \alpha(\xi_P) = \xi, \forall \xi \in g \}
\]

where \( \xi_P \in \text{Vect}(P) \) is the corresponding vector field. The space \( \mathcal{A}(P) \) is an affine space modelled on \( \Omega^1(\Sigma, g) \), and has a natural symplectic structure given by

\[
\Omega^1(\Sigma, g)^2 \to \mathbb{R}, \quad (a_1, a_2) \mapsto \int_\Sigma (a_1 \wedge a_2).
\]

Here \( (a_1 \wedge a_2) \in \Omega^2(\Sigma) \) is the result of composition

\[
\Omega^1(\Sigma, g)^{\otimes 2} \to \Omega^2(\Sigma, g^{\otimes 2}) \to \Omega^2(\Sigma, g^2) \to \Omega^2(\Sigma, \mathbb{R})
\]

where the latter is induced by an invariant inner product \( g \times g \to \mathbb{R} \). The action of the group \( G(P) \) of gauge transformations on \( \mathcal{A}(P) \) by pullback is Hamiltonian with moment map given by the curvature:

\[
\mathcal{A}(P) \to \Omega^2(\Sigma, \mathcal{P}(g)), \quad A \mapsto F_A.
\]

The symplectic form on \( \mathcal{A}(P) \) descends to a closed two-form on the symplectic quotient

\[
\mathcal{R}(\Sigma) := \{ A \in \mathcal{A}(P) \mid F_A = 0 \}/G(P)
\]

the moduli space of flat connections on the trivial bundle. The tangent space to \( \mathcal{R}(\Sigma) \) at the isomorphism class of a connection \( A \) has a natural identification

\[
T_{[A]} \mathcal{R}(\Sigma) \cong H^1(d_A)
\]

with the cohomology \( H^1(d_A) \) of the associated covariant derivative \( d_A \) in the adjoint representation. The Hodge star furnishes a Kähler structure on the moduli space.

Extensions to the case with boundary are given in, for example, Mehta-Seshadri [79]. Suppose \( \Sigma \) is a compact oriented surface of genus \( g \) with \( n \) boundary components. That is, \( \Sigma \) is obtained from a closed compact oriented surface of genus \( g \) by removing \( n \) disjoint disks. Let \( Z_k \subset \Sigma \) be the \( k \)-th boundary circle, and \( \{ Z_k \} \in \pi_1(\Sigma) \) the class defined by a small loop around the \( k \)-th boundary component for \( k = 1, \ldots, n \). Let \( G = SU(2) \) denote group of special unitary \( 2 \times 2 \) matrices. The space of conjugacy classes \( G/\text{Ad}(G) \) is naturally parametrized by an interval:

\[
[0, 1/2] \to G/\text{Ad}(G), \quad \lambda \mapsto \text{diag}(\exp(\pm 2\pi i \lambda)).
\]

Let \( \lambda_1, \ldots, \lambda_n \in [0, 1/2] \) be *labels* attached to the boundary components. Choose a base point in \( \Sigma \) and let \( \pi_1(\Sigma) \) denote the fundamental group of homotopy classes of based loops. Each loop \( Z_k \) defines an element \( [Z_k] \in \pi_1(\Sigma) \), by connecting \( Z_k \) to a base point, which is well-defined up to conjugacy. For numbers \( \mu_1, \mu_2 \) we denote by \( \text{diag}(\mu_1, \mu_2) \) the diagonal \( 2 \times 2 \) matrix with diagonal entries \( \mu_1 \) and \( \mu_2 \). Let \( \mathcal{R}(\lambda_1, \ldots, \lambda_n) \) denote the moduli space of isomorphism classes flat bundles with holonomy around the boundary circles given by \( \exp(2\pi i \text{diag}(\lambda_k, -\lambda_k)) \), \( k = 1, \ldots, n \).
Since any flat bundle is described up to isomorphism by the associated holonomy representation of the fundamental group, we have the explicit description

\[
R(\lambda_1, \ldots, \lambda_n) = \left\{ \varphi \in \text{Hom}(\pi_1(\Sigma), SU(2)) \mid \varphi([Z_k]) \in SU(2) \exp(2\pi i \text{diag}(\lambda_k, -\lambda_k)) \right\} / SU(2).
\]

By Mehta-Seshadri [79], the moduli space of flat bundles may be identified with the moduli space of parabolic bundles with weights \(\lambda_1, \ldots, \lambda_n\). Here a parabolic bundle means a holomorphic bundle on closed Riemann surface with markings \(z_1, \ldots, z_n\) with the additional datum of one-dimensional subspace in the fiber at each marking.

In the case of rank two bundles there is a simple interpretation of these moduli spaces in terms of spherical polygons. Namely \(\pi_1(\Sigma)\) is generated by homotopy classes of paths \(\gamma_1, \ldots, \gamma_n : S^1 \to \Sigma\) with the single relation

\[
[\gamma_1] \cdots [\gamma_n] = 1 \in \pi_1(\Sigma).
\]

Thus a representation of the fundamental group corresponds to a tuple

\(g_1, \ldots, g_n \in SU(2),\ g_1 \cdots g_n = 1\).

Consider the polygon in \(SU(2)\) with vertices

\(e,\ g_1,\ g_1 g_2, \ldots,\ g_1 \cdots g_n = e\),

where \(e \in SU(2)\) is the identity. Because the metric on \(SU(2)\) is invariant under the right action, the distance between the \(j-1\)-th and \(j\)-th vertices is the distance between \(e\) and \(g_j\). Using invariance again it suffices to assume that \(g_j = \text{diag}(2\pi i (\lambda_j, -\lambda_j))\) in which case the distance is \(\lambda_j\), once the metric is normalized so the maximal torus has volume one. Using the identification of \(S^3\) with \(SU(2)\) any representation gives rise to a polygon in \(S^3\) with edge lengths \(\lambda_1, \ldots, \lambda_n\) and one sees from the description that this correspondence is bijective up to isometries of the three-sphere.

For generic weights the moduli space of flat bundles has a smooth running of the mmp given by varying the labels in a uniform way. First, a result of Boden-Hu [15] and Thaddeus [102, Section 7] shows that varying the labels leads to a generalized flips in the sense that all conditions are satisfied except the condition that the
morphisms to the singular quotient are relatively ample. For this the variation of Kähler class should be in the canonical direction.

In the case without boundary, the anticanonical class was computed by Drezet-Narasimhan [32] and in the case of parabolic bundles by Biswas-Raghavendra [13]; see Meinrenken-Woodward [72] for a symplectic perspective. The anticanonical class is expressed in terms of the symplectic class and the line bundles \(L_j\) associated to the eigenspaces of the holonomy around the boundary components by

\[
c_1(\mathcal{R}(\lambda_1, \ldots, \lambda_n)) = 4[\omega_{\mathcal{R}(\lambda_1, \ldots, \lambda_n)}] - \sum_{j=1}^{n}(4\lambda_j - 1)2c_1(L_j).
\]

In particular, if all weights \(\lambda_i = 1/4\) then the moduli space is Fano. The moduli space has a smooth running of the mmp given by the sequence of moduli spaces

\[
\mathcal{R}(\lambda_1 - t, \ldots, \lambda_n - t)_{1 - 4t}.
\]

This family can be produced as a variation of symplectic quotient using the construction of [72] as follows: Let \(LG\) denote the loop group of \(G = SU(2)\), \(\mathcal{R}\) denote the moduli space of flat \(G\)-connections with framings on the boundary equipped with its natural Hamiltonian action of \(LG^n\), and \(\mathcal{O}_{\lambda_1}, \ldots, \mathcal{O}_{\lambda_n}\) the \(LG\)-coadjoint orbits corresponding to \(\lambda_1, \ldots, \lambda_n\). Then \(\mathcal{R}(\lambda_1, \ldots, \lambda_n)\) has a realization as a symplectic quotient

\[
\mathcal{R}(\lambda_1, \ldots, \lambda_n) = (\mathcal{R} \times \mathcal{O}_{\lambda_1} \times \ldots \times \mathcal{O}_{\lambda_n})/LG^n.
\]

Consider the product of anticanonical bundles

\[
K_{\mathcal{R}}^\vee \boxtimes K_{\mathcal{O}_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{\mathcal{O}_{\lambda_n}}^\vee \to \mathcal{R} \times \mathcal{O}_{\lambda_1} \times \ldots \times \mathcal{O}_{\lambda_n}
\]

in the sense of [72]. Its total space minus the zero section has closed two form given by

\[
\pi^*\omega_{\mathcal{R} \times \mathcal{O}_{\lambda_1} \times \ldots \times \mathcal{O}_{\lambda_n}} + d(\alpha, \phi) \in \Omega^2((K_{\mathcal{R}}^\vee \boxtimes K_{\mathcal{O}_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{\mathcal{O}_{\lambda_n}}^\vee) - \{0\})
\]

where \(\alpha\) is a connection one-form and \(\phi\) is the logarithm of the norm on the fiber. This two-form is non-degenerate on the region defined by \((\lambda_i - \phi)/(1 - 4\phi) \in (0, 1/2)\) for each \(i\). The action of \(S^1\) by scalar multiplication in the fibers is Hamiltonian with moment map \(\phi\) and the quotient

\[
\tilde{\mathcal{R}}(\lambda_1, \ldots, \lambda_n) := (K_{\mathcal{R}}^\vee \boxtimes K_{\mathcal{O}_{\lambda_1}}^\vee \boxtimes \ldots \boxtimes K_{\mathcal{O}_{\lambda_n}}^\vee - \{0\})/LG^n
\]

has a residual action of \(S^1\) whose quotients are the family given above:

\[
\tilde{\mathcal{R}}(\lambda_1, \ldots, \lambda_n)/\mathcal{S}^1 = R\left(\frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t}\right).
\]

At any fixed point the action of \(S^1\) on the anticanonical bundle has positive weight, by definition. Note that the case \(\lambda_1 = \ldots = \lambda_n = 1/4\) has a trivial mmp. One should think of the markings as moving away from the “center” 1/4 of the Weyl alcove [0, 1/2] under the mmp.
The flips or blow-downs occur at transition times at which there are reducible
(abelian) bundles. More precisely, the set of transition times
\[ T = \left\{ t \mid \exists \epsilon_1, \ldots, \epsilon_n \in \{0, 1\}, \quad \sum_{i=1}^n \epsilon_i \frac{\lambda_i - t}{1 - 4t} \in \mathbb{Z}/2 \right\}. \]

The projective bundles involved in the flip can be explicitly described as follows
using the description of the moduli space as loop space quotient in [71]; we focus on
the genus zero case and omit the proofs. Let \( I_{\pm} = \{ i, \epsilon_i = \pm 1 \} \). Fix a decomposition
of the curve \( \Sigma \) into Riemann surfaces with boundary \( \Sigma_+, \Sigma_- \) such that \( \Sigma_{\pm} \) contains
the markings in \( I_{\pm} \). Let \( S_{\pm} \) denote the moduli space of bundles that are abelian on
\( \Sigma_{\pm} \):
\[ S_{\pm} = \left\{ [A] \in \mathcal{R} \left( \frac{\lambda_1 - t_{\pm}}{1 - 4t}, \ldots, \frac{\lambda_n - t_{\pm}}{1 - 4t} \right) \mid \dim(\mathcal{G}_A|\Sigma_{\pm}) = 1 \right\}. \]

Then the flip replaces \( S_+ \) with \( S_- \):
\[
\begin{array}{ccc}
S_- & \xrightarrow{\mathcal{R}} & \mathcal{R}^{ab} \left( \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right) \\
& \downarrow & \downarrow \\
\mathcal{R} \left( \frac{\lambda_1 - t_{\pm}}{1 - 4t}, \ldots, \frac{\lambda_n - t_{\pm}}{1 - 4t} \right) & & \mathcal{R} \left( \frac{\lambda_1 - t_{\pm}}{1 - 4t}, \ldots, \frac{\lambda_n - t_{\pm}}{1 - 4t} \right) \\
& \downarrow & \downarrow \\
S_+ & \xrightarrow{\mathcal{R}} & \mathcal{R}^{ab} \left( \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right)
\end{array}
\]

where \( \mathcal{R}^{ab} \left( \frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t} \right) \) is the moduli space of abelian representations. Thus a
projective bundle over the Jacobian gets replaced with another projective bundle.

As in the case of polygon spaces, there are two ways of obtaining Mori fibrations:
First, fibrations with \( \mathbb{P}^1 \)-fiber occur whenever one of the markings \( \lambda_i - t \) reaches 0
or 1/2, with base the moduli space of flat bundles with one less marking and labels
\[
\frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_{i-1} - t}{1 - 4t}, \frac{\lambda_{i+1} - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t}
\]
resp.
\[
\frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_{i-1} - t}{1 - 4t}, \frac{\lambda_{i+1} - t}{1 - 4t}, \ldots, \frac{1/2 - \lambda_n - t}{1 - 4t}
\]
if the marking \( \lambda_i \) reaches 0 resp. 1/2. Second, in genus zero the moduli space can
become empty before any of the markings reach 0 or 1/2. By a special case of
Agnihotri-Woodward [6], proved earlier by Treloar [101] we have
\[
\mathcal{R}(\lambda_1, \ldots, \lambda_n) = \emptyset \iff \exists I = \{ i_1 \neq \ldots \neq i_{2k+1} \}, \quad \sum_{i \in I} \lambda_i > k + \sum_{i \not\in I} \lambda_i.
\]

Thus in the last stage there is either a fibration over a moduli space with one
less parabolic weight, with two-sphere fiber, or in genus zero one can also have a
projective space at the last stage if the moduli space becomes empty. One can
then continue with the base to obtain an extended running. The minimal model
program of this moduli space is discussed in greater detail in Moon-Yoo [82] as well
as Boden-Hu [15] and Thaddeus [102, Section 7].
The analog of the bending flow was introduced by Goldman [42]. First one constructs a densely defined circle action on the moduli space of bundles associated to any circle on the surface. Given any circle $C \subset \Sigma$ disjoint from the boundary, the holonomy $\varphi(C)$ of the flat bundle $P$ around $C$ is given by an element $\exp(\text{diag}(\pm 2\pi i \mu))$ up to conjugacy. After gauge transformation, the holonomy is exactly $\exp(\text{diag}(\pm 2\pi i \mu))$. Given an element $\exp(2\pi i \tau) \in U(1)$, one may construct a bundle $P_\tau$ by cutting $\Sigma$ along $C$ into pieces $\Sigma_+, \Sigma_-$ and gluing back the restrictions $P|\Sigma_+, P|\Sigma_-$ together using the transition map $e(\tau) := \text{diag}(\exp(2\pi i \tau))$:

$$P_\tau := (P|\Sigma_+) \bigcup_{e(\tau)} (P|\Sigma_-).$$

See Figure 7.

![Figure 7. Twisting a bundle along a circle](image)

The automorphism given by $\text{diag}(\exp(2\pi i \tau))$ commutes with the holonomy so the resulting bundle has a canonical flat structure, whose holonomies around loops $\Sigma_+, \Sigma_-$ are equal, but parallel transport from $\Sigma_+$ to $\Sigma_-$ is twisted by $\text{diag}(\exp(2\pi i \tau))$. Let $\mathcal{R}(\lambda_1, \ldots, \lambda_n)^C$ denote the locus where $\mu \notin \{0, 1/2\}$, for which the construction $[P] \mapsto [P_\tau]$ is well-defined and independent of all choices. The map

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n)^C \to \mathcal{R}(\lambda_1, \ldots, \lambda_n)^C, \quad [P] \mapsto [P_\tau]$$

defines a circle action. By Goldman [42], see also Meinrenken-Woodward [71] the action has moment map given by

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n)^C \to (0, 1/2), \quad [\varphi] \mapsto \mu$$

where $\varphi(C) = \text{diag}(\exp(\pm 2\pi i \mu))$ up to conjugacy. Furthermore, if $C_1, C_2$ are disjoint circles then the circle actions defined above commute on the common locus

$$\mathcal{R}(\lambda_1, \ldots, \lambda_n)^{C_1} \cap \mathcal{R}(\lambda_1, \ldots, \lambda_n)^{C_2}.$$ 

Recall that a pants decomposition of a surface is a decomposition into three-holed spheres. Any compact oriented Riemann surface with boundary admits a finite pants decomposition, by choosing sufficiently many separating surfaces so that each piece has Euler characteristic one. Choose a pants decomposition $\mathcal{P}$ that refines the
decomposition into pieces $\Sigma_+, \Sigma_-$. Given a pants decomposition, one repeats the construction for each interior circle in the pants decomposition to obtain a map

$$\Psi_P : \mathcal{R}(\lambda_1, \ldots, \lambda_n) \to [0, 1/2]^{n-3}.$$ 

In the genus zero case, the generic fibers are Lagrangian tori. For each pairs of pants $P$ in the pants decomposition with labels $\mu_1, \mu_2, \mu_3$, define the looseness of $P$ by

$$l(P) := \min \left( \min_{i \neq j \neq k} (\mu_i + \mu_j - \mu_k), 1 - \mu_1 - \mu_2 - \mu_3 \right).$$

As before, this is a tropical version of L’Huilier’s generalization of Heron’s formula: for a triangle $T$ in the unit sphere with edge lengths $2\pi \mu_1, 2\pi \mu_2, 2\pi \mu_3$ the area $A(T)$ is determined by

$$\tan(\frac{A(T)}{4}) = \sqrt{\frac{\tan(2\pi(1 - \mu_1 - \mu_2 - \mu_3))}{\prod_{i \neq j \neq k, i < j} \tan(2\pi(\mu_i + \mu_j - \mu_k))}}.$$ 

We say that a labelling $\mu \in [0, 1/2]^{n-3}$ is uniformly loose if the looseness $l(P)$ is the same for each pair of pants $P \in \mathcal{P}$, 

$$\#\{l(P) | P \in \mathcal{P}\} = 1$$

and if the first fibration in the running occurs at a time greater than $l(P)$. See Figure 2 for two examples in the case $n = 5$.

The regular Lagrangians are described as follows. Suppose that $R\left(\frac{\lambda_1 - t}{1 - 4t}, \ldots, \frac{\lambda_n - t}{1 - 4t}\right)$ is an mmp running as above. The transition times $T$ are the times $t$ for which there is an abelian representation. Given such a representation with holonomies $\text{diag}(\exp(\pm \epsilon_j \mu_j))$ define a partition of the surface $\Sigma$ into pieces $\Sigma_+, \Sigma_-$ containing the markings $\mu_j$ for which $\epsilon_j$ is positive resp. negative. We claim that if $\mu$ is uniformly loose and $l(\mu)$ is sufficiently small then the Goldman fiber

$$L_\mu := \Psi_P^{-1}(\mu)$$

is regular. The Goldman bending flow induces a toric structure on $\mathcal{R}(\lambda_1, \ldots, \lambda_n)$. The image of the Goldman map (4) is given by

$$\Psi_P(\mathcal{R}(\lambda_1, \ldots, \lambda_n)) = \{ \mu \in [0, 1/2]^{n-3} | \forall P \in \mathcal{P}, l(P) \geq 0 \};$$

that is, for each pair of pants in the pants decomposition the looseness is non-negative. It follows that if $l$ is a uniformly loose pants decomposition then $\Psi^{-1}(l)$ is toric and there are $n - 2$ homotopy classes of disks with Maslov index two and boundary in $\Psi^{-1}(l)$, of equal area.

2.5. Mmp for flag varieties. We end with, in some sense, the most trivial example: flag varieties. Once again, there is a trivial mmp running. Other runnings are given by fibrations over partial flag varieties. Let $X$ be the variety of complete flags in a vector space of dimension $n$ with polarization corresponding to a dominant weight $\lambda$. The space $X$ has a natural transitive action of the unitary group which induces a diffeomorphism $X \cong U(n)/U(1)^n$. We identify the Lie algebra with
\( \mathbb{R}^n \), the weight lattice with \( \mathbb{Z}^n \) and let \( \epsilon_1, \ldots, \epsilon_n \in \mathbb{Z}^n \) denote the standard basis of weights for \( U(1)^n \). The tangent bundle of \( X \) is the associated fiber bundle

\[
TX \cong U(n) \times_{U(1)\n} \bigoplus_{1 \leq i < j \leq n} C_{\epsilon_i - \epsilon_j}
\]

where \( C_{\epsilon_i - \epsilon_j} \) is the space on which \( U(1)^n \) acts with weight \( \epsilon_i - \epsilon_j \). Hence the canonical bundle of \( X \) is

\[
\Lambda^{\text{top}} TX \cong U(n) \times_{U(1)\n} \mathbb{C}_{2\rho}
\]

where \( \mathbb{C}_{2\rho} \) denotes the one-dimensional representation of \( U(1)^n \) with weight

\[
2\rho := ((n - 1)\epsilon_1 + (n - 3)\epsilon_2 + \ldots + (1 - n)\epsilon_n).
\]

An extended running of the minimal model program is the sequence of partial flag varieties corresponding to the piecewise linear path \( \lambda_t \) starting with \( \lambda_t = \lambda - \rho t \), where \( \rho \) and whenever \( \lambda_t \) hits a wall \( \sigma \) of the positive chamber \( \lambda_t \in \sigma \) one continues with the path \( \lambda_t = \lambda_t - (t - t_i)\pi_{\sigma}\rho \), where \( \pi_{\sigma} \) is the projection onto \( \sigma \). Each transition is a Mori fibration with Grassmann fiber and base the partial flag variety corresponding to the element \( \lambda_t \).

A simple example is the variety of complex flags in a three-dimensional complex vector space which admits the structure of a Mori fibration in two ways. For example, let

\[
X = \text{Fl}(\mathbb{C}^3) := \{ V_1 \subset V_2 \subset \mathbb{C}^3 \mid \dim(V_k) = k, k = 1, 2 \}
\]

be the variety of full flags in \( \mathbb{C}^3 \). Equip \( X \) with the symplectic class such that the \( \mathbb{P}^1 \)-fiber of the natural fibration

\[
X \to \mathbb{P}^2, \quad (V_1, V_2) \mapsto V_1
\]

is small. A running of the minimal model program is given by \( X, \mathbb{P}^2, \text{pt.} \) There are no flips or divisorial contractions in this case, so no regular Lagrangians.

### 3. Fukaya algebras

The Fukaya algebra of a Lagrangian brane in a symplectic manifold is defined by counts of holomorphic disks with boundary in the Lagrangian. Fukaya and collaborators have introduced several versions of this algebra. Here review a version developed jointly with Charest [24] based on the notion of treed disks appearing in e.g. Cornea-Lalonde [28] with additional generators added to give strict units.

#### 3.1. A-infinity algebras

We begin by introducing notation for homotopy associative algebras introduced and the family of spaces called associahedra from Stasheff [99]. Let \( g > 0 \) be an even integer. Recall that a \( \mathbb{Z}_g \)-graded non-unital \( A_{\infty} \) algebra consists of a \( \mathbb{Z}_g \)-graded vector space

\[
A = \bigoplus_{d \in \mathbb{Z}_g} A_d
\]

together with for each \( d \geq 0 \) a multilinear composition map

\[
\mu^d : A^{\otimes d} \to A[2-d]
\]
satisfying the $A_{\infty}$-associativity equations

$$0 = \sum_{n+m \leq d} (-1)^{n+\sum_{i=1}^{n} |a_i|} \mu^{d-m+1}(a_1, \ldots, a_n, a_m(a_{n+1}, \ldots, a_{n+m}), a_{n+m+1}, \ldots, a_d)$$

for any tuple of homogeneous elements $a_1, \ldots, a_d$ with degrees $|a_1|, \ldots, |a_d| \in \mathbb{Z}_g$. The signs are the shifted Koszul signs, that is, the Koszul signs for the shifted grading in which the structure maps have degree one as in Kontsevich-Soibelman [60]. The element $\mu^0(1) \in A$ is the curvature of the algebra. The $A_{\infty}$ algebras considered in this paper are always finitely generated over the base ring, in comparison with those in Fukaya et al. [39] which are often infinitely generated.

A strict unit for the algebra $A$ is an element $e \in A$ such that

$$\mu^1(a) = 0, \quad \mu^2(e, a) = (-1)^{|a|}\mu^2(a, e) = a, \quad \forall a \in A$$

and all higher compositions involving the identity $e$ vanish:

$$0 = \mu^3(a_1, a_2, e) = \mu^3(a_1, e, a_2) = \mu^3(e, a_1, a_2),$$

$$0 = \mu^4(a_1, a_2, a_3, e) = \mu^4(a_1, a_2, e, a_3) = \ldots, \forall a_1, a_2, \ldots \in A.$$  

An $A_{\infty}$ morphism $\mathcal{F}$ from $A_0$ to $A_1$ with composition maps $\mu^n_0$ resp. $\mu^n_1$ consists of a collection of maps

$$\mathcal{F}^d : A_0^d \rightarrow A_1[1-d], \quad d \geq 0$$

such that the following holds:

$$\sum_{i+j \leq d} (-1)^{i+\sum_{i=1}^{i} |a_i|} \mathcal{F}^{d-j+1}(a_1, \ldots, a_i, \mu^j_0(a_{i+1}, \ldots, a_{i+j}), a_{i+j+1}, \ldots, a_d) =$$

$$\sum_{i_1+\ldots+i_m = d} \mu^m_1(\mathcal{F}^{i_1}(a_1, \ldots, a_{i_1}), \ldots, \mathcal{F}^{i_m}(a_{i_1+\ldots+i_{m-1}+1}, \ldots, a_d))$$

where the first sum is over integers $i, j$ with $i + j \leq d$, the second is over partitions $d = i_1 + \ldots + i_m$. To any flat $A_{\infty}$ algebra $A$ is associated an ordinary cohomology algebra

$$H(A) := \ker(\mu^1)/\text{im}(\mu^1)$$

with product given by

$$[a_1 \circ a_2] = (-1)^{|a_1|}|\mu^2(a_1, a_2)|.$$  

Any $A_{\infty}$ morphism $\mathcal{F} = (\mathcal{F}^d) : A_0 \rightarrow A_1$ defines an morphism of cohomology groups

$$H(\mathcal{F}) : H(A_0) \rightarrow H(A_1), \quad [a] \mapsto (-1)^{|a|}[\mathcal{F}^1(a)].$$

If $A_0, A_1$ are strictly unital $A_{\infty}$ algebras with units $e_0, e_1$ then a morphism $\mathcal{F} : A_0 \rightarrow A_1$ is unital iff $\mathcal{F}^1(e_0) = e_1$ and all other maps involving the identity vanish:

$$0 = \mathcal{F}^2(a_1, e_0) = \mathcal{F}^2(e_0, a_1), \quad 0 = \mathcal{F}^3(e_0, a_1, a_2) = \ldots.$$  

Later we will be especially interested in deformations of $A_{\infty}$ algebras which lead to flat $A_{\infty}$ algebras. The space of such deformations is known as the moduli space of the $A_{\infty}$ algebra, as we now explain. Let $A$ be a $A_{\infty}$ algebra defined over $\Lambda_{\geq 0}$ with
strict unit \( e \). Suppose that the vector space underlying \( A \) is a finite rank and free \( \Lambda \geq 0 \)-module and furthermore admits a \( \mathbb{Z} \)-grading
\[
A = \bigoplus_{d \in \mathbb{Z}} A_d.
\]
We write
\[
A = \bigoplus_{d \geq 0} A_d, \quad A_{\leq 0} = \bigoplus_{d < 0} A_d
\]
for some integers \( d_0 \leq d_+ \); define \( A_{\geq 0} \) and \( A_{>0} \) similarly. Write the map \( \mu^n \) in terms of its homogeneous components,
\[
\mu^n = \sum_{m} \mu^{n,m}, \quad \mu^{n,m}(A^{(\geq n)}_i) \subset A_{i+2-n+m}.
\]
We say that an \( A_{\infty} \) algebra \( A \) as above is convergent iff \( \mu^{0}(1) \in q^{E_0} A \) for some \( E_0 > 0 \) and there exists a sequence \( E_m \to \infty \) such that
\[
(15) \quad \mu^{n,m}(A) \subset q^{E_m} A, \quad \forall n \geq 0.
\]
Given a convergent \( A_{\infty} \) algebra, define
\[
(16) \quad A_+ := A_{\leq 0} + \Lambda_{>0} A_{>0}, \quad A_{++} := A_{<0} + \Lambda_{>0} A_{\geq 0}.
\]
For \( b \in A_+ \) the sum
\[
(17) \quad \mu_0^0(1) := \mu^{0}(1) + \mu^{1}(b) + \mu^{2}(b,b) + \ldots
\]
is well-defined. More generally the same argument implies convergence of the deformed composition map
\[
\mu^n_b(a_1, \ldots, a_n) = \sum_{i_1, \ldots, i_{n+1}} \mu^{n+i_1+\ldots+i_{n+1}}(b, \ldots, b, a_1, \ldots, b, a_2, \ldots, b, a_n, \ldots, b)
\]
over all possible combinations of insertions of the element \( b \in A_+ \) between (and before and after) the elements \( a_1, \ldots, a_n \), is convergent for similar reasons. The maps \( \mu^n_b \) define an \( A_{\infty} \) structure on \( A \). In particular
\[
(\mu^1_b)^2(a) = \mu^2_b(\mu^0_b(1), a_1) - (-1)^{|a|} \mu^2_b(a, \mu^0_b(1))
\]
for all homogeneous \( a \in A \). The weak Maurer-Cartan equation for \( b \in A_+ \) is
\[
(18) \quad \mu^0_b(1) = \mu^{0}(1) + \mu^{1}(b) + \mu^{2}(b,b) + \ldots \in \Lambda e.
\]
Denote by \( MC(A) \) the space of solutions to the weak Maurer-Cartan equation (18). Any solution to the weak Maurer-Cartan equation defines an \( A_{\infty} \) algebra such that \( (\mu^1_b)^2 = 0 \) and so has a well-defined cohomology
\[
H(\mu^1_b) = \frac{\ker(\mu^1_b)}{\text{im}(\mu^1_b)}.
\]
If \( F : A_0 \to A_1 \) is a strictly unital \( A_{\infty} \) morphism then there is a natural map
\[
MC(F) : MC(A_0) \to MC(A_1), \quad a \mapsto \sum_{n \geq 0} F_n(a, \ldots, a).
\]
If $F$ is a homotopy equivalence satisfying a suitable convergence property then $MC(F)$ induces a bijection on the Maurer-Cartan moduli spaces after modding out by a suitable notion of gauge equivalence, and the cohomology of $\mu^b_1$ for $b \in MC(A)$ is isomorphic to the cohomology of $\mu^b_{MC(F)}$.

3.2. Treed holomorphic disks. Roughly speaking the idea of the definition of the Fukaya algebra is quite simple: to define the composition maps one takes some model for cochains on the Lagrangian, pulls back cochains to the moduli space of holomorphic disks via evaluation maps for markings on the boundary and pushes forward using a final evaluation map. However reality is not as simple as the idea not only because of singularities in the moduli spaces but also because of the difficulty in pushing forward chains. In order to simplify the difficulty of push-forward we take very simple model for cochains on the Lagrangian, namely the Morse cochains which are formal combinations of critical points of a Morse function.

We briefly review the construction of Fukaya algebras for Lagrangians in symplectic manifolds with rational symplectic classes from [24]. Let $L$ be a compact connected Lagrangian in $X$. Fix a metric $G$ on $L$ and a Morse function $F : L \to \mathbb{R}$. Let $I \subset \mathbb{R}$ be an open or closed interval. The gradient vector field of $F$ is defined by

$$G(\text{grad}_F, \cdot) = dF \in \Omega^1(L).$$

A gradient flow line for $F$ is a map

$$u : I \to L, \quad \frac{d}{ds} u = -\text{grad}_F(u)$$

where $s$ is a coordinate on $I$. Given a time $s \in \mathbb{R}$ let $\phi_s : L \to L$ denote the time $s$ gradient flow of $F$. Denote by

$$\text{crit}(F) = \{ l \in L \mid dF(l) = 0 \}$$

the space of critical points of $F$. Taking the limit of the gradient flow determines a discontinuous map

$$L \to \text{crit}(F), \quad y \mapsto \lim_{s \to \pm \infty} \phi_s(y).$$

By the stable manifold theorem each $l \in \text{crit}(F) \subset L$ determines stable and unstable manifolds

$$S^\pm_l := \left\{ y \in L \mid \lim_{s \to \pm \infty} \phi_s(y) = l \right\} \subset L$$

consisting of points whose downward resp. upwards gradient flow converges to $x$. The pair $(F, G)$ is Morse-Smale if the intersections

$$S^+_l \cap S^-_k \subset L$$

are transverse for all $l, k \in \text{crit}(F)$. Let $J : TX \to TX$ be an almost complex structure on $X$. A metric tree is a tree equipped with a finite number (called length) for each finite edge and a root semiinfinite edge; the remaining semiinfinite edges are called leaves. A treed holomorphic disk is obtained from a metric tree by replacing each vertex by a disk with number of boundary markings given by the valence of the vertex, and also some interior markings. A treed holomorphic disks is stable if it is obtained from a stable tree by replacing each vertex with a stable disk. Thus
the valence of each vertex of the tree is at least three and each disk has at least three special boundary points or one special boundary point and one special interior point. An example of a moduli space of stable treed disks is shown in Figure 8.

![Figure 8. Moduli of stable treed disks with one interior marking](image)

In order to obtain Fukaya algebras with strict units we require weightings, similar to the construction of homotopy units in [39]. A weighted treed holomorphic disk is a treed holomorphic disks together with a subset of the leaves called weighted leaves and an assignment of an element of $[0, \infty]$ called a weight for each weighted leaves. (To jump ahead a bit, the moduli spaces of treed holomorphic disks are defined so that when the weight is infinity then forgetful maps exist, while for zero weight they do not.) A weighted holomorphic treed disk consists of a weighted treed disk $C = S \cup T$ together with a continuous map $u: C \rightarrow X$ such that

(a) $u$ maps the boundary $\partial S$ of the surface part $S \subset C$ and the tree part $T \subset C$ so the Lagrangian $L$, that is, $u(\partial S) \subset L$, $u(T) \subset L$.

(b) $u$ is holomorphic on the surface part $S$, that is, $\bar{\partial}f(u|_S) = 0$.

(c) $u$ is a gradient segment on each edge in the tree part $T$, that is, $\text{grad}_f(u|_T) = \partial_s(u|_T)$, where $s$ is a local coordinate.

A holomorphic treed disk is stable if it has only finitely many automorphisms, except for translational automorphisms of infinite length segments with one weighted end and one unweighted end. Two weighted holomorphic treed disks $u_0 : C_0 \rightarrow X, u_1 : C_1 \rightarrow X$ are equivalent if there exists an equivalence of $C_0$ and $C_1$ (a combination of isomorphisms of disks and collapsing zero length edges) intertwining the maps $u_0, u_1$, and forgetting constant gradient segments with one weighted and one unweighted end.
The moduli spaces break up into components depending on the asymptotic limit of the map along the semi-infinite edges. We denote by $l^*_{m}$ the maximum of the Morse function $F$, assumed unique. Introduce two new symbols denoted $l^*_{m}, l^0_{m}$.

$$\mathcal{I}(L) = \text{crit}(F), \quad \tilde{\mathcal{I}}(L) := \mathcal{I}(L) \cup \{l^*_{m}, l^0_{m}\}.$$ 

The extra symbols will be used to obtain an $A_\infty$ algebra with strict units. A labelling of the semi-infinite edges is a collection $l_0, \ldots, l_n \in \mathcal{I}(L)$ such that whenever an edge is weighted with weighting in $(0, \infty)$, then the edge is labelled $l^*_{m}$ or $l^0_{m}$ resp. $l^*_{m}$ or $l^0_{m}$. We denote by $\mathcal{M}_n(X, L, l)$ the moduli space of equivalence classes of stable weighted treed holomorphic disks with limits $l$:

$$\mathcal{M}_n(X, L, l) = \left\{ [u : C \to X] \in \mathcal{M}_n(X, L),\lim_{s \to \pm \infty} u(\phi_i(s)) = l_i \right\}$$

where $\phi_i : \pm(0, \infty) \to T$ are coordinates on the semi-infinite edges, and the sign is positive resp. negative if the edge is outgoing resp. incoming.

In order to achieve regularity for the moduli spaces of holomorphic disks with Lagrangian boundary conditions we use the stabilizing divisors framework introduced for Lagrangian submanifolds in [23]. We assume that the symplectic manifold is rational in the sense that the class $[\omega] \in H^2(X)$ is rational, or equivalently, if there exists a line bundle $\tilde{X} \to X$ with connection $\alpha$ such that $\text{curv}(\alpha) = (2\pi k/i) \omega$ for some integer $k > 0$. A Lagrangian $L \subset X$ is rational if there exists a line bundle $\tilde{X} \to X$ with curvature $\text{curv}(\alpha) = (2\pi k/i) \omega$ so that the restriction $\tilde{X}|L$ has a covariant constant section

$$\sigma : L \to \tilde{X}|_{L}, \quad \nabla_\alpha \sigma = 0.$$ 

A stabilizing divisor for $L$ is symplectic codimension two submanifold $D \subset X - L$ and $L$ is exact in $X - D$:

$$\exists \beta \in \Omega^1(X - D), \phi \in \Omega^0(L), \quad d\beta = \omega|_{X - D}, \quad d\beta|_{L} = d\phi.$$ 

Stabilizing divisors $D$ with

$$[D]^\vee = k[\omega] \in H^2(X, \mathbb{Q})$$

for $k$ sufficiently large exist for rational Lagrangians by Charest-Woodward [23] and Cieliebak-Mohnke [22]. In fact the results of [23] show that stabilizing divisors exist in a weak sense for any compact Lagrangian in a rational symplectic manifold; however here we will always assume that $L$ is rational.

Regularity for moduli spaces of holomorphic disks may be achieved using domain-dependent almost complex structures with additional markings mapping to the divisor. For each combinatorial type of weighted stable treed disk let $\mathcal{M}_\Gamma$ denote the corresponding moduli space. Let $U_\Gamma \to \mathcal{M}_\Gamma$ denote the universal stable weighted treed disk, whose fiber over an isomorphism class $[C]$ of stable treed disk is isomorphic to $C$. The decomposition $C = S \cup T$ into a surface part $S$ and a tree part $T$
induces a decomposition of the universal curve \( \mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma \) into surface and tree parts.

Associated to any operation (Collapsing edges, Making edge lengths or weights finite and non-zero, cutting edges of infinite length) resulting in any lower dimensional type \( \Gamma' \), there is a morphism of moduli spaces \( \mathcal{U}_{\Gamma'} \to \mathcal{U}_\Gamma \). In addition, if \( \Gamma = \Gamma_1 \sqcup \Gamma_2 \) is disconnected then there is a product expression \( \mathcal{M}_\Gamma = \mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2} \) and a homeomorphism \( \mathcal{U}_\Gamma \cong \pi_1^* \mathcal{U}_{\Gamma_1} \sqcup \pi_2^* \mathcal{U}_{\Gamma_2} \), where \( \pi_1, \pi_2 \) are the projections on the factors, see [24].

Our perturbation data will be certain maps on the universal spaces of stable weighted treed disks. Let \( \mathcal{J}(X) \) denote the space of \( \omega \)-tamed almost complex structures on \( X \), and \( \mathcal{G}(L) \) the space of Riemannian metrics on \( L \). A perturbation datum for \( L \) of type \( \Gamma \) is a datum \( P_\Gamma = (J_\Gamma, F_\Gamma, G_\Gamma) \), consisting of a domain-dependent almost complex structure \( J_\Gamma \) on \( X \) and a domain-dependent Morse function \( F_\Gamma \) and metric \( G_\Gamma \) on \( L \). A perturbation datum for \((X, L)\) consists of a divisor \( D \) and a collection of perturbation data \( P = (P_\Gamma) \) for each type \( \Gamma \) of stable disk.

Given such a perturbation datum we define the moduli spaces as follows. Recall that for any weighted treed marked disk \( C \) there is a stable weighted treed marked disk \( f(C) \) obtained by collapsing unstable tree components. If \( C \) has type \( \Gamma \) let \( f(\Gamma) \) denote the type of \( f(C) \). The stabilization map induces a map \( C \to \mathcal{U}_{f(\Gamma)} \) which is an immersion on stable components of the domain. Given a perturbation datum \( P = (P_\Gamma) \), in particular \( P_{f(\Gamma)} \), we obtain a perturbation system denoted \( P_\Gamma \) on \( C \) by pull-back, necessarily constant on the unstable components of the domain. An adapted stable (weighted treed) disk with boundary in \( L \) of type \( \Gamma \) consists of a stable weighted treed marked disk \( C \) and a \( J_\Gamma \)-holomorphic map \( C \to X \) with boundary in \( L \) such that each interior marking maps to \( D \), and each component of \( u^{-1}(D) \) contains an interior marking, and the limit for any weighted marking is \( l_m \). Let

\[
\mathcal{M}_n(X, L, D) = \bigcup_{\Gamma \in T_n} M_\Gamma(X, L, D)
\]

denote the moduli space of stable disks, where \( \Gamma \) ranges over types \( T_n \) with \( n \) semi-infinite edges. As before, \( \mathcal{M}_n(X, L, D, l) \) denote the subset of \( \mathcal{M}_n(X, L, D) \) with limits \( l = (l_0, \ldots, l_n) \) along the semi-infinite edges. The moduli space \( \mathcal{M}_n(X, L, D) \) is cut out locally as the zero set of a smooth map of Banach spaces, see [24]. For any \([u] \in \mathcal{M}_n(X, L, D)\) we denote by \( D_u \) the corresponding linearized Fredholm operator (c.f. 4.7 below) and call the index of \( D_u \) the expected dimension of \( \mathcal{M}_n(X, L, D) \) at \([u]\). Denote by \( \mathcal{M}_n(X, L, l)_d \) the subset of expected dimension \( d \).

In order to obtain moduli space with good compactness properties the perturbations for the various combinatorial types must be chosen coherently. A perturbation system \( P = (P_\Gamma) \) is coherent if it is

(a) (Compatible) compatible with the operations (Collapsing edges, Making edge lengths or weights finite and non-zero, cutting edges of infinite length) in the
sense that for any lower dimensional type \(\Gamma'\), the restriction of \(P_\Gamma\) to \(U_{\Gamma'}\) is obtained by pull-back from \(P_{\Gamma'}\).

(b) (Product form) on any disconnected type \(\Gamma = \Gamma_1 \cup \Gamma_2\) obtained by cutting an edge of infinite length, If \(P_{\Gamma_1}\) and \(P_{\Gamma_2}\) are regular perturbation data for the types \(\Gamma_1, \Gamma_2\), then one obtains regular perturbation data for type \(\Gamma\) by pulling back under the maps \(U_\Gamma = \pi_1^*U_{\Gamma_1} \cup \pi_2^*U_{\Gamma_2}\), where \(\pi_k : M_\Gamma \to M_{\Gamma_k}\) are the projections onto the components.

(c) (Forgetting tails with infinite weighting) Suppose that \(\Gamma\) is a type with semi-infinite edge labelled by the degree zero element \(l_m \in I(L)\). Let \(\Gamma'\) denote the edge obtained by forgetting the \(i\)-th semi-infinite edge and stabilizing. We obtain perturbation data \(P_{\Gamma}\) for type \(\Gamma\) by pull-back of \(P_{\Gamma'}\), and \(P_{\Gamma}\) is regular if \(P_{\Gamma'}\) is. We choose a base almost complex structure and metric so that no holomorphic spheres are contained in the divisor and the unstable and stable manifolds for the Morse function on the Lagrangian meet transversally.

A stabilizing property for the almost complex structures ensures that the moduli spaces are compact. Choose a domain-independent almost complex structure

\[ J_D \in \mathcal{J}(X) \]

with the property that the hypersurface contains no holomorphic spheres of any energy:

\[ (u : \mathbb{P}^1 \to D, \quad \bar{\partial}_J(u) = 0) \implies du = 0. \]

By [27, Proposition 8.14], see also [23, Lemma 8.4], for each energy \(E > 0\) there exists a neighborhood

\[ \mathcal{J}^*(J_D, E, D) \subset \mathcal{J}(X) \]

of almost complex structures preserving \(D\) taming \(\omega\) so that if \(J' \in \mathcal{J}^*(J_D, E, D)\) then there are no non-constant \(J'\)-holomorphic spheres in \(D\):

\[ (u : \mathbb{P}^1 \to D, \quad \bar{\partial}_J(u) = 0, \quad E(u) < E) \implies du = 0. \]

Absence of non-constant holomorphic spheres in the stabilizing divisor is guaranteed as follows. For any component \(C_i\) of \(C\) at infinite distance from all other disk components, let \(\Gamma_i\) is the subtree of \(\Gamma\) corresponding to \(C_i\) with a single vertex and \(n(\Gamma_i)\) the number of markings on \(C_i\). Recall from (19) that \(k\) represents the multiple of \([\omega]\) representing \([D]^{\lor} \in H^2(X)\). Thus any disk of energy at least \(E\) has at least \(n(\Gamma_i)/k\) intersection points with \(D\) on \(C_i\), counted with multiplicity. We say that a domain-dependent almost complex structure \(J_{\Gamma}\) is stabilizing if \(J_{\Gamma}\) takes values in \(\mathcal{J}^*(J_D, n(\Gamma_i)/k, D)\):

\[ J_{\Gamma}(z) \in \mathcal{J}^*(J_D, n(\Gamma_i)/k, D), \quad \forall z \in C_i. \]

This guarantees the absence of non-constant \(J_{\Gamma}(z)\)-holomorphic spheres for any \(z \in C_i\):

\[ ((u : \mathbb{P}^1 \to D, \quad \bar{\partial}_{J_{\Gamma}(z)}(u) = 0) \implies du = 0) \forall z \in C_i. \]

We state the compactness and regularity properties of the moduli spaces in the theorem below. We remark that one cannot achieve moduli spaces of expected dimension for all strata of the moduli spaces because of the following problem: Recall that a *ghost bubble* is a component of the domain on which the map is constant. If
a configuration has a marking on a ghost bubble, then by adding more markings on
the same bubble one obtains a configuration that lies in a stratum of the moduli
space of negative expected dimension; this problem is closely related to the problem
that in higher genus the moduli spaces of constant maps are not expected dimension.
An adapted stable map is **uncrowded** if each maximal ghost bubble contains at most
one marking:

\[(C' \subset C \text{ connected, } \text{du}|_{C'} = 0) \implies \#\{z_i \in C'\} \leq 1.\]

A perturbation datum is **regular** if each uncrowded moduli space of expected dimen-
sion at most one is cut out transversally.

**Theorem 3.1.** [24] Let \( \Gamma \) be an uncrowded type of stable treed disk of expected di-

mension at most one and suppose that regular, stabilizing perturbation data \( P_{\Gamma'} \)

have been chosen for all boundary strata \( U_{\Gamma'} \subset U_{\Gamma} \) are regular. There exists a comeager

subset of the space of regular perturbation data \( P_{\Gamma} \) equal to the given perturbation
data on lower-dimensional strata and sufficiently close to the base almost complex
structure and metric such that

(a) (Transversality) Every element of \( M_{\Gamma}(X, L, D) \) is regular.

(b) (Compactness) The closure \( \overline{M}_{\Gamma}(X, L, D) \) is compact and contained in the

uncrowded, stable locus.

(c) (Tubular neighborhoods) If \( M_{\Gamma}(X, L, D) \) has expected dimension zero then it

has a tubular neighborhood in any adjoining uncrowded strata \( M_{\Gamma'}(X, L, D) \)

of one higher dimension.

(d) (Orientations) The uncrowded strata \( M_{\Gamma}(X, L, D) \) of formal dimension one

or two are equipped with orientations satisfying universal gluing signs for

inclusions of boundary strata, see [24]; in particular we denote by \( \epsilon([u]) \in \{\pm 1\} \)
the orientation sign associated to the zero-dimensional moduli spaces.

(e) (One-leaf forgetful morphisms) among the regular perturbations there exist a

comeager subset that are divisorial, and for these there exist forgetful mor-

phisms \( M_{\Gamma}(X, L, D) \rightarrow M_{f(\Gamma)}(X, L, D) \) where \( \Gamma \) is any type with a single

leaf and \( f(\Gamma) \) the type with no leaves obtained by forgetting the leaf.

Lagrangian Floer homology with gradings is defined for compact spin (or pin)

Lagrangians in symplectic manifolds with certain additional structure which we now

introduce. Let \( X \) be a compact symplectic manifold. Let \( \text{Lag}(X) \) denote the fiber

bundle over \( X \) whose fiber \( \text{Lag}(X)_x \) at \( x \) consists of Lagrangian subspaces of \( T_xX \).

Let \( g \) be an even integer. A **Maslov cover** is an \( g \)-fold cover \( \text{Lag}^g(X) \rightarrow \text{Lag}(X) \)
such that the induced two-fold cover \( \text{Lag}^2(X) := \text{Lag}^g(X)/\mathbb{Z}_g/2 \rightarrow \text{Lag}(X) \) is the

oriented double cover. A **background class** is an element \( b \in H_2(X, \mathbb{Z}_2) \). The tuple

\( (X, \text{Lag}^g(X), b) \) will be called a **symplectic background**.

The space of Floer cochains is defined as follows. Given a symplectic background

a Lagrangian brane is a compact oriented Lagrangian with the following additional

structure. A **grading** on \( L \) is a lift of the canonical map \( L \rightarrow \text{Lag}(X), l \mapsto T_lL \)
to \( \text{Lag}^g(X) \). A **relative spin structure** for an oriented Lagrangian \( L \subset X \) is a lift

of the \( \check{\text{C}} \)ech class of its tangent bundle to relative non-abelian cohomology with
values in Spin relative to the map \( L \to X \), with cohomology class equal to the given background class \( b \in H_2(X) \).

The Floer cohomology is in general defined over various formal power series rings called *Novikov rings*. Let \( q \) be a formal variable and \( \Lambda \) the *universal Novikov field* of formal sums with rational coefficients

\[
\Lambda = \left\{ \sum_i c_i q^{\rho_i}, \quad c_i \in \mathbb{Q}, \quad \rho_i \in \mathbb{R}, \quad \rho_i \to \infty \right\}
\]

Let \( \Lambda_{\geq 0} \subset \Lambda \) denote the subset with only non-negative powers of \( q \):

\[
\Lambda_{\geq 0} = \left\{ \sum_i c_i q^{\rho_i}, \quad c_i \in \mathbb{Q}, \quad \rho_i \in [0, \infty), \quad \rho_i \to \infty \right\}
\]

Let \( \Lambda^\times \) denote the subset of \( \Lambda_{\geq 0} \) with leading coefficient in \( \mathbb{C}^\times \):

\[
\Lambda^\times = \left\{ c_0 + \sum_{i \geq 0} c_i q^{\rho_i}, \quad c_0 \neq 0, c_i \in \mathbb{Q}, \quad \rho_i \in (0, \infty), \quad \rho_i \to \infty \right\}
\]

Then \( \Lambda \) is a field, while \( \Lambda_{\geq 0} \) is a ring and \( \Lambda^\times \) is a group.

The structure coefficients of the Fukaya algebra also depend on a choice of local system on the Lagrangian. Any stable disk \( u : (C, \partial) \to (X, L) \) defines, by restriction to the boundary, an element \([\partial u] \in \pi_1(L)\). Given a local system \( y \in \mathcal{R}(L) := \text{Hom}(\pi_1(L), \Lambda^\times) \) and an isomorphism class of treed stable disk \([u : C \to X]\) denote by

\[
y([\partial u]) \in \Lambda^\times
\]

the evaluation of \( y \) on the class of the boundary \([\partial u] \in \pi_1(L)\) (since \( \Lambda^\times \) is abelian, this is independent of the choice of the base point.) A *brane structure* on \( L \) is a relative spin structure with the given background class, a local system, and a grading for the given Maslov cover. A *Lagrangian brane* is a compact oriented Lagrangian submanifold equipped with a brane structure.

Floer cochains will be defined as formal combinations of critical points of the Morse function, with additional generators added to achieve strict units. Let \( L \) be an admissible Lagrangian brane. Define the space of Floer cochains

\[
CF(X, L) = \bigoplus_{x \in \mathcal{I}(L)} \Lambda <x>, \quad \widehat{CF}(X, L) = \bigoplus_{x \in \mathcal{I}(L)} \Lambda <x>.
\]

Using the regularized moduli spaces of holomorphic disks we define higher composition maps as follows. Define

\[
\mu^n : \widehat{CF}(X, L)^{\otimes n} \to \widehat{CF}(X, L)
\]

on generators by

\[
\mu^n(<l_1>, \ldots, <l_n>) = \sum_{l_0, [u] \in \mathcal{M}_n(X, L, D, l_0 u)} (-1)^{\hat{\omega}([l_0])} y([\partial u])q^{E([u])} \epsilon([u]) <l_0>
\]
where \( \bigodot = \sum_{i=1}^{n} i |l_i| \) the evaluation \( y([u]) \) is the local system evaluated on the boundary of \( u \) as in (20), and \( E([u]) \) is the sum of the energies (or equivalently symplectic areas) of the holomorphic disks. If we wish to emphasize the dependence on \( y \), we write \( \mu^n_y \) for \( \mu^n \).

**Theorem 3.2.** Let \( L \) be a rational Lagrangian brane in a symplectic background \( X \).

(a) (Fukaya algebras for Lagrangians via stabilizing divisors \([24]\)) For any coherent regular stabilizing divisorial perturbation system \( P = (P_\tau) \) the maps \( (\mu^n)_{n \geq 0} \) satisfy the axioms of a (possibly curved) \( A_\infty \) algebra \( \hat{CF}(X, L) \) with strict unit and weak divisor axiom. The subspace \( CF(X, L) \) is a subalgebra without unit, and an \( A_\infty \) bimodule over \( \hat{CF}(X, L) \). The homotopy type of \( \hat{CF}(X, L) \) is independent of all choices.

(b) (Floer cohomology for Lagrangians via stabilizing divisors) For perturbations sufficiently \( C^2 \)-close to the base almost complex structure and metric, the Fukaya algebra \( \hat{CF}(X, L) \) is a convergent \( A_\infty \) algebra in the sense of \([24]\). Define \( MC(L) = MC(\hat{CF}(X, L)) \) as in (18) and for any \( b \in MC(L) \) define the Floer cohomology \( HF(X, L, b) := H(\mu^1_b) \). Non-vanishing of the Floer cohomology is independent of the choice of perturbations.

### 3.3. Spectral sequence and divisor equation.

Various computational techniques for Floer cohomology involve the Oh spectral sequence, induced by the energy filtration on holomorphic disks, and a version of the divisor equation for insertions of codimension one cocycles; combining these techniques leads to a well-known procedure for computing the Floer cohomology of Lagrangian tori via critical points of potentials, see for example Fukaya-Oh-Ohta-Ono \([40]\).

**Remark 3.3.** (a) (Oh spectral sequence) In the case that the Floer cohomology is weakly unobstructed there is a Morse-to-Floer spectral sequence which relates the Floer cohomology with the Morse cohomology, discussed in the monotone case in Oh \([87]\). Let

\[
\mu^{0,n} : CF(X, L)^{\otimes n} \to CF(X, L)
\]

denote the \( q^0 \) terms in the higher composition map \( \mu^n \). These are, by construction, the higher composition maps in the Morse \( A_\infty \) algebra for \( L \). In particular, the cohomology of \( \mu^{1,0} \) is the Morse cohomology and so isomorphic to the singular cohomology of \( L \):

\[
H(\mu^{1,0}) \cong H(L).
\]

Consider the \( \mathbb{R} \)-filtration by \( q \)-degree on \( CF(X, L, \Lambda_{\geq 0}) \), induced by the filtration on \( \Lambda_{\geq 0} \):

\[
CF(X, L, \Lambda_{\geq 0}) = \bigcup_{\rho > 0} q^\rho CF(X, L, \Lambda_{\geq 0}).
\]
Define an associated $\mathbb{Z}$-filtration induced by $q^k \mathcal{E}$, where $E < h$

$$\mathcal{C}(X, L, \Lambda_{\geq 0}) = \bigcup_{k \geq 0} q^k \mathcal{E}(X, L, \Lambda_{\geq 0}).$$

Configurations of zero energy and a single leaf contain no disks: any disk would have to have two incoming edges, and following the edge not connecting the leaf to the root one would eventually have to arrive at a constant disk with only a single outgoing leaf, necessarily unstable, which gives a contradiction. It follows that the differential on the associated graded

$$\text{Gr}(\mathcal{C}(X, L, \Lambda_{\geq 0})) = \bigcup_{k \geq 0} q^k \mathcal{E}(X, L, \Lambda_{\geq 0}) / q^{k+1} \mathcal{E}(X, L, \Lambda_{\geq 0})$$

is the associated graded of the Morse differential $\mu^{1,0}$:

$$\text{Gr}(\mu^1) = \text{Gr}(\mu^{1,0})$$

The filtration leads to a spectral sequence $E^r$ with first page equal to the Morse homology

$$E^1 = H(\mu^{1,0}, \Lambda_{\geq 0})$$

and such that the spectral sequences converges to the Floer cohomology:

$$E^r \Rightarrow \mathcal{H}(X, L, \Lambda_{\geq 0}).$$

Convergence of the spectral sequence is discussed in in Fukaya-Oh-Ohta-Ono [39, Theorem 16.3.28]. The two assumptions needed are that $\mathcal{C}(X, L)$ is finite (which follows since $L$ is compact) and the differential is “gapped” in the language of [39], which follows from energy quantization for disks.

(b) (Divisor equation) The moduli spaces do not admit forgetful morphisms in general, hence the divisor axiom is not satisfied. However, there is a weak version given as follows. For any type $\Gamma$ of holomorphic treed disk of expected dimension 0 with one incoming and one outgoing edge, for generic perturbation data $P_\Gamma = (J_\Gamma, F_\Gamma, G_\Gamma)$ that is equal to the base metric and Morse function $(F, G)$ on the tree part, that is,

$$(F_\Gamma, G_\Gamma)|_T = (F, G)$$

the moduli space $\mathcal{M}_\Gamma(X, L)$ is smooth of expected dimension. Indeed, one can take the perturbations on the incoming and outgoing edges to be trivial. The moduli spaces not involving holomorphic disks are already transverse, by the Morse-Smale assumption, while the moduli spaces involving non-trivial holomorphic disks are transverse for a comeager subset of perturbation data by a standard Sard-Smale argument.

We say that a divisor cycle is a Morse cocycle $c \in \mathcal{C}(X, L)$ of degree one:

$$c = \sum c_x <x> \in \ker(\mu_{1,0}) \subset \mathcal{C}^1(X, L).$$

For any Morse cocycle $c$ let

$$S^+(c) = \sum c_x S^+(x) \in C^1(L)$$
denote the cycle given as the union of unstable manifolds for components of \( c \) and
\[
[S^+(c)] \in H^1(L)
\]
the corresponding cohomology class.

We say that a perturbation datum \( P_{\Gamma} \) for types with a single incoming edge is divisorial if \( F_{\Gamma} \) is equal to \( F \) on the incoming edge and \((J_{\Gamma}, F_{\Gamma}, G_{\Gamma})\) is pulled back from the type \( f(\Gamma) \) obtained by forgetting tail. For divisorial perturbations we have a forgetful map
\[
(21) \quad f_{\Gamma}' : \mathcal{M}_{\Gamma}(X, L, D, l_1, l_0) \to \mathcal{M}_{\Gamma'}(X, L, D, l_0)
\]
obtained by forgetting the map on the incoming edge. The fiber over an element \([u : C \to X] \in \mathcal{M}_{\Gamma'}(l_0)\) is the set of points in the boundary \( \partial C \) mapping to \( S^+(l_1) \),
\[
(f_{\Gamma}')^{-1}([u]) \cong (u|\partial C)^{-1}(S^+(l_1)).
\]
If the perturbation data for types with a single incoming edge are divisorial then the structure coefficients satisfy a version of the divisor equation: Note that the tangent space to the space \( \mathcal{R}(L) \) of local systems is \( H^1(L, \Lambda_{\geq 0}) \). For \( c \in H^1(L, \Lambda_{\geq 0}) \) we denote by \( \partial_c \mu^0_y \) the derivative of \( \mu^0_y \) in the direction of \( c \). Then
\[
(22) \quad \mu^1_y(c) = \partial_c \mu^0_y = \sum_{u \in \mathcal{M}_1(X, L, D, l_0)} q^{A(u)} \epsilon(u) \sigma(u)([\partial u], [S^+(c)]) y([\partial u]).
\]
Indeed the forgetful map preserves the orientations, holonomies, and areas of the disks, hence the claim.

To see that one can choose the perturbations \( P_{\Gamma} \) for a single incoming edge to be divisorial, note that the pair \((F, G)\) is already assumed to be Morse-Smale. Thus transversality for configurations with no disk components are already regular without perturbation. Next consider the space of isomorphism classes of pairs
\[
\mathcal{M}_{\Gamma}(l_0; l_1) = \{[u : C \to X, P_{f(\Gamma)}], z] \}
\]
where \( C \) is a treed holomorphic disk of type \( f(\Gamma) \), \( P_{f(\Gamma)} \) is a perturbation datum for type \( f(\Gamma) \), \( z \in \partial S \) is a point on the boundary distinct from the tree part \( T \), and \( z \) maps to \( S^+(l_1) \) under \( u \). An argument similar to that of Theorem 3.1 implies that for a comeager set of perturbations \( P_{f(\Gamma)} \), the moduli space \( \mathcal{M}_{\Gamma}(l_0; l_1) \) is regular of expected dimension. Indeed perturbations of the map which vanish at the marking \( z \) are enough to force an element \( \eta \) in the cokernel of the linearized operator to satisfy \( D_u^* \eta = 0 \). Then the pullback \( P_{\Gamma} \) of \( P_{f(\Gamma)} \) by the forgetful map is divisorial. More complicated perturbations are necessary for more than one leaf because configurations with constant disks correspond to intersections of the unstable manifolds of the unperturbed Morse function, which are not transversally intersecting for obvious reasons (e.g. they may be equal.)
(c) (Disk potentials) In our set-up the disk potential is the coefficient of the identity in the curvature, defined on the space of local systems on the Lagrangian brane. (If one had a true divisor equation, one would obtain an equivalent function on the space of solutions to the weak Maurer-Cartan equation in degree one.) Let

$$\mathcal{R}(L) := \text{Hom}(\pi_1(L), \Lambda^\infty)$$

denote the space of local systems on $L$. We suppose we are given $b \in \widehat{CF}(L)$ such that $b$ is a weak solution of the Maurer-Cartan equation for every $y \in \mathcal{R}(L)$. Set

$$W_b : \mathcal{R}_b(L) \to \Lambda_{\geq 0}, \quad W_b(y)|^+_L = \mu^0_{b,y}(1)$$

where $\mu^0_{b,y}(1)$ is the zeroth higher composition map for the Lagrangian brane with boundary deformation $b$ and local system $y$. For Lagrangian tori with $\mathcal{R}_b(L) = \mathcal{R}(L)$, the critical values of the potential $W_b$ determine the non-vanishing of the associated Floer cohomology: Let $l_1 \in \ker(\mu^1)$ be a cocycle in the first page of the Morse-to-Floer spectral sequence and $[S^+(l_1)]$ the homology class of the unstable manifolds of the critical points appearing in $l_1$. Then

$$\mu^1(l_1) = \sum_{u \in M_1(l_0)} q^A(u) \epsilon(u) \sigma(u)([\partial u], [S^+(l_1)])(y([\partial u])) = \partial_{[l_1]} \mu^0(1).$$

In particular, suppose that for some $y \in \mathcal{R}(L)$, for all $b \in H^1(L, \Lambda_{\geq 0})$, $\partial_b W_0(y) = 0$, and the classical cohomology is generated by $H^1(L)$. Then the Floer cohomology is the ordinary cohomology,

$$H(\mu^1_0, \Lambda_{\geq 0}) = H(L, \Lambda_{\geq 0}),$$

by the following argument. In the first page of the Morse-to-Floer spectral sequence we show by induction that $\mu^1(c) = 0$ for all classes $c \in H(L)$. As in Fukaya-Oh-Ohta-Ono [40, Lemma 13.1] we write

$$\mu^1(c) = \sum_{\beta} q^\beta \mu^{1,\beta}(c)$$

for some rational numbers $\mu^{1,\beta}$. Then comparing terms with coefficient $\beta$ in the $A_\infty$ associativity relation we obtain

$$\mu^{1,\beta}(\mu^{2,0}(c_1, c_2)) = \sum_{\beta_1 + \beta_2 = \beta} \pm \mu^{2,\beta_1}(\mu^{1,\beta_2}(c_1), c_2)$$

$$+ \sum_{\beta_1 + \beta_2 = \beta} \pm \mu^{2,\beta_1}(c_1, \mu^{1,\beta}(c_2)) + \sum_{\beta_1 + \beta_2 = \beta, \beta_2 > 0} \pm \mu^{1,\beta_1}(\mu^{2,\beta_2}(c_1, c_2)).$$

The first two terms on the right vanish by the inductive hypothesis, since $\max(\beta_1, \beta_2) < \beta$, while the last term vanishes by the inductive hypothesis as well since $\beta_1 < \beta$. 
It follows that if for some $y \in \mathcal{R}(L)$
\[ \forall c \in H^1(L, \Lambda^\times), \quad \partial_c W_b(y) = 0 \]
and the classical cohomology is free and generated by $H^1(L)$, then the Floer cohomology
\[ H(\mu_{b,y}^1, \Lambda) = H(L, \Lambda) \]
is non-trivial. Indeed, by the above discussion $H(\mu_{b,y}^1, \Lambda_{\geq 0}) = H(L, \Lambda_{\geq 0})$. The result for the Novikov field $\Lambda$ follows from the universal coefficient theorem.

(d) (Invariance under perturbation) We say that a Lagrangian torus $L$ is non-degenerate if the leading order part $W_0$ of the potential $W$ (that is the sum of terms with lowest $q$-degree) has a non-degenerate critical point at some element $c \in \mathcal{R}(L)$. Existence of a non-degenerate critical point is invariant under perturbation by Fukaya-Oh-Ohta-Ono [40, Theorem 10.4], and in particular there exists a bijection between the set $\text{Crit}(W_0)$ of critical points of $W_0$ and critical points $\text{Crit}(W)$ of $W$. This statement is a version of the formal implicit function theorem adapted to the setting Novikov rings with real coefficients; essentially one may, given a critical point $y_0$ of $W_0$ one may solve for a critical point $y$ of $W$ order by order, using non-degeneracy of the Hessian of $W$ at $y_0$; and conversely any critical point of $W$ determines a critical point of $W_0$. By the previous paragraph, existence of a non-degenerate critical point implies that $L$ has non-trivial Floer cohomology.

Example 3.4. (Potential for the projective line) Suppose that $X = S^2$ with area $A$ and $L \cong S^1$ separates $X$ into pieces of areas $A_1, A_2$. Thus the primitive disks have areas $A_1, A_2$ with opposite boundary homotopy classes and the potential is
\[ W(y) = q^{A_1}y + q^{A_2}/y. \]
The Floer differential vanishes iff
\[ 0 = y\partial W/\partial y = q^{A_1}y - q^{A_2}/y \]
for some $y$. The equation $y^2 = q^{A_2-A_1}$ has a solution in $\mathcal{R}(L) \cong \Lambda^\times$ iff $A_1 = A_2$. Thus only in this case (the case that the Lagrangian is Hamiltonian isotopic to the equator) one has non-trivial Floer cohomology.

4. Broken Fukaya algebras

In this section we introduce a version of the Fukaya algebra for Lagrangians in broken symplectic manifolds. What this means is that the symplectic manifold is degenerated, through neck stretching, to a broken symplectic manifold; this is a special case of symplectic field theory as introduced by Eliashberg-Givental-Hofer [34]. A sequence of pseudoholomorphic curves with respect to the degenerating almost complex structure converges to a broken pseudoholomorphic map: a collection of pseudoholomorphic curves in the pieces as well as a maps to the neck region. In the version that we consider here introduced by Bourgeois [20], these components are connected by gradient trajectories of possibly finite or infinite length; the limit is explained in the following section.
4.1. Broken curves. First we describe the kind of domains that appear in the particular kind of broken limit we will consider, following Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [19]. The domains are nodal curves, except that the nodes are replaced with line segments and additionally segments are attached to markings on the boundary.

**Definition 4.1.** Let $n, m, s \geq 0$ be integers.

(a) (Level) A *level of a broken curve* with $n$ boundary markings, $m$ interior markings, and $s$ sublevels consists of

- (i) a sequence $C_1, \ldots, C_s$ of treed nodal curves with boundary, called *sublevels*; in our situation only the first piece $C_1$ is allowed to have non-empty boundary;
- (ii) interior markings $z^\pm_i \subset C_i$ for $i = 1, \ldots, s$;
- (iii) a collection of *finite and semi-infinite edges attached to boundary points* intervals connecting boundary points in components of $C_i$ for $i = 1, \ldots, s$; since in our situation only $C_1$ is allowed to have non-empty boundary, these edges only connecting components of $C_1$; and
- (iv) a sequence of *possibly broken* intervals $I_0, \ldots, I_{k_+}$ attached to interior points in $C_i$ with lengths $\lambda_{0,1}, \ldots, \lambda_{s,1}$ (that is, each collections $I_i$ has lengths $\lambda_{j,1}, \lambda_{j,2}, \ldots$), where $\lambda_{0,k}$ and $\lambda_{s,k}$ are required to be infinite for all $k$.

Out of the data above one constructs a topological space $C$ by removing the nodes and gluing in the intervals.

(b) (Broken curve) A *broken curve* with $k$ levels is obtained from $k$ curves at a single level $C_1, \ldots, C_k$, where the first $C_1$ has only outgoing edges and the last only incoming edges, by gluing together the endpoints at infinity. The *combinatorial type* of a broken curve $C$ is the graph $\Gamma$ whose

- (i) vertices $\text{Vert}(\Gamma)$ are irreducible components of the level curves $C_i$, together with the segments connecting the levels, and

![Figure 9. A broken disk](image-url)
(ii) edges $\text{Edge}(\Gamma)$ are boundary or interior nodes or joining points of the segments or incoming or outgoing markings or boundary markings. A broken curve is stable if the marked curve obtained by collapsing the intervals is stable.

(c) (Broken disk) A broken disk is defined similarly to a broken curve but the first level $C_1$ is a disjoint union of treed disk and sphere components with segments attached, while the other pieces have surface parts that are spheres; furthermore, the combinatorial type is a tree.

Thus each broken disk $C$ is the union of a surface part $S$, a tree part $T$ which further decomposes into segments $T_b$ connecting boundary components of $C$ and segments $T_i$ connecting interior points of $C$.

Let $\mathcal{M}_{n,m,k}$ denote the moduli space of stable weighted treed broken disks with $n$ boundary markings, $m$ interior markings and $k$ levels. The space $\mathcal{M}_{n,m,k}$ is naturally a Hausdorff stratified space with a stratification by combinatorial type

$$\mathcal{M}_{n,m,k} = \bigcup_{\Gamma \in \mathcal{T}_{n,m,k}} \mathcal{M}_{\Gamma}$$

where $\mathcal{T}_{n,m,k}$ denotes the set of possible combinatorial types. Each stratum fibers over a stratum in the moduli space of curves with boundary without tree parts, with fibers given by the possible assignments of lengths corresponding to nodes connecting curves at different levels. Note that the curve part of any broken curve has two kinds of nodes, those joining sublevels and those joining levels.

For each combinatorial type $\Gamma$ of broken curve we denote by $\mathcal{M}_\Gamma$ the closure of the moduli space $\mathcal{M}_\Gamma$ and by $\mathcal{U}_\Gamma$ the universal broken (weighted, treed, stable) disk, whose fiber over $[C] \in \mathcal{M}_\Gamma$ is isomorphic to $C$. We have a decomposition into curve and tree parts

$$\mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma$$

and a further decomposition of the tree parts into parts connected to the boundary of disk and those parts connecting to the interior of the curve:

$$\mathcal{T}_\Gamma = \mathcal{T}^b_\Gamma \cup \mathcal{T}^i_\Gamma.$$

4.2. Broken maps. A broken map is a map from a broken curve into a broken symplectic manifold, defined as follows.

**Definition 4.2.** (a) (Broken symplectic manifold) Let $X$ be a compact rational symplectic manifold. Let $Z \subset X$ be a coisotropic hypersurface separating $X \setminus Z$ into components with cylindrical ends $X^+ \subset X^-$. Suppose that $Z$ has null foliation which is a circle fibration over a symplectic manifold $Y = Z/U(1)$. By a construction of Lerman [66] the unions

$$X_\pm := X^\pm \cup Y$$

have the structure of symplectic submanifolds. Let $N_\pm \to Y$ denote the normal bundle of $Y$ in $X_\pm$, and $N_\pm \oplus \mathbb{C}$ the sums with the trivial bundle. Denote

$$\mathbb{P}(N_+ \oplus \mathbb{C}) \cong \mathbb{P}(N_- \oplus \mathbb{C})$$
the projectivized normal bundle, where the isomorphism is induced from
\[ \mathbb{P}(N_+ \oplus \mathbb{C}) \cong \mathbb{P}((N_+ \oplus \mathbb{C}) \otimes N_-) = \mathbb{P}(N_+ \oplus N_-). \]

The broken symplectic manifold arising from the triple \((X_-, X_+, Y)\) is the topological space \(X = X_- \cup_Y X_+\).

(b) (Multiply broken symplectic manifold) For an integer \(m \geq 1\) define the \(m-1\)-broken symplectic manifold
\[ X[m] = X_- \cup_Y \mathbb{P}(N_+ \oplus \mathbb{C}) \cup_Y \mathbb{P}(N_+ \oplus \mathbb{C}) \cup_Y \ldots \cup_Y X_+, \]
where there are \(m-2\) copies of \(\mathbb{P}(N_+ \oplus \mathbb{C})\) called broken levels. Define
\[ X[m]_0 = X_- \quad X[m]_1 = \mathbb{P}(N_+ \oplus \mathbb{C}) \quad \ldots \quad X[m]_m = X_+. \]

There is a natural action of the complex torus \((\mathbb{C}^\times)^{l-2}\) on \(\mathbb{P}(N_+ \oplus \mathbb{C})\) given by scalar multiplication on each projectivized normal bundle:
\[ \mathbb{C}^\times \times \mathbb{P}(N_+ \oplus \mathbb{C}) \to \mathbb{P}(N_+ \oplus \mathbb{C}), \quad (z, [n, w]) \mapsto z[n, w] := [zn, w]. \]

The fixed points of the \(\mathbb{C}^\times\) action are the divisors at 0 and \(\infty\):
\[ \mathbb{P}(N_+ \oplus \mathbb{C})^{\mathbb{C}^\times} = \{[n, 0] \cup \{0, w\}\} \]
where \(n\) reps. \(w\) ranges over vectors in \(N_+\) resp. \(Y \times \mathbb{C}\).

(c) An almost complex structure \(J \in \mathcal{J}(\mathbb{R} \times Z)\) is of cylindrical form if there exists an almost complex structure \(J_Y\) on \(Y\) such that the projection \(\pi_Y: \mathbb{R} \times Z \to Y\) is almost complex and \(J\) is invariant under the \(\mathbb{C}^\times\)-action on \(\mathbb{R} \times Z\) induced from the embedding in \(\mathbb{P}(N_+ \oplus \mathbb{C})\) given by
\[ \mathbb{C}^\times \times \mathbb{P}(N_+ \oplus \mathbb{C}) \to \mathbb{P}(N_+ \oplus \mathbb{C}), \quad s \exp(it)(s_0, z) = (s_0 + s, \exp(it)z). \]

That is,
\[ D\pi_Y J = J_Y D\pi_Y, \quad J \in \mathcal{J}(\mathbb{R} \times Z)^{\mathbb{C}^\times}. \]

We denote by \(\mathcal{J}^{\text{cyl}}(\mathbb{R} \times Z)\) the space of tamed almost complex structures of cylindrical form, and by
\[ \mathcal{J}(X) = \mathcal{J}(X_-^\circ) \times \mathcal{J}^{\text{cyl}}(\mathbb{R} \times Z) \mathcal{J}(X_+^\circ) \]
the fiber product consisting of tamed almost complex structures of cylindrical form on the ends. (Note that this definition differs from the one in [19], which does not suffice for our purposes.)

We now define broken maps. We first treat the unperturbed case.

**Definition 4.3.** (Broken maps) Let \(X\) be a broken symplectic manifold as above, and \(L \subset X_-\) a Lagrangian disjoint from \(Y\). Let \(J \in \mathcal{J}(X)\) be an almost complex structure on \(X\) of cylindrical form, \(H\) be a Morse function on \(Y\) and \((F, G)\) a Morse-Smale pair on \(L\). A broken map to \(X\) with boundary values in \(L\) consists of:

(a) (Broken curve) a broken curve \(C = (C_0, \ldots, C_l)\);
(b) (Broken map) a map \(u: C \to X\), that is, collection of maps \(C_k\) to \(X[p]_k\) (notation from (24)) for \(k = 0, \ldots, p\) satisfying the following non-linear partial differential equations:
(i) (Pseudoholomorphicity) On the two-dimensional part $S \subset C$, the map $u$ is $J$-holomorphic with respect the identification of $C$ with a fiber of the universal curve;

(ii) (Gradient flow in the Lagrangian) On the one-dimensional part $T_0 \subset C$ connecting boundary nodes, $u$ is a segment of a gradient trajectory on each interval component $I_k \subset T_0$ for the Morse function $F$ on $L$;

(iii) (Intersection multiplicity) If a pseudoholomorphic map $u : C \to X$ has isolated intersections with an almost complex codimension two submanifold $Y \subset X$ then at each point $z \in u^{-1}(Y)$ there is a positive intersection multiplicity $s(u, z) \in \mathbb{Z}_{>0}$ describing the winding number of a small loop counterclockwise around $Y$:

$$s(u, z) = \exp(2\pi i \theta) \mapsto u(z + r \exp(2\pi i \theta)) \in \pi_1(U - (U \cap Y)) \cong \mathbb{Z}$$

where $U$ is a contractible open neighborhood of $z$ and $r$ is sufficiently small so that $u(z + r \exp(2\pi i \theta)) \in U$ for all $\theta \in [0, 1]$.

(iv) (Gradient flow in the manifold) On the one-dimensional part $T_i \subset C$ connecting interior nodes, $u$ is a segment of a gradient trajectory on each interval component $I_k \subset T_i$ for the Morse function $H$ on $Y$;

and satisfying the following:

(i) (Matching condition) For the nodes of the domain on either end of a Morse trajectory, the intersection multiplicities $s_{i,j}^\pm$ with the hypersurface $Y$ are equal.

We denote by $\ell(s)$ the number of contact points with the divisor $Y$, that is, the number of multiplicities $s_{i,j}$.

(c) (Isomorphisms of broken maps) An isomorphism between broken curves $u_i : C_i \to X[k], i = 0, 1$ is an automorphism of the domain $\phi : C_0 \to C_1$ together with an element $g \in (\mathbb{C}^\times)^{k-1}$ such that $u_1 \circ \phi = gu_0$, and the automorphism is trivial on any infinite segment with one weighted end and one unweighted end. A broken map is stable if it has only finitely many automorphisms, except for automorphisms of infinite length segments with one weighted end and one unweighted end. This means in particular at least one component at each level is not a trivial cylinder.

The combinatorial type of a broken map is the combinatorial type of the underlying curve, but with the additional data of the homotopy class of each component (as a labelling of the vertices) and the intersection multiplicities with the stabilizing divisor (for each semi-infinite edge corresponding to an interior marking or edge connecting a non-ghost component to a ghost component.) Denote by $\overline{M}_\Gamma(X, L)$ the moduli space of stable weighted treed broken disks to $X$ of type $\Gamma$. Let $\Gamma$ be a type with $n$ leaves (corresponding to trajectories of the Morse function on the Lagrangian) and $l$ broken Morse trajectories on the degenerating divisor. An admissible labelling for a $\Gamma$ is a collection $l \in \mathbb{I}(L)^{n+1}$ such that whenever a leaf has weight 0 resp. $(0, \infty)$ resp. $\infty$, the corresponding label is $l_m^\bullet$ or $l_m^\circ$ resp. $l_m^\circ$ or $l_m^\circ$. Given an admissible labelling we denote by $\overline{M}_\Gamma(X, L, l) \subset \overline{M}_\Gamma(X, L)$ the locus of
maps with limits given by $l$ (replacing each $l^0_m, l^m_m, l^m_0$ with $l_m$) along the semi-infinite edges.

Broken maps may be viewed as pseudoholomorphic maps of curves with cylindrical ends; this leads to a natural notion of convergence in which the moduli space of broken maps of any given combinatorial type is compact. This is essentially a special case of compactness in symplectic field theory [19], [1], although the particular set-up we use here has not been considered before.

First we recall terminology for the type of cylindrical ends we consider. First we introduce notation for the symplectic manifolds with cylindrical ends: Let $X^\circ_\pm$ denote the manifold obtained by removing the divisor, or more generally, for the intermediate pieces $P(N_{\pm} \oplus \mathbb{C})^\circ \simeq R \times \mathbb{Z}$ the manifold obtained by removing the divisors at zero and infinity, isomorphic to $Y$. We identify a neighborhood of infinity in $P(N_{\pm} \oplus \mathbb{C})$ with $R_0 \times \mathbb{Z}$ with the almost complex structure induced from a connection on $Z$ and the given almost complex structure on $Y$.

Recall that the notion of Hofer energy makes sense for stable Hamiltonian structures. A Hamiltonian structure on a manifold $Z$ is a closed two-form $\omega \in \Omega^2(Z)$ of rank $\dim(Z) - 1$. A stable Hamiltonian structure is a one-form $\alpha \in \Omega^1(Z)$ with the property that

$$\ker(d\alpha) \subset \ker(\omega), \quad \ker(\alpha) \cap \ker(\omega) = \{0\}.$$

The second condition means that $\alpha$ is non-vanishing on the non-zero vectors in $\ker(\omega) \subset TZ$. Any circle-fibered coisotropic submanifold of a symplectic manifold has a stable Hamiltonian structure by taking $\alpha$ to be a connection on the circle bundle, and $\omega$ to be the restriction of the symplectic form.

For stable Hamiltonian structures a suitable notion of energy is introduced in [19]. We recall the definitions of action and energy of holomorphic curves in $\mathbb{R} \times Z$, where $Z$ is equipped with Hamiltonian structure $\omega_Z$ and connection form $\alpha$;

**Definition 4.4.** (Action and energy)

(a) (Horizontal energy) The **horizontal energy** of a holomorphic map $u = (\phi, v) : (C, j) \to (\mathbb{R} \times Z, J)$ is ([19, 5.3])

$$E^h(u) = \int_C v^*\omega_Z.$$

(b) (Vertical energy) The **vertical energy** of a holomorphic map $u = (\phi, v) : (C, j) \to (\mathbb{R} \times Z, J)$ is ([19, 5.3])

$$E^v(u) = \sup_{\zeta} \int_C (\zeta \circ \phi) d\phi \wedge v^*\alpha$$

where the supremum is taken over the set of all non-negative $C^\infty$ functions $\zeta : \mathbb{R} \to \mathbb{R}$ having compact support and satisfying the condition

$$\int_{\mathbb{R}} \zeta(s) ds = 1.$$

(c) (Hofer energy) The **Hofer energy** of a holomorphic map $u = (\phi, v) : (C, j) \to (\mathbb{R} \times Z, J)$ is ([19, 5.3]) is the sum

$$E(u) = E^h(u) + E^v(u).$$
(d) (Generalization to manifolds with cylindrical ends) Suppose that $X^\circ$ is a symplectic manifold with cylindrical end modelled on $\mathbb{R}_{>0} \times \mathbb{Z}$. The vertical energy is defined as before in (25) and the Hofer energy of a map $u : C^\circ \to X^\circ$ from a surface $C^\circ$ with cylindrical ends to $X^\circ$ is defined by dividing $X^\circ$ into a compact piece $X^{\text{com}}$ and a cylindrical end $\mathbb{R}_{>0} \times \mathbb{Z}$, and defining $E(u) = E(u|X^{\text{com}}) + E(u|\mathbb{R}_{>0} \times \mathbb{Z})$.

**Theorem 4.5.** Any sequence of finite energy broken pseudoholomorphic maps $u_\nu : C_\nu \to X^\circ_\pm$ resp. $u_\nu : C_\nu \to X[k]^\circ$ with bounded Hofer energy $\sup_\nu E(u_\nu) < \infty$ has a convergent subsequence, and any such convergent sequence has a unique limit.

**Proof.** We begin with some historical remarks. In the case that the almost complex structure is compatible and preserves the horizontal subspace, Theorem 4.5 is essentially a special case of the compactness result in symplectic field theory [19, Section 5.4] (with further details and corrections in Abbas [1] and alternative approach given in [21]) with the additional complication of Lagrangian boundary conditions. Since the Lagrangian is compact in $X^\circ -$, these Lagrangian boundary conditions do not affect any of the arguments except that an additional argument for energy quantization for curves with cylindrical ends and boundary in $L$ is required.

For transversality reasons later we need the case of tamed almost complex structures not necessarily preserving the horizontal subspace. As noted in Wendl [104], there is gap in the literature on exponential decay and convergence results in symplectic field theory for such almost complex structures. Fortunately our particular setup corresponds to the case of relative stable maps in Ionel-Parker [52] and Li-Ruan [74], as explained in Bourgeois et al. [19, Remark 5.9]. In particular, asymptotic convergence follows from asymptotic convergence for holomorphic maps to $Y$; energy quantization for disks in $X_-$ implies energy quantization for finite energy holomorphic maps of half-cylinders to $X_-^\circ$, where the boundary of the cylinder maps to the Lagrangian $L$. Energy quantization for holomorphic maps of spheres to $Y$ implies energy quantization for maps of holomorphic spheres to $\mathbb{P}(N_\pm \oplus \mathbb{C})$ with non-trivial projection to $Y$ has energy at least $h$. Matching of intersection multiplicities is [19, Remark 5.9], Tehrani-Zinger [100, Lemma 6.6]: By removal of singularities, there is a one-to-one correspondence between finite energy holomorphic curves in $X^\circ_+$ resp. $X[k]^\circ$ and those in $X_+^\circ$ resp. $X[k]_+$ that are not contained in the divisor $Y$ resp. divisors at zero and infinity. Thus the intersection multiplicity is the degree of the cover of the Reeb orbit at infinity.

We sketch the argument. It suffices to show that on each tree or surface part of the domain, a subsequence converges to some limit in the Gromov sense. Since each domain is stable, each surface part has a unique hyperbolic metric so that the boundary is totally geodesic, see Abbas [1, I.3.3]. We denote by $r_\nu : C_\nu \to \mathbb{R}_{>0}$ the injectivity radius. The argument of Bourgeois et al. [19, Chapter 10], see also Abbas [1], shows that after adding finitely many sequences of points to the domain we may assume that the domain $C_\nu$ converges to a limit $C$ such that the first derivative $\sup|du_\nu|/r_\nu$ is bounded with respect to the hyperbolic metric on the surface part, and with respect to the given metric on the tree part. This implies that there
exists a limiting map $u : C^\infty \to X$ on the complement $C^\infty$ of the nodes so that on compact subsets of the complement of the nodes a subsequence of $u_\nu$ converges to $u$ in all derivatives. Removal of singularities and matching conditions then follows from the corresponding results for holomorphic maps: the matching condition for nodes mapping into the cylindrical end is simply the matching condition for the maps to $Y$, in addition to matching of intersection degrees which is immediate from the description as a winding number. Convergence on the tree part of the domain follows from uniqueness of solutions to ordinary differential equations.

Remark 4.6. (Exponential decay) In fact any holomorphic map $u : C^\circ \to X^\circ$ with finite Hofer energy converges exponentially fast to the corresponding Reeb orbit: In coordinates $s,t$ on the cylindrical end diffeomorphic to $\mathbb{R}_{>0} \times \mathbb{Z}$, there exists a constant $C$ and constants $s_0, s_1 > 0$ such that for $s > s_1$

$$\text{dist}(u(s,t), (\mu(s-s_0), \gamma(t))) < C \exp(-s).$$

This follows from the correspondence with holomorphic maps to the compactification. This property will be used later for the gluing result.

4.3. Broken perturbations. Each stratum of the moduli space of broken maps is smooth under a certain regularity condition involving the surjectivity of a suitable linearized operator. Because the situation is Morse-Bott, there are zero modes in the tangential operator at infinity and the linearized Cauchy-Riemann operator in standard Sobolev spaces is not Fredholm. Instead, the linearized operator acts on Sobolev spaces involving a choice of Sobolev weight for the cylindrical ends which is not in the spectrum of the tangential operator at infinity.

Remark 4.7. (Linearized operators) We first introduce suitable Sobolev spaces of maps modelled on multiple covers of Reeb orbits at infinity. Let $X, Y$ be as above. Let $\lambda \in (0,1)$ be a Sobolev weight. Choose a cutoff function

$$\beta \in C^\infty(\mathbb{R}), \quad \begin{cases} \beta(s) = 0 & s \leq 0 \\ \beta(s) = 1 & s \geq 1 \end{cases}.$$  

We equip the complement $X^\circ = X - Y$ with a cylindrical end modelled on $\mathbb{R}_{>0} \times \mathbb{Z}$. Let

$$\pi_\mathbb{R} : \mathbb{R}_{>0} \times \mathbb{Z} \to \mathbb{R}_{>0}$$

denote projection on the first factor. We fix a multiplicity $\mu \in \mathbb{Z}$. For the moment we assume that the domain $C^\circ$ has no tree parts.

First we define a weighted Sobolev space of maps asymptotic to a Reeb orbit. For $p > 2$ and any map $u : C^\circ \to X^\circ$ asymptotic to $(\mu(s-s_0), \gamma(t-t_0))$ as $s \to \infty$ in the sense that

$$\lim_{s \to \infty} \text{dist}(u(s,t) - (\mu(s-s_0), \gamma(t))) = 0$$

define

$$\|u\|_{1,p,\mu,\lambda} := \int_C (\|\text{d}u - \mu \beta(\alpha + \text{d}\pi_\mathbb{R})\|^p + \|\beta(\pi_\mathbb{R} \circ u - \mu(s-s_0))\|^p) e^{\beta(s)ps\lambda} \text{d}Vol_C.$$  

Here $s,t$ are cylindrical coordinates on the cylindrical end of $C$, $\mu \beta \alpha$ and $\beta(s)ps\lambda$ are by definition equal to 0 on the complement of the support of $\beta$, where the function
where $u$ is zero exactly if $u$ is a $\mu$-fold cover of a Reeb orbit $\gamma(t)$ on the cylindrical end. Let

$$\text{Map}(C^0, X^0)_{1,p,\mu,\lambda} \subset W^{k,p}_{\text{loc}}(C^0, X^0)$$

denote the space of $W^{1,p}_{\text{loc}}$ maps from $C^0$ to $X^0$ with finite $1,p,\mu,\lambda$-norm (27). For any smooth map $u : C^0 \to X^0$ with finite $1,p,\lambda$ norm we denote by $\Omega^0(C^0, u^*TX^0)$ the space of smooth sections of $u^*TX^0$. For any section $\xi : C^0 \to u^*TX^0$ vanishing sufficiently rapidly at infinity we denote by

$$\|\xi\|_{1,p,\lambda} := \int_{C^0} (\|\nabla\xi\|^p + \|\xi\|^p) e^{p\lambda} d\text{Vol} C^0.$$ 

More generally if $\xi$ is asymptotic to $\xi_\infty \in \ker(\alpha) \subset TZ$ we denote by

$$\|\xi\|_{1,p,\lambda} := \left(\|\xi - \xi_\infty\|^p_{1,p,\lambda} + |\xi_\infty|^p\right)^{1/p}. \tag{28}$$

The space of $W^{1,p}_{\text{loc}}$-sections with finite $1,p,\lambda$-norm (28) is denoted $\Omega^0(C^0, u^*TX^0)_{1,p,\lambda}$; these are sections which are equal to a constant valued in $\ker(\alpha) \cong \pi^*TY$ plus an exponentially decaying term. For any smooth map $u$ with finite $(1,p,\lambda)$-norm, pointwise geodesic exponentiation defines a map (using Sobolev multiplication estimates)

$$\exp_u : \Omega^0(C^0, u^*TX^0)_{1,p,\lambda} \to \text{Map}(C^0, X)_{1,p,\mu,\lambda}. \tag{29}$$

The maps (29) provide charts on $\text{Map}(C,X)_{1,p,\mu,\lambda}$ making $\text{Map}(C,X)_{1,p,\mu,\lambda}$ into a smooth Banach manifold. More generally, in the case that $C = S^0 \cup T$ is the union of surface and tree parts we define similar Sobolev spaces by taking the standard $W^{1,p}$ spaces on the tree parts. Thus, for example, we denote for simplicity

$$\Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda} := \Omega^{0,1}(S^0, u^*TX^0)_{0,p,\lambda} \oplus \Omega^{1}(T_b, u^*TL)_{0,p} \oplus \Omega^{1}(T_t, u^*TY)_{0,p}.$$ 

The pseudoholomorphic maps are cut out locally by a smooth map of Banach spaces: Define

$$\|\eta\|_{0,p,\lambda} := \int_{C^0} \|\eta\|^p e^{p\lambda} d\text{Vol} C^0$$

and let $\Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda}$ denote the space of $I^p_{\text{loc}}$ sections with finite $0,p,\lambda$-norm. The Cauchy-Riemann operator defines a map

$$\mathcal{F}^\omega_u : \Omega^0(C^0, u^*TX^0, (\partial u)^*TL)_{1,p,\lambda} \to \Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda},$$

$$\xi \mapsto (T_u(\xi)^{-1}\mathcal{J}\exp_u(\xi), \frac{d}{ds} - \text{grad}(F_\Gamma))(\xi|_T), \frac{d}{ds} - \text{grad}(H_\Gamma))(\xi|_T))$$

where

$$T_u(\xi) : \Omega^0(C^0, u^*TX^0) \to \Omega^0(C^0, (\exp_u(\xi))^*TX^0).$$

denotes parallel transport using an almost complex connection. The linearization of $\mathcal{F}^\omega_u$ at 0 is

$$D^\omega_u : \Omega^0(C^0, u^*TX^0)_{1,p,\lambda} \to \Omega^{0,1}(C^0, u^*TX^0)_{0,p,\lambda},$$

$$\xi \mapsto \left(\nabla\xi^{0,1} - (1/2)(J\nabla\xi J)\partial_u, \frac{d}{ds}\xi - \nabla\xi|_T \text{grad}(F_\Gamma), \frac{d}{ds}\xi - \nabla\xi|_T \text{grad}(H_\Gamma)\right).$$
For sufficiently small Sobolev weight, the linearized operator above has the same kernel as the kernel of the standard linearized operator. Consider the map on the compact curve

\[ D_u : \Omega^0(C, u^*TX)_{1,p} \to \Omega^{0,1}(C, u^*TX)_{0,p}. \]

We claim that for \( \lambda \in (0, 1) \), the kernel resp. cokernel of \( D_u \) is isomorphic to the kernel resp. cokernel of the linearized operator \( D_u^\circ \). Indeed, elliptic regularity implies that any element of \( \ker(D_u) \) is smooth and so is equal to a constant plus a term that is at most linear in the local coordinate, hence exponentially decaying in the coordinates \( s, t \); while conversely any element \( \xi \) of \( \ker(D_u^\circ) \) lifts to a finite energy element of \( D_u \) on \( L^p \) sections; by removal of singularities \( \xi \) extends to an element of \( D_u \). A similar argument gives the identification of cokernels: the adjoint \( (D_u^\circ)^* \) acts on the space \((D_u^\circ)^* : \Omega^{0,1}(C^\circ, u^*TX^\circ)_{0,p', -\lambda} \to \Omega^0(C^\circ, u^*TX^\circ)_{1,p', -\lambda}\) with small negative exponential weight \(-\lambda\), allowing elements with small exponential growth in the domain and range. However for \( \lambda \) small standard analysis implies that any element in the cokernel converges exponentially to a constant along the cylindrical end (c.f. [3, 4.7].) The map \( u \) is regular if the linearized operator is surjective.

In order to achieve transversality we introduce stabilizing divisors satisfying a compatibility condition with the degeneration and introduce domain-dependent almost complex structures. By a broken divisor we mean a divisor which arises from degeneration of a divisor in the original manifold via neck stretching.

**Definition 4.8.** (Broken divisors) A broken divisor for the broken almost complex manifold \( \mathbb{X} := X_- \cup_Y X_+ \) consists of a pair \( \mathbb{D} = (D_-, D_+) \) where \( D_{\pm} \subset X_\pm \) is a codimension two almost complex submanifold such that each intersection \( D_- \cap Y = D_Y \) is a codimension two almost complex submanifold in \( Y \). Given a broken divisor we may view the space \( \mathbb{D} = D_- \cup_{D_Y} D_+ \) as a subspace of the broken manifold \( \mathbb{X} \).

Given a broken divisor \( D_{\pm} \) as above we obtain a divisor

\[ D_N := \mathbb{P}(N|D_Y + C) \subset \mathbb{P}(N_+ \oplus C). \]

We suppose that each \([D_{\pm}]\) is dual to a large multiple of the symplectic class on \( X_\pm \), that is, \([D_{\pm}] = k[\omega_{\pm}]\). Then \([D_N] = k\pi_Y^*[\omega_Y]\), where \( \pi_Y \) is projection onto \( Y \), and as a result does not represent a multiple of any symplectic class on \( \mathbb{P}(N_+ \oplus C) \). Thus the divisor \( D_N \) can be disjoint from non-constant holomorphic spheres in \( \mathbb{P}(N_+ \oplus C) \), namely the fibers. However, holomorphic spheres whose projections to \( Y \) are non-constant automatically intersect \( D_N \).

As in the unbroken case, to achieve transversality we use almost complex structures equal to a fixed almost complex structure on the stabilizing divisor. We introduce the following notations. For a symplectic manifold \( X^\circ \) with cylindrical end, denote by \( \mathcal{J}(X^\circ) \) the space of tamed almost complex structures on \( X^\circ \) that are of cylindrical form on the end. Define

\[ \mathcal{J}(\mathbb{X}) := \mathcal{J}(X^\circ) \times_{\mathcal{J}(\mathbb{R} \times Z)} \mathcal{J}(X_+^\circ) \]

the space of almost complex structures on \( X^\circ, \mathbb{R} \times Z, X_+^\circ \) that are cylindrical form on the cylindrical ends and induce the same almost complex structure on \( Y \). Given
$J_{D_{\pm}} \in \mathcal{J}(D_{\pm})$, we denote by $\mathcal{J}(\mathcal{X}, J_{D})$ the space of almost complex structures in $\mathcal{J}(\mathcal{X})$ that agree with $J_{D_{\pm}}$ on $D_{\pm}$:

$$\mathcal{J}(\mathcal{X}, J_{D}) = \{(J_{-}, J_{+}) \in \mathcal{J}(\mathcal{X}), J_{\pm}|_{D_{\pm}} = J_{D_{\pm}}\}.$$ 

Fix a tamed almost complex structure $J_{D_{\pm}}$ such that $D_{\pm}$ contains no non-constant $J_{D_{\pm}}$-holomorphic spheres of any energy and any holomorphic sphere meets $D_{\pm}$ in at least three points, as in [27, Proposition 8.14]. By [23, Proposition 8.4], for any energy $E > 0$ there exists a contractible open neighborhood $\mathcal{J}^*(X_{\pm}, J_{D_{\pm}}, E)$ of $J_{D_{\pm}}$ agreeing with $J_{D_{\pm}}$ on $D_{\pm}$ with the property that $D_{\pm}$ still contains no non-constant holomorphic spheres and any holomorphic sphere of energy at most $E$ meets $D_{\pm}$ in at least three points. We denote by $\mathcal{J}^*(X, J_{D}, E)$ the fiber product $\mathcal{J}^*(X_{-}, J_{D_{-}}, E) \times_{\mathcal{J}^*(\mathbb{R} \times Z)} \mathcal{J}^*(X_{+}, J_{D_{+}}, E)$.

Given a broken divisor define perturbation data for a broken symplectic manifold, as before, by allowing the almost complex structure etc. to depend on the additional intersection points with the divisor. The new data in comparison with Charest-Woodward [24] is the choice of perturbations of the Morse function on the degenerating divisor. For base almost complex structures $J_{D, \pm}$ agreeing on $Y$, a perturbation datum for type $\Gamma$ of broken maps is datum $P_{\Gamma} = (J_{\Gamma}, F_{\Gamma}, G_{\Gamma}, H_{\Gamma})$ where

$$J_{\Gamma} : \mathcal{S}_{\Gamma} \to \mathcal{J}(\mathcal{X}, J_{D}), \quad F_{\Gamma} : T^b_{\Gamma} \to C^\infty(L)$$

$$G_{\Gamma} : T^b_{\Gamma} \to \mathcal{G}(L), \quad H_{\Gamma} : T^i_{\Gamma} \to C^\infty(Y)$$

and $J_{\Gamma}$ is equal to the fixed almost complex structures $J_{D, \pm}$ on $D_{\pm}$.

In order to define perturbation data we have to keep in mind that the domain of a broken curve is not necessarily stable because of Morse trajectories of infinite length. However, given a broken curve $C$ we obtain a stable broken curve $f(C)$ by collapsing unstable components, and a perturbation system $P_{\Gamma}$ for curves of such type by pulling back $P_{f(\Gamma)}$ under the stabilization map $C \to U_{f(C)}$.

**Definition 4.9.** (Adapted broken maps) Given a type $\Gamma$ of broken disk and a perturbation datum $P_{\Gamma}$, an adapted stable broken map is a stable map $u : C \to \mathcal{X}[k]$ from a broken weighted treed disk $C$ to $\mathcal{X}[k]$ for some $k$ such that

(a) (Stable domain) The surface part $S$ of the domain $C$ is stable: each disk component resp. sphere component on which $u$ is constant has at least three boundary special points or one boundary special point and one special interior point resp. three special points;

(b) (Marking property) each interior marking maps to $\mathbb{D}$ and each component of $u^{-1}(\mathbb{D}) \cap S$ contains an interior marking; and

(c) (Sphere property) any holomorphic sphere contained in $D_{\pm}$ is constant, while any holomorphic sphere contained in $D_Y$ is contained in a fiber of $\mathbb{P}(N_{\pm} \oplus \mathbb{C}) \to Y$.

Let $\mathcal{M}_{\Gamma}(\mathcal{X}, L, \mathbb{D})$ denote the moduli space of stable adapted broken weighted treed disks with boundary in $L$ of type $\Gamma$. 
We generalize the various conditions on perturbation data from the unbroken case in Theorem 3.1:

(a) A perturbation system $P = (P_\Gamma)$ is coherent if it satisfies natural conditions with respect to (Collapsing edges, Making edge lengths or weights finite and non-zero, cutting edges of infinite length) and on any disconnected type $\Gamma = \Gamma_1 \cup \Gamma_2$, $P_\Gamma$ is obtained as the product of perturbation data $P_{\Gamma_1}$ and $P_{\Gamma_2}$.

(b) We say that a perturbation datum $P_{\Gamma}$ is regular if every broken map of uncrowded, stable type $\Gamma$ is regular. Under this assumption, $\mathcal{M}_\Gamma(X, L, D)$ is smooth of expected dimension. Note in particular that regularity means that, for each type of expected dimension zero, each Morse trajectory occurring in the configuration is disjoint from the stabilizing divisor, $u_T^{-1}(D) = \emptyset$, since the condition of meeting the stabilizing divisor is codimension one in the space of Morse trajectories.

(c) A perturbation system $P = (P_\Gamma)$ is stabilizing if for each type $\Gamma$, $J_\Gamma$ takes values in $J^*(X, J_D, n(\Gamma_i)/k)$ on any component $C_i$ having $n(\Gamma_i)$ interior markings, where $\Gamma_i$ is the single-vertex graph corresponding to $C_i$.

(d) A perturbation system $P_\Gamma$ of type $\Gamma$ with one leaf is divisorial if it is pulled back under the forgetful map (21) forgetting the incoming leaf.

The following generalizes the compactness and transversality results from Charest-Woodward [24] to the broken case:

**Theorem 4.10.** Let $\Gamma$ be an uncrowded type of stable broken treed disk of expected dimension at most one and suppose that regular, stabilizing perturbation data $P_\Gamma'$ have been chosen for all uncrowded boundary strata $U_\Gamma' \subset U_\Gamma$. There exists a comeager subset of the space of regular perturbation data $P_\Gamma$ equal to the given perturbation data on lower-dimensional strata and sufficiently close to the base almost complex structure and metric such that

(a) (Transversality) Every element of $\mathcal{M}_\Gamma(X, L, D)$ is regular.

(b) (Compactness) The closure $\overline{\mathcal{M}}_\Gamma(X, L, D)$ is compact and contained in the stable, uncrowded locus.

(c) (Tubular neighborhoods) Each uncrowded stratum $\mathcal{M}_\Gamma(X, L, D)$ of dimension zero has a tubular neighborhood of dimension one in any adjoining uncrowded strata of one higher dimension.

(d) (Orientations) The uncrowded strata $\mathcal{M}_\Gamma(X, L, D)$ of formal dimension one or two are equipped with orientations satisfying the standard gluing signs for inclusions of boundary strata as in [24]; in particular we denote by $\epsilon([u]) \in \{\pm 1\}$ the orientation sign associated to the zero-dimensional moduli spaces.

(e) (One-leaf forgetful morphisms) among the regular perturbations there exist a comeager subset that are divisorial, and for these there exist forgetful morphisms

$$\mathcal{M}_\Gamma(X, L, D) \to \mathcal{M}_{f(\Gamma)}(X, L, D)$$

where $\Gamma$ is any type with a single leaf and $f(\Gamma)$ the type with no leaves obtained by forgetting the leaf.
Proof. Without the domain-dependent perturbations, compactness is Theorem 4.5. The case with domain-dependent perturbations is proved similarly, using the coherence condition on the perturbations. To see that the limit is stable and uncrowded, note that the stabilization condition on the divisor implies that any sphere bubble resp. disk bubble appearing in the limit has at least three resp. one interior intersection points $u^{-1}(D)$ with the stabilizing divisor $D$. Furthermore, by preservation of intersection multiplicity with the divisor, each maximal ghost component $C_i \subset C$ mapping to the divisor $D$ must contain at least one marking $z_j \in C_i$. Now $C_i$ must be adjacent either to at least two non-ghost components $C_j, C_k$, a single non-ghost component $C_j$, or adjacent to two tree segments $T_j, T_j \subset C$. Since strata with a component with a point with intersection multiplicity two, or mapping the node to the divisor are codimension at least two, they do not occur in the limit. Hence the $C_i$ contains at most one marking, so the limit is of uncrowded type.

Transversality is an application of Sard-Smale as in Charest-Woodward [23] on the universal space of maps, after removing maximal ghost components. We say a ghost component in $\mathbb{P}(N_\pm \oplus \mathbb{C})$ is one that projects to a point in $Y$. Such maps are automatically multiple covers of a fiber of $\mathbb{P}(N_\pm \oplus \mathbb{C}) \rightarrow Y$ and so automatically regular. We begin by covering the universal treed broken disk $U_{\Gamma} \rightarrow M_\Gamma$ by local trivializations $U_{\Gamma}^i \rightarrow M_{\Gamma}^i, i = 1, \ldots, N$. For each local trivialization consider an moduli space defined as follows. Let $\text{Map}_{k,p}^{k,p}(C, X, L, \mathbb{D})$ denote the space of maps of class $k, p$ mapping the boundary of $C$ into $L$, the interior markings into $\mathbb{D}$, and constant (or constant after projection to $Y$, if the component maps to a neck piece) on each disk with no interior marking. Let $\mathcal{U}_{\Gamma}^{\text{hin}}$ be a small neighborhood of the nodes and attaching points in the edges $\mathcal{U}_{\Gamma}$, so that the complement in each edge and surface component is open. Let $\mathcal{P}_\Gamma^i(X, L, \mathbb{D})$ denote the space of perturbation data $P_\Gamma = (J_\Gamma, F_\Gamma, G_\Gamma, H_\Gamma)$ of class $C^i$ equal to the given pair $(J, F, G, H)$ on $\mathcal{U}_{\Gamma}^{\text{hin}}$, and such that the restriction of $P_\Gamma$ to $\mathcal{U}_{\Gamma'}$ is equal to $P_{\Gamma'}$, for each boundary type $\Gamma'$. Let $l \gg k$ be an integer and and

$$\mathcal{B}_{k,p,l,\Gamma} := M_{\Gamma}^i \times \text{Map}_{k,p}^{k,p}(C, X, L, \mathbb{D}) \times \mathcal{P}_\Gamma^i(X, L, \mathbb{D}).$$

Consider the map given by the local trivialization

$$M_{\Gamma}^\text{univ},i \rightarrow J(S), \ m \mapsto j(m).$$

Let $S_{\text{nc}} \subset S$ be the union of disk and sphere components with interior markings. Let $m(e)$ denote the function giving the intersection multiplicities with the stabilizing divisor, defined on edges $e$ corresponding to intersection points, and consider the fiber bundle $\mathcal{E}^i = \mathcal{E}^i_{k,p,l,\Gamma}$ over $\mathcal{B}_{k,p,l,\Gamma}$ given by

$$\begin{align*}
(\mathcal{E}^i_{k,p,l,\Gamma})_{m,u,J} \subset \Omega_{j,l,\Gamma}^{0,1}(S_{\text{nc}}, (u|_S)^*TX)_{k-1,p} \\
\quad \oplus \Omega(T_b, (u|_{T_b})^*TL)_{k-1,p} \oplus \Omega(T_i, (u|_{T_i})^*TY)_{k-1,p}
\end{align*}$$

the space of 0, 1-forms with respect to $j(m), J$ which vanish to order $m(e) - 1$ at the node or marking corresponding to each contact edge $e$. The Cauchy-Riemann and shifted gradient operators applied to the restrictions $u_S$ resp. $u_T$ of $u$ to the two
resp. one dimensional parts of $C = S \cup T$ define a $C^q$ section
\begin{equation}
(33) \quad \partial_T: \mathcal{B}_{k,p,l,\Gamma}^i \to E_{k,p,l,\Gamma}, \quad (m, u, J, F) \mapsto \left( \partial_{j(m), Ju}, \left( \frac{d}{ds} - \text{grad}_F \right) u_{T_b}, \left( \frac{d}{ds} - \text{grad}_H \right) u_{T_i} \right)
\end{equation}
where
\begin{equation}
(34) \quad \partial_{j(m), Ju} := \frac{1}{2} (Ju_S - du_S j(m)),
\end{equation}
and $s$ is a local coordinate with unit speed. The local universal moduli space is
\[
\mathcal{M}^{\text{univ},i}_{\Gamma}(X, L, \mathbb{D}) = \overline{\partial}^{-1} \mathcal{B}_{k,p,l,\Gamma}^i
\]
where $\mathcal{B}_{k,p,l,\Gamma}^i$ is embedded as the zero section. This subspace is cut out transversally: by [27, Lemma 6.5, Proposition 6.10], the linearized operator is surjective on the two-dimensional part of the domain mapping to $X_\pm$ on which $u$ is non-constant, while at any point $z$ in the interior of an edge in $C$ with $du(z) \neq 0$ the linearized operator is surjective by a standard argument. It remains to deal with components that map to a neck piece $\mathbb{P}(N_\pm \oplus \mathbb{C})$ that project to a constant map to $Y$. Let $\eta \in \Omega^{\bullet,1}(u^* T(\mathbb{P}(N_\pm \oplus \mathbb{C})))$ be a one-form on one of the intermediate broken pieces $S_i$ such that $\eta$ lies in the cokernel of the universal linearized operator
\[
D_{u,i}(\xi, K) = D_u \xi + \frac{1}{2} K Du_j.
\]
for the universal moduli space. Variations of tamed almost complex structure of cylindrical type are $J$-antilinear maps $K: T\mathbb{P}(N_\pm \oplus \mathbb{C}) \to T\mathbb{P}(N_\pm \oplus \mathbb{C})$ which vanish on the vertical subbundle. Since the horizontal part of $D_z u$ is non-zero at some $z \in C$, we may find an infinitesimal variation $K$ of almost complex structure of cylindrical type by choosing $K(z)$ so that $K(z)D_z u_j(z)$ is an arbitrary $(j(z), J(z))$-antilinear map from $T_z C$ to $T_{u(z)} \mathbb{P}(N_\pm \oplus \mathbb{C})$. Choose $K(z)$ so that $K(z)D_z u_j(z)$ pairs non-trivially with $\eta(u(z))$ and extend $K(z)$ to an infinitesimal almost complex structure $K$ by a cutoff function. By the implicit function theorem, $\mathcal{M}^{\text{univ},i}_{\Gamma}(X, L, \mathbb{D})$ is a Banach manifold of class $C^q$, and the forgetful morphism
\[
\varphi_i: \mathcal{M}^{\text{univ},i}_{\Gamma}(X, L, \mathbb{D})|_{k,p,l} \to \mathcal{P}_{\Gamma}(X, L, \mathbb{D})|_l
\]
is a $C^q$ Fredholm map. Let
\[
\mathcal{M}^{\text{univ},i}_{\Gamma}(X, L, \mathbb{D})_d \subset \mathcal{M}^{\text{univ},i}_{\Gamma}(X, L, \mathbb{D})
\]
denote the component on which $\varphi_i$ has Fredholm index $d$. By the Sard-Smale theorem, for $k,l$ sufficiently large the set of regular values $\mathcal{P}^{i,\text{reg}}_{\Gamma}(X, L, D)_l$ of $\varphi_i$ on $\mathcal{M}^{\text{univ},i}_{\Gamma}(X, L, \mathbb{D})_d$ in $\mathcal{P}_{\Gamma}(X, L, \mathbb{D})_l$ is comeager. Let
\[
\mathcal{P}^{i,\text{reg}}_{\Gamma}(X, L, \mathbb{D})_l = \cap_i \mathcal{P}^{i,l,\text{reg}}_{\Gamma}(X, L, \mathbb{D})_l.
\]
A standard argument shows that the set of smooth domain-dependent $\mathcal{P}^{\text{reg}}_{\Gamma}(X, L, \mathbb{D})$ is also comeager. Fix $(J_\Gamma, F_\Gamma) \in \mathcal{P}^{\text{reg}}_{\Gamma}(X, L, \mathbb{D})$. By elliptic regularity, every element
of $\mathcal{M}^f\Gamma(\mathcal{X}, L, \mathbb{D})$ is smooth. The transition maps for the local trivializations of the universal bundle define smooth maps
\[
\mathcal{M}^f\Gamma(\mathcal{X}, L, \mathbb{D})|_{\mathcal{M}^f\Gamma\cap\mathcal{M}^f\Gamma} \rightarrow \mathcal{M}^f\Gamma(\mathcal{X}, L, \mathbb{D})|_{\mathcal{M}^f\Gamma\cap\mathcal{M}^f\Gamma}.
\]
This construction equips the space
\[
\mathcal{M}^\Gamma(\mathcal{X}, L, \mathbb{D}) = \bigcup_i \mathcal{M}^f\Gamma(\mathcal{X}, L, \mathbb{D})
\]
with a smooth atlas. Since $\mathcal{M}^\Gamma$ is Hausdorff and second-countable, so is $\mathcal{M}^\Gamma(\mathcal{X}, L, \mathbb{D})$ and it follows that $\mathcal{M}^\Gamma(\mathcal{X}, L, \mathbb{D})$ has the structure of a smooth manifold.

Existence of orientations and tubular neighborhoods for codimension one strata involving broken Morse trajectory is standard. However, for strata corresponding to a trajectory of length zero, there is a new gluing result necessary which is proved in Section 4.6.

**Figure 10.** Broken disk with a boundary node

**Remark 4.11.** (True and fake boundary components) The codimension one strata are of the following types:

(a) Strata of maps to $X[1]$ such that one component has an interior node or boundary, that is, connected by a gradient trajectory of $H$ or $F$ of length 0. See Figure 10.

(b) Strata of maps with pieces mapping to $X[2]$, that is, with three levels connected by gradient trajectories of $H$. See Figure 9.

(c) Strata of maps to $X[1]$ with a broken Morse trajectory for $F$ passing through a critical point. See Figure 11.

Of these three types, the first two are *fake* boundary types in the sense that they do not represent points in the topological boundary. In the first case one can make the length of either the first gradient trajectory or the second finite, but not both (since the total length must be infinite). In the second case, one can either make the length finite and non-zero or resolve the node; this shows that the stratum is in the closure of two strata of top dimension. The last type is a true boundary component since the only deformation is that which deforms the length of the trajectory to a finite real number.
Using the regularized moduli spaces of broken maps we define the higher composition maps of the broken Fukaya algebra
\[ \mu^{n,y} : \widehat{CF}(L)^{\otimes n} \to \widehat{CF}(L) \]
on generators by
\[ \mu^{n,y}(\langle l_1 \rangle, \ldots, \langle l_n \rangle) = \sum_{[u] \in \mathcal{M}_\Gamma(X,L,D)} (-1)^\heartsuit (\sigma([u])!)^{-1} y(\partial u) q^{E([u])} e([u]) \langle l_0 \rangle \]
where \( \heartsuit = \sum_{i=1}^n i |l_i| \).

**Theorem 4.12.** (Broken Fukaya algebra) For any regular coherent stabilizing divisorial perturbation system \( P = (P_\Gamma) \) as above sufficiently \( C^2 \) close to the base datum \((J,G,H)\), the maps \( (\mu^{n,y})_{n \geq 0} \) satisfy the axioms of a convergent \( A_\infty \) algebra \( \widehat{CF}(X,L,D) \) with strict unit and weak divisor axiom. The homotopy type of \( \widehat{CF}(X,L,D) \) and non-vanishing of the broken Floer cohomology is independent of all choices up to homotopy equivalence.

**Proof.** The proof is similar to that of Theorem 3.2. Note that boundary components corresponding to broken stable maps with three levels are negative expected dimension. The estimate needed for the definition of convergence follows for a fixed \((J,G,H)\) by finiteness of the set of homotopy classes for a given energy, by compactness; then the same finiteness holds for perturbations sufficiently \( C^2 \) small. \( \square \)

4.4. **Broken divisors.** In the rest of the section we show that broken stabilizing divisors exist. The result is an analog of a result in algebraic geometry that follows from Kodaira vanishing: Let \( X \) be a smooth complex projective variety equipped with an ample line bundle \( \mathcal{E} \) and \( i : Y \subset X \) a smooth subvariety of codimension one. Let \( \mathcal{E}(Y) \) denote the sheaf of sections vanishing on \( Y \). The exact sequence of sheaves
\[ 0 \to \mathcal{E}(Y) \to \mathcal{E} \to i_* i^* \mathcal{E} \to 0 \]
induces a long exact sequence of cohomology groups including the sequence
\[ 0 \to H^0(\mathcal{E}(Y)) \to H^0(\mathcal{E}) \to H^0(i_* i^* \mathcal{E}) \to H^1(\mathcal{E}(Y)) \to \ldots . \]
By Kodaira vanishing $H^1(\mathcal{E}(Y))$ vanishes for sufficiently positive $\mathcal{E}$ and furthermore $\mathcal{E}(Y)$ is generated by its global sections. By the long exact sequence $H^0(\mathcal{E}) \to H^0(i_*i^*\mathcal{E})$ is surjective. By Sard’s theorem, the relative version of Bertini holds: For any $s_Y \in H^0(i^*\mathcal{E})$ there exists a section $s \in H^0(\mathcal{E})$ restricting to $s_Y$ and cutting out a smooth divisor.

The symplectic version of this statement is obtained by a modification of Donaldson’s argument in [31]. Let $\hat{X} \to X$ be a line-bundle with connection $\alpha$ over $X$ whose curvature two-form $\text{curv}(\alpha)$ satisfies $\text{curv}(\alpha) = (2\pi/i)\omega$; since our symplectic manifolds are rational we may always assume this to be the case after taking a suitable integer multiple of the symplectic form.

**Definition 4.13.** (Asymptotically holomorphic sequences of sections) Let $(s_k)_{k \geq 0}$ be a sequence of sections of $\hat{X}^k \to X$.

(a) The sequence $(s_k)_{k \geq 0}$ is **asymptotically holomorphic** if there exists a constant $C$ and integer $k_0$ such that for $k \geq k_0$,

\begin{equation}
|s_k| + |\nabla s_k| + |\nabla^2 s_k| \leq C,
\end{equation}

\begin{equation}
|\partial s_k| + |\nabla \partial s_k| \leq Ck^{-1/2}.
\end{equation}

(b) The sequence $(s_k)_{k \geq 0}$ is **uniformly transverse** to 0 if there exists a constant $\eta$ independent of $k$ such that for any $x \in X$ with $|s_k(x)| < \eta$, the derivative of $s_k$ is surjective and satisfies $|\nabla s_k(x)| \geq \eta$.

In both definitions the norms of the derivatives are evaluated using the metric $g_k = k\omega(\cdot, J \cdot)$.

**Theorem 4.14.** For $k \gg 0$ there exist approximately holomorphic codimension two submanifolds $D_{\pm} \subset X_{\pm}$ such that $D_+ \cap Y = D_- \cap Y = D_Y$ is asymptotically holomorphic and codimension two, and each $D_-, D_+, D_Y$ represents $k[\omega_-]$ resp. $k[\omega_+]$ resp. $k[\omega_Y]$.

The proof will be given after two lemmas below.

**Lemma 4.15.** (Extension of asymptotically holomorphic sequences) Given any asymptotically holomorphic sequence $s_{Y,k}$ of $\hat{Y}^k \to Y$, there exists an asymptotically holomorphic sequence $s_{k,\pm}$ of $\hat{X}^k \to X$ such that $s_{k,\pm}|Y = s_{Y,k}$.

**Proof.** We may identify $X_\pm$ near $Y$ with the normal bundle $N_{\pm}$ of $Y$ in $X_\pm$ on a neighborhood $U_{\pm}$ of the zero section. Let $\pi_{\pm} : N_{\pm} \to Y$ denote the projection and $\phi : N_{\pm} \to \mathbb{R}_{\geq 0}$ the norm function. We may assume that the linearization $\hat{X}$ is given by a pull-back

$\hat{X}^k|U_{\pm} \cong \pi_{\pm}^* \hat{Y}^k$

and the connection in the normal direction is given to leading order by the expression

$\frac{k}{4}(z_1 \overline{d}z_1 - \overline{z}_1 dz_1)$

where $z_1$ is a coordinate on the fiber of the normal bundle. Define a Gaussian sequence

$s_{k,\pm} = \pi_{\pm}^* s_{Y,k} \exp(-k|z_1|^2/4)$
independent of the local trivialization of the normal bundle (local coordinate) and multiply by a cutoff function supported in a neighborhood of size $k^{-1/6}$ of $Y$. The bounds

$$|s_k| + |\nabla s_k| + |\nabla^2 s_k| \leq C$$

follows immediately from the fact that the derivatives of the Gaussian are bounded, and the derivatives are with respect to the metric $g_k$; the bound

$$|\bar{\partial}s_k| + |\nabla \bar{\partial}s_k| \leq Ck^{-1/2}$$

follows from the fact that the Gaussian is holomorphic to leading order as in [31, (10)]: Consider the splitting $TN_\pm$ into the vertical part $N_\pm$ and its orthogonal complement $T_{\text{hor}}N_\pm$, the difference between the almost complex structures $\pi^*J_Y \oplus J_{\text{vert}}$ and $J_\pm$ is represented by a map $\mu : \Lambda^{1,0}TN_\pm \to \Lambda^{0,1}N_\pm$. Then as on [31, p. 10] we have

$$\bar{\partial}s_k = \sum_{i \geq 2} \mu(dz_i) \partial_i s_k + \sum_{i \geq 2} (\bar{\partial}_i \pi^* s_{Y,k}) e^{-k|z_1|^2/4} + \pi^* s_{Y,k} \mu(z_1 dz_1) e^{-|k^{1/2}z_1|^2/4}$$

where the last term represents the derivative in the normal direction. Since $\mu$ and all its derivatives are uniformly bounded this implies

$$|\bar{\partial}s_k| \leq Ck^{-1/2}|k^{1/2}z_1| e^{-k|z_1|^2/4} \sup \|\bar{\partial}s_{Y,k}\| \leq Ck^{-1/2}$$

and

$$|\nabla \bar{\partial}s_k| \leq C(\|\nabla \mu\|_1 k^{1/2}z_1| e^{-|k^{1/2}z_1|^2/4} + |\mu|\|\nabla k^{1/2}z_1 dz_1 e^{-k|z_1|^2/4}$$

$$\leq Ck^{-1/2}(|z_1| + |z_1|^3) e^{-k|z_1|^2/4} + \sup \|\nabla \bar{\partial}s_{Y,k}\|.$$

**Lemma 4.16.** For any $p \in X_\pm$, there exists an asymptotically holomorphic sequence $s_{p,k}$ of $\hat{X}_\pm$ concentrated at $p$ with the property that $s_{p,k}$ vanishes on $Y$. Furthermore the estimates in (35) are uniform in $p$ as long as the first coordinate $p_1$ satisfies $|p_1| \geq k^{-1/2}$.

**Proof.** Given an asymptotically holomorphic sequence $s'_{p,k}$ with sufficiently small support (for example, a Gaussian $s(z) = \exp(-k|z|^2)$ in a local Darboux chart and trivialization of $\hat{X}^k$) we may assume that $Y$ is described locally by $z_1 = 0$ and $p_1 \neq 0$ is the first coordinate of the point $p$. Then the section $s_{p,k}(z) = s'_{p,k}(z)z_1/p_1$ is also asymptotically holomorphic, uniformly in $p$ as long as $|p_1| > k^{-1/2}$. The bound

$$|s_{k,p}| + |\nabla s_{k,p}| + |\nabla^2 s_{k,p}| \leq C$$

is immediate, and uniform if $|p_1| > k^{-1/2}$ since $s'_{p,k}$ is Gaussian in $k^{1/2}z_1$. The bound

$$|\bar{\partial}s_{k,p}| + |\nabla \bar{\partial}s_{k,p}| \leq Ck^{-1/2}$$

follows from the fact that $z_1/p_1$ is holomorphic to leading order.
Proof of Theorem 4.14. The section \( s_{\pm,k} \) from Lemma 4.15 is already asymptotically holomorphic and uniformly transverse in a neighborhood of size \( \sqrt{k} \) around \( Y \). Therefore we may use sections \( s_{p,k} \) in Lemma 4.16 below for \( p \) of distance at least \( k^{-1/2} \) to achieve transversality off of \( Y \). More precisely, in Donaldson’s construction [31, p. 681] taking \( V_0 \) to be, rather than empty, a neighborhood of size \( k^{-1/2} \) around \( Y \).

\[ \square \]

Proposition 4.17. For any type \( \Gamma \) with \( k \geq 1 \) components of the surface part joined by \( e \) cylindrical ends and Morse trajectories and limits \( l \) along the \( n \) semi-infinite edges mapping to the Lagrangian, the expected dimension of the moduli space \( \mathcal{M}_\Gamma(X, L, \mathbb{D}, l) \) of stable adapted broken maps of combinatorial type \( \Gamma \) limits \( l \) is given by

\[
\dim T_{[u]} \mathcal{M}_\Gamma(X, L, \mathbb{D}, l) = \dim(L) + 2I(u) - \dim(S^+_{l_0}) - \sum_{i=1}^n \dim(S^-_{l_i}) + n - 3 + (k - 1) \dim(X) - \sum_{i=1}^e (\dim(Y) - 4s_i)
\]

where \( s_i \) are the multiplicities at the intersection points with \( Y \).

Proof. This is a consequence of Riemann-Roch for Cauchy-Riemann operators on surfaces with boundary, [78, Appendix] giving the basic sum \( \dim(L) + (k - 1) \dim(X) + 2I(u) \) for the sum of the indices on the various surface components, with corrections \( \dim(S^-_{l_i}) \) appearing from the constraints at the joining points between the tree parts mapping to \( L \) and the disk parts, the \( n - 3 \) equal to the dimension of the moduli space of stable disks, and finally the contribution \( \sum_{i=1}^e (\dim(Y) - 4s_i) \) from the matching and tangency conditions for the Morse trajectories in \( Y \). \[ \square \]

4.5. The case of a blow-up or reverse flip. Now we specialize to the case that the symplectic manifold is obtained by a small simple reverse flip or blow-up, and the broken manifold is obtained by stretching a hypersurface separating the exceptional locus. We will show that in this case the broken Fukaya algebra of regular Lagrangian branes is unobstructed and the Floer cohomology is non-trivial.

As mentioned in the introduction we want to restrict to the case of flips occurring in smooth runnings of the mmp, where the center of the flips are trivial. These flips are called simple:

Definition 4.18. (a) (Smooth flips) \( X \) is obtained from a smooth reverse flip if the local model \( \tilde{V} \) in (2.6) has positive weights all equal to 1. It follows that \( X_+ \) is free of orbifold singularities.

(b) (Simple flips) \( X \) is obtained from a small reverse simple flip or blow-up iff the local model \( \tilde{V} \) in 2.6 has all weights equal to \( \pm 1 \). In this case there exists an embedded projective space \( \mathbb{P}^{n+1} \) in \( X \) and a tubular neighborhood of \( \mathbb{P}^{n+1} \) in \( X \) symplectomorphic to \( U^\oplus m \) where \( n_+ \oplus n_- = n + 1 \) and \( B \) is the tautological bundle over \( \mathbb{P}^{n+1} \).

Remark 4.19. (Exceptional and unexceptional pieces) Let \( Z \) denote the unit sphere bundle in \( B^n \). Then \( Z \) is circle-fibered coisotropic fibering over the projectivized
bundle $Y = \mathbb{P}(\mathcal{R}^-) = \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. The variety $X$ degenerates to the union of a toric variety

$$X_\epsilon = \mathbb{P}(U_{\mathbb{R}} \oplus \mathbb{C})$$

and the blow-up $X_\epsilon$ of $X$ at $\mathbb{P}^{n-1}$. We call $X_\epsilon$ resp. $X_\nu$ the exceptional resp. unexceptional piece of $X$.

**Remark 4.20.** (Exceptional piece as a symplectic quotient) The space $X_\epsilon$ may also be realized as a symplectic quotient

$$X_\epsilon = (\mathbb{C}^n_{-1} \oplus \mathbb{C}^n_{+1} \oplus \mathbb{C})/(U(1) \times U(1))$$

where the action is by weights $\mathbb{C}^n_{-1}$ with weight $(-1, 1)$ and on $\mathbb{C}^n_{+1}$ with weight $(+1, 1)$, and on the last factor of $\mathbb{C}$ with weight $(0, -1)$. The flip is obtained by variation of git quotient in the above local model, which changes the unstable locus from $\{0\} \oplus \mathbb{C}^n$ to $\mathbb{C}^n \oplus \{0\}$. It follows that under the flip $((\mathbb{C}^n \oplus \mathbb{C}^n) \mathbb{C}^x \cong \mathbb{P}^{n-1}$ is replaced by $(\mathbb{C}^n \oplus \{0\}) \mathbb{C}^x \cong \mathbb{P}^{n-1}$.

**Remark 4.21.** (Exceptional piece as a toric manifold) The description of $X_\epsilon$ as a symplectic quotient from 2.7 implies that $X_\epsilon$ is a symplectic toric manifold obtained by symplectic quotient of $\mathbb{C}^{n+1}$ by $(\mathbb{C}^x)^2$. We denote by $T = (S^1)^{n+1}/(S^1 \times S^1)$ the residual torus acting on $X_\epsilon$. Denote the moment map

$$\Phi : X_\epsilon \rightarrow \mathfrak{t}^\vee.$$ 

The moment polytope $\Phi(X_\epsilon)$ of $X_\epsilon$ has facets defined by normal vectors arising from projection of the standard basis vectors in $\mathbb{R}^{n+1} = \text{Lie}(S^1)^{n+1}$. Using the parametrization of

$$S^1 \rightarrow T, \quad (z_1, \ldots, z_{n-1}) \mapsto [z_1, \ldots, z_{n-1}, 1, 1]$$

we have in terms of the standard basis vectors

$$\nu_1 := \epsilon_1, \nu_2 := \epsilon_2, \ldots, \nu_{n-1} := \epsilon_{n-1} \in \mathfrak{t}^\vee$$

while from the description of the weights we have

$$\nu_n := \epsilon_1 + \ldots + \epsilon_{n-1} - \epsilon_{n+1} - \ldots - \epsilon_{n-1}, \nu_{n+1} := -\epsilon_1 + \ldots - \epsilon_{n-1}.$$ 

Let $P_-$ denote its moment polytope

$$P_- = \{ \mu \mid \langle \mu, \nu_k \rangle \geq c_k, \ k = 1, \ldots, n+1 \}$$

where

$$c_1 = c_2 = \ldots c_{n-1} = 0, \ c_n = \epsilon, \ c_{n+1} \gg 0$$

and $\epsilon > 0$ represents the size of the exceptional divisor. Let

$$\Phi_- : X_\epsilon \rightarrow P_-$$

denote the moment map for the action of the $n$-torus $T^n$. The fiber over

$$\lambda = \epsilon(1, \ldots, 1)/(n_- - n_+ - 1)$$

is a Lagrangian torus $L$. For example, if $n_+ = 2, n_- = 1$ then the corresponding transition is a blow-down of curve in a surface. The moment polytope has normal vectors $\epsilon_1, \epsilon_2, \epsilon_1 - \epsilon_2, -\epsilon_1$. The moment polytope is

$$\{(\lambda_1, \lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 - \lambda_2 \geq \epsilon, \lambda_1 \leq c_4\}$$
where $c_4 \gg 0$. Then $L$ is the fiber over $(\epsilon, \epsilon)$. See Figure 12 where the point $\lambda$ is shown as a shaded dot inside the moment polytope, which is a trapezoid.

We compute the Floer cohomology of the torus (40). Recall from Remark 4.19 that the pieces $X_{\pm}$ are joined along the hypersurface

$$Y = \mathbb{P}(U^{n-}) = \mathbb{P}^{n_+ - 1} \times \mathbb{P}^{n_- - 1}.$$  

Let $H : Y \to \mathbb{R}$ be the standard Morse function on $Y$ obtained as the pairing of the moment map with a generic vector. For $i \leq k$ let

$$\epsilon_i := [0, \ldots, 1, 0, \ldots, 1] \in \mathbb{P}^k$$

denote the points whose homogeneous coordinates are all zero except for the $i$-th coordinate; these are the fixed points for the standard torus action on $\mathbb{P}^k$. The critical points are the fixed points for the torus action

$$\text{crit}(H) = \{(\epsilon_{i_-}, \epsilon_{i_+}) \in \mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1}, \ i_{\pm} \leq n_{\pm}\}.$$  

The choice of $H$ corresponds to a choice of one-parameter subgroup $\mathbb{C}^\times$ inside the standard torus of dimension $n_- \times n_+$ acting on $\mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1}$. The Morse cycles consist of those points that flow to $(\epsilon_{i_-}, \epsilon_{i_+})$ under the $\mathbb{C}^\times$-action:

$$S^-_{([\epsilon_i], [\epsilon_{\bar{i}}])} = \left\{(z_1, z_2) \in \mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1} \mid \lim_{z \to 0} z(z_1, z_2) = (\epsilon_{i_-}, \epsilon_{i_+}) \right\}.$$  

Here it suffices to consider the one-parameter subgroup generated by the element $(1, \ldots, n_-), (1, \ldots, n_+)$; it is standard that the Morse cycles are those points $(z_-, z_+)$ such that the homogeneous coordinates of $z_{\pm}$ above index $i_{\pm}$ vanish. That is,

$$S^-_{([\epsilon_i], [\epsilon_{\bar{i}}])} \cong \mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1}, \text{ for } i_- \leq n_-, i_+ \leq n_+.$$  

For example, when $n_- = 1, n_+ = 2$ so the transition corresponds to the blow-down of a curve in a surface, $Y = \mathbb{P}^1$ and the Morse cycles are either all of $\mathbb{P}^1$ or a point. This ends the Remark.

**Lemma 4.22.** The exceptional piece $X_e$ is a Fano toric variety.

**Proof.** Since $X_e$ is a toric variety it suffices to check that the anticanonical degree of rational invariant holomorphic curves is positive. By Remarks 2.7, 4.21 $X_e$ is a $\mathbb{P}^{n_-}$-bundle over $\mathbb{P}^{n_+ - 1}$, obtained by git from quotienting $\mathbb{C}^{n_- + n_+ + 1}$ by the action of $(\mathbb{C}^\times)^2$ with weights $(-1, 1)$ with multiplicity $n_-$, and $(1, 1)$ with multiplicity $n_+ + 1$.
Hence the anticanonical bundle of $X_e$ is the quotient of the trivial bundle with weight $(n_+ - n_- + 1, n_+ + n_- + 1)$:

$$K_{X_e}^{-1} \cong \left( \mathbb{C}^{n_-+n_+ + 1} \times \mathbb{C}^{2(n_+ - n_- + 1, n_+ + n_- + 1)} \right) / (\mathbb{C}^n)^2.$$ 

By standard toric arguments the irreducible invariant holomorphic curves either lie in a fiber of the projection to $\mathbb{P}^{n_+ - 1}$, the zero section $\mathbb{P}^{n_- - 1}$ or the divisor at infinity $\mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1}$. We denote by $\mathcal{O}(k)$ the $k$-th tensor power of the hyperplane line bundle. The restriction of $K_{X_e}^{-1}$ to the zero section $\mathbb{P}^{n_-}$ is $\mathcal{O}(n_+ - n_- + 1)$, while the restriction to the divisor at infinity $\mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1}$ is

$$K_{X_e}^{-1}|_{\mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1}} \cong \pi_- \mathcal{O}(n_- + 1) \otimes \pi_+ \mathcal{O}(n_+ + 1)$$

where $\pi_{\pm}$ are the projections on the factors. Finally the anticanonical bundle on the fibers $\mathbb{P}^{n_-}$ is positive. The claim follows from positivity on the zero section, divisor at infinity, and fibers. \qed

Next we describe the broken holomorphic disks corresponding to the decomposition into exceptional and unexceptional pieces.

**Corollary 4.23.** There exist regular perturbation data for the broken manifold $\mathcal{X}$ with the property that the almost complex structure on $X_e \subset \mathcal{X}$ is the standard one.

**Proof.** The Fano condition implies that any constant holomorphic spheres have positive Chern number. Hence the only configurations of broken weighted treed disks with index zero or one consist of broken curves whose piece mapping to $X_e$ is a single disk. As in Cho-Oh [26], this implies that all holomorphic disks (necessarily given by the Blaschke products in (6)) are regular. \qed

The following is the main result of this section; it states that the broken Floer theory of the Lagrangian torus is unobstructed and non-trivial.

**Theorem 4.24.** Let $L \subset X_e$ be a regular Lagrangian brane. If $(P_\Gamma)$ is a collection of (regular, coherent, stabilizing, divisorial) perturbations such that each almost complex structure is constant equal to the standard complex structure on $X_e$ and the Morse function on $Y$ for the nodes attaching to the disk components are the standard one on $Y \cong \mathbb{P}^{n_- - 1} \times \mathbb{P}^{n_+ - 1}$ then

(a) The broken Fukaya category $\hat{CF}(\mathcal{X}, L)$ is weakly unobstructed: there exists $b(y) \in CF(\mathcal{X}, L)$ depending on the local system $y \in \mathcal{R}(L)$ such that $(\mu_{y,b(y)}^1)^2 = 0$.

(b) there exists a local system $y \in \mathcal{R}(L)$ such that the Floer cohomology is non-vanishing: $H(\mu_{y,b(y)}^1) = H(L) \neq 0$.

**Proof.** As mentioned in the introduction, moduli spaces of disks in toric varieties with boundary on Lagrangian torus orbits generally have “excess dimension”; for example, the smallest dimension moduli space has dimension that of the Lagrangian. More generally, let $l \in \text{crit}(F)$ and $(x_1, \ldots, x_m) \in \text{crit}(H)$. Denote by $\mathcal{M}(X_e, L, \underline{x}, l)$ the moduli space of disks with a single level and no leaves, limiting to critical points $\underline{x} \in \text{crit}(H)$ along the semi-infinite edges attached to interior points and $l$ along the semi-infinite edge attached to the boundary point. (That is, $\mathcal{M}(X_e, L, \underline{x}, l)$ is
the “disk part” of a component of the moduli space of broken disks.) In order that the moduli space \( M(X_e, L, x, l) \) is non-empty and of expected dimension zero for perturbations with almost complex structures close to the standard complex structure, we must have

\[
I(u) - \sum_i (\deg(x_i) + 2) + \deg(l) - 2 = 0
\]

for any element \([u]\). Indeed, the moduli space of disks with boundary in \( L \) is the moduli space of Blaschke products (6) which is regular and has evaluation map at any interior marking transverse to the Morse cycles \( P_i \times P_j \). Thus we may in fact assume that the almost complex structure on \( X_e \) is the standard torus-invariant complex structure. The requirement that \( u(z) \) meet \( P_i \times P_j \) at a marking \( z_k \) implies that \( u \) has a zero at \( z_k \) for the codim(\( P_i \times P_j \)) coordinates defining \( P_i \times P_j \). Each such zero contributes two to the Maslov index \( I(u) \) of \( u \), hence for the moduli space to contain some element \([u]\) we must have

\[
I(u) - 2 \geq \sum_i (\deg(x_i) + 2).
\]

Combining (42) with (41) it follows that the moduli space can be non-empty only when \( \deg(l) = 0 \), in which case \( \mu^0(1) \) is a count of Maslov index two disks and so a multiple of the maximum \( \mu^0(1) = Wl_m^* \in \Lambda_l^* \). As explained in [24, Lemma 2.36], this implies that \( \hat{CF}(X, L) \) is weakly unobstructed via the solution \( b = Wl_m^* \) to the weak Maurer-Cartan equation.

The leading order terms in the potential \( W(y) \) are as in the toric case and can be read off from (38), (39):

\[
W(y_1, \ldots, y_n) = q^\epsilon \left( y_1 + \ldots + \frac{y_1 \cdots y_{n+} \cdots y_n}{y_{n+1} \cdots y_n} \right) + \text{higher order}
\]

where the higher order terms have \( q \)-exponent at least \( \epsilon \). The leading order potential

\[
W_0(y_1, \ldots, y_{n-1}) = y_1 + \ldots + \frac{y_1 \cdots y_{n+}}{y_{n+1} \cdots y_n}.
\]

Its partial derivatives are for \( i \leq n_+ \)

\[
y_i \partial_{y_i} W_0(y_1, \ldots, y_{n-1}) = y_i + \frac{y_1 \cdots y_{n+}}{y_{n+1} \cdots y_{n-1}}.
\]

For \( i > n_+ \) the partial derivatives are

\[
y_i \partial_{y_i} W_0(y_1, \ldots, y_{n-1}) = y_i - \frac{y_1 \cdots y_{n+}}{y_{n+1} \cdots y_n}.
\]

Setting all partial derivatives equal to zero we obtain

\[
y_1 = \ldots = y_{n+} = -y_{n+1} = \ldots = -y_n
\]

and

\[
y_1^{n_+ - n_-} = (-1)^{n_-}.
\]

Hence \( W_0(y) \) has a non-degenerate critical point at certain roots of unity. Now one can solve for a critical point of \( W = W_0 + \) higher order using the implicit function
theorem in [40, Theorem 10.4] by varying the local system. This gives a local system for the broken Fukaya algebra $\hat{CF}(X, L)$ for which the Floer cohomology is non-vanishing.

4.6. Getting back together. The existence of a tubular neighborhood for strata in Theorem 4.10 involving a bubble on a cylindrical end requires a gluing theorem which is similar but slightly different from that in Bourgeois-Oancea [18]. Much more complicated gluing theorems in symplectic field theory have been proved in Hutchings-Taubes [51] and Hofer-Wysocki-Zehnder (see e.g. [49]) both of which involve obstructions arising from multiple branched covers of Reeb orbits. Here any such cover corresponds to a fiber of the bundle $\mathbb{P}(N_{\pm} \oplus \mathbb{C})$, and so one has transversality automatically (although one also has to achieve transversality with the diagonal at the nodes).

**Theorem 4.25.** Suppose that $u : C \to X$ is a regular adapted broken map with either a broken Morse trajectory or a Morse trajectory of length zero. Then there exists an $\delta > 0$ and a family of adapted broken maps $u_{\delta} : C_{\delta} \to X$ such that $\lim_{\delta \to 0} [u_{\delta}] = [u]$.

**Proof.** Given a broken curve $C$ with two sublevels $C_+, C_-$, a standard gluing procedure creates, for any small gluing parameter $\delta \in \mathbb{C}$, a curve $C_{\delta}$ obtained by removing small disks around the node and gluing in using a map given in local coordinates by $z \mapsto \delta/z$. Similarly, given a broken Morse trajectory a similar gluing procedure replaces the trajectory with one of finite length by removing a small ball around the breaking and gluing together the pieces on either side. The gluing procedure for the second kind of gluing (replacing a broken gradient trajectory with one of finite length) is rather standard so we focus on the first kind. For simplicity, we focus on the case that $C$ has a single level and two sublevels $C_+, C_-; the general case is similar.

The proof is an application of a quantitative version of the implicit function theorem. The steps are: construction of an approximation solution; construction of an approximate inverse to the linearized operator; quadratic estimates; application of the contraction mapping principle. We assume for simplicity that $X[k_{\pm}] = \mathbb{P}(N_{\pm} \oplus \mathbb{C})$. We think of any curve in $X[k_{\pm}]$ as a curve with cylindrical ends in $\mathbb{R} \times \mathbb{Z}$ where $Z$ is the circle-fibered coisotropic with base $Y$.

We will need local trivializations of the universal treed disk and the associated families of complex structures and metrics on the domains. If $\Gamma_{\pm}$ denote the combinatorial types of $u_{\pm}$ let

$$\mathcal{U}_{\Gamma_{\pm}} \to \mathcal{M}_{\Gamma_{\pm}} \times S_{\pm}, i = 1, \ldots, l$$

be a local trivializations of the universal treed disk, identifying each nearby fiber with $(C^0_{\pm}, z, w)$ such that each point in the universal treed disk is contained in one of these local trivializations. We may assume that $\mathcal{M}_{\Gamma_{\pm}}$ is identified with an open ball in Euclidean space so that the the fiber $C^0_{\pm}$ correspond to 0. Similarly, we assume we have a local trivialization of the universal bundle near the glued curve giving rise to a family of complex structures

$$\mathcal{M}_{\Gamma} \to J(S_{\delta})$$
of complex structures on the two-dimensional locus $S^2_\delta \subset C^0_\delta$, which are constant on the neck region.

An approximate solution is given by gluing together the two solutions $u_\pm$ using a cutoff function. With $\beta$ as in (26) let $y \in Y$ be the evaluation of $u_\pm$ at the node and $\gamma(t)$ a Reeb orbit in the fiber over $y$ in $Z$ with multiplicity $\mu$ so that $u_\pm$ considered locally as maps to $X^\circ$ are asymptotic to $\gamma(t)$:

$$\lim_{s \to -\infty} d(u_+(s, t), (\mu s, \gamma(t))) = \lim_{s \to -\infty} d(u_-(s, t), (\mu s, \gamma(t))) = 0.$$

Any point $x \in X$ let

$$\exp_x : T_x X \to X$$

denote the map given by geodesic exponentiation. We write in cylindrical coordinates near the divisor at infinity $Y$, $x \mapsto \exp_x(\mu \gamma(t))(\zeta_{\pm}(s, t)).$

Define $u^{\text{pre}}_\delta$ to be equal to $u_\pm$ away from the neck region, while on the neck region of $C_\delta$ with coordinates $s, t$ define

$$u^{\text{pre}}_\delta(s, t) = \exp_{(\mu s, \gamma(t))}(\zeta_\pm(s, t)).$$

In other words, one translates $u_+, u_-$ by some amount $|\ln(\delta)|$, and then patches them together using the cutoff function and geodesic exponentiation.

We set up a map between suitable Banach spaces whose zeroes describe pseudoholomorphic maps near to the approximate solution. Since $C_\delta$ satisfies a uniform cone condition, one has uniform Sobolev embedding estimates and multiplication estimates for the spaces. We will need Sobolev spaces on the curves $C_\delta$ which approximate the weighted Sobolev spaces appearing in Section 4.2. We denote by

$$(s, t) \in [-|\ln(\delta)|/2, |\ln(\delta)|/2] \times S^1$$

the coordinates on the neck region created by the gluing. Define a Sobolev weight function

$$\kappa_\lambda : C_\delta \to [0, \infty), \quad (s, t) \mapsto \beta(|\ln(\delta)|/2 - |s|)p\lambda(|\ln(\delta)|/2 - |s|)$$

where $\beta(|s| - |\ln(\delta)|/2)p\lambda(|s| - |\ln(\delta)|/2))$ is by definition zero on the complement of the neck region.

Pseudoholomorphic maps near the pre-glued solution are cut out locally by a smooth map of Banach spaces. Given a smooth map $u : C_\delta \to X$, element $m \in \mathcal{M}_1^p$, and a section $\xi : C_\delta \to u^*TX$ define

$$(44) \quad \| (m, \xi) \|_{1, p, \lambda} := \left( \| m \|^p + \int_{C_\delta} (\| \nabla \xi \|^p + \| \xi \|^p) \exp(\kappa_\lambda) \, d \text{Vol}_{C_\delta} \right)^{1/p}.$$ 

Let $\Omega^0(C_\delta, u^*TX)_{1, p, \delta}$ be the space of $W^{1, p}_{\text{loc}}$ sections with finite norm (44). Pointwise geodesic exponentiation defines a map (using Sobolev multiplication estimates)

$$(45) \quad \exp_{\text{pre}} : \Omega^0(C_\delta, (u^{\text{pre}}_\delta)^*TX)_{1, p, \lambda} \to W^{1, p}(\text{Map}(C_\delta, X)).$$

The space of pseudoholomorphic maps is cut out locally by a smooth map of Banach spaces. For a 0, 1-form $\eta \in \Omega^{0, 1}(C_\delta, u^*TX)$ define

$$\| \eta \|_{0, p, \lambda} = \left( \int_{C_\delta} \| \eta \|^p \exp(\kappa_\lambda) \, d \text{Vol}_{C_\delta} \right)^{1/p}.$$
Parallel transport using an almost-complex connection defines a map
\[ T_{u^\text{pre}}(\xi) : \Omega^{0,1}(C,(u^\text{pre})^*TX)_{0,p,\lambda} \to \Omega^{0,1}(C,(\exp u^\text{pre}(\xi))^*TX)_{0,p,\lambda}. \]

We write \( C = S \cup T_b \cup T_i \) where \( T_b \) are the tree parts mapping to \( L \) and \( T_i \) are the tree parts mapping to \( Y \).

(46) \( \mathcal{F}_\delta : \mathcal{M}_T \times \Omega^0(C_\delta,(u^\text{pre})^*TX)_{1,p} \to \Omega^{0,1}(C_\delta,(u^\text{pre})^*TX)_{0,p} \)
\[ \xi \mapsto \left( T_{u^\text{pre}}(\xi|\delta)^{-1} J_{\delta,j(m)} \exp u^\text{pre}(\xi|\delta), \right. \]
\[ \left. \frac{d}{dt} - \text{grad}(H_\delta)(\xi_T), \left( \frac{d}{dt} - \text{grad}(F_\delta) \right)(\xi_U) \right). \]

Treed pseudoholomorphic maps close to \( u^\text{pre}_\delta \) correspond to zeroes of \( \mathcal{F}_\delta \). In addition, because we are working in the adapted setting, our curves \( C_\delta \) have a collection of markings \( z_1, \ldots, z_n \) and we work subject to the constraint
\[ (\exp u^\text{pre}_\delta(\xi))(z_i) \in D, \quad i = 1, \ldots, n. \]

By choosing local coordinates near \( u(z_i) \), these constraints may be incorporated into the map \( \mathcal{F}_\delta \) to produce a map
(47) \( \mathcal{F}_\delta^D : \mathcal{M}_T \times \Omega^0(C_\delta,(u^\text{pre})^*TX)_{1,p} \)
\[ \to \Omega^{0,1}(C_\delta,(u^\text{pre})^*TX)_{0,p} \oplus \bigoplus_{i=1}^n T_{u(z_i)}X/T_{u(z_i)}D \]
whose zeroes correspond to adapted pseudoholomorphic maps near the preglued map \( u^\text{pre}_\delta \).

We first consider the failure of the approximate solution to be an exact solution. The one-form \( \mathcal{F}_\delta^D(0) \) has contributions created by the cutoff function as well as the difference between \( J_{u_\pm} \) and \( J_{u^\text{pre}} \):
\[ \| \mathcal{F}_\delta^D(0) \|_{0,p,\lambda} = \| \exp_{(\mu_s,\gamma(t))}(\beta(-s)\zeta_\pm(-s + \ln(\delta)/2, t) + \beta(s)\zeta_\pm(s - \ln(\delta)/2, t)) \|_{0,p,\lambda} \]
\[ = \| (D \exp_{(\mu_s,\gamma(t))})(\beta(-s)\zeta_\pm(-s + \ln(\delta)/2, z) + \beta(s)\zeta_\pm(s - \ln(\delta)/2, t)) + \]
\[ (\beta(-s)\zeta_\pm(-s + \ln(\delta)/2, z) + \beta(s)\zeta_\pm(s - \ln(\delta)/2, t)) \|_{0,0,\lambda} \]
\[ \leq Ce^{\left| \ln(\delta) \right|(1-\lambda)} = C\delta^{\lambda-1}, \]
c.f. Abouzaid [3, 5.10]. Similarly from the terms involving the derivatives of the cutoff function and exponential convergence of \( \zeta_\pm \) to 0 we obtain an estimate
(48) \[ \mathcal{F}_\delta^D(0) \|_{0,p,\lambda} < C \exp(-\left| \ln(\delta) \right|(1-\lambda)) = C\delta^{\lambda-1}. \]

Next one constructs a uniformly bounded right inverse for the linearized operator of the approximate solution from the given right inverses of the pieces. Given
an element \( \eta \in \Omega^{0,1}(u^{\text{pre}}) T(\mathbb{R} \times \mathbb{Z})_{0,p} \), one obtains elements \( \underline{\eta} = (\eta_-, \eta_+) \) by multiplication with the cutoff function:

\[
\eta_+ = \beta \eta, \quad \eta_- = (1 - \beta) \eta.
\]

By Remark 4.7, there exist right inverse \( Q \) to the linearized operators \( D_{u^{\phi}} \) for the broken map \( u^\phi \), that is, for any \( \underline{\eta} = (\eta_+, \eta_-) \in \Omega^{0,1}(C^\infty, u^\phi T X^\phi)_{0,p} \) there exists

\[
\xi = (\xi_+, \xi_-) \in \Omega^{0}(C^\infty, u^\phi T X^\phi)_{1,p}, \quad D_{u^{\phi}} \xi_{\pm} = \eta_{\pm}, \quad \xi_+(w_{\mp}) = \xi_-(w_{\pm})
\]

where \( w_{\pm} \in C_\pm \) are the nodal points, and furthermore,

\[
\xi(z_i) \in T_{u(z_i)} D_i, \quad i = 1, \ldots, n.
\]

We write

\[
u_{\pm} = \exp_{u^\phi_{\text{pre}}} (\zeta_{\pm})
\]

and denote by

\[
\xi = (Q^\delta_-(\eta), Q^\delta_+(\eta)).
\]

The limits of the elements \( \xi_{\pm} \) are by assumption equal to some element \( \xi_{\infty} \in \Omega^0(S^1, \gamma^* TX) \). Therefore we may define \( Q^\delta \eta \) equal to \( (Q^\delta_-(\eta_-, \eta_+), Q^\delta_+(\eta_-, \eta_+)) \) away from \( Z \) and near \( Z \) by patching these solutions together using a cutoff function

\[
Q^\delta \eta := \beta(-s - 1/4 | \ln(\delta)|)(Q^\delta_-(\eta) - \xi_{\infty}) + \beta(s + 1/4 | \ln(\delta)|)(Q^\delta_-(\eta) - \xi_{\infty}) + \xi_{\infty} \in \Omega^{0,1}(C^\infty, (u^\phi_{\text{pre}})^* TX)_{1,p,\lambda}.
\]

Then for any \( \rho \) sufficiently small, there exists \( \delta_0 \) such that for \( \delta > \delta_0 \)

\[
\| D_{u^\phi_{\text{pre}}} Q^\delta \eta - \eta \|_{1,p,\lambda} = \| D_{u^\phi_{\text{pre}}} Q^\delta \eta - T_- D_{u^\phi_{\text{pre}}} Q^\delta \eta - T_+ D_{u^\phi_{\text{pre}}} Q^\delta \eta \|_{1,p,\lambda} \leq \rho \| \beta(s + | \ln(\delta)/4) \zeta_+ \|_{1,p,\lambda} \| \eta \|_{0,p,\lambda} + \rho \| \beta(-s - | \ln(\delta)/4) \zeta_- \|_{1,p,\lambda} \| \eta \|_{0,p,\lambda} + \| \beta(s - | \ln(\delta)/4) Q^\delta \eta \|_{0,p,\lambda} + \| \beta(-s + | \ln(\delta)/4) Q^\delta \eta \|_{0,p,\lambda}
\]

where \( T_\pm \) denote parallel transport from \( u^\phi_{\pm} \) to \( u \). The difference in the exponential factors in the definition of the Sobolev norms implies that possibly after changing the constant \( C \), we have

\[
\| \beta(s - | \ln(\delta)/4) Q^\delta_\pm \eta \|_{1,p,\lambda} < Ce^{-\lambda \delta}.
\]

Hence one obtains an estimate as in Fukaya-Oh-Ohta-Ono \cite[7.1.32]{FOOO}, Abouzaid \cite[Lemma 5.13]{Abouzaid}: for some constant \( C > 0 \), for any \( \delta > 0 \)

\[
\| D_{u^\phi_{\text{pre}}} Q^\delta - \text{Id} \| < Ce^{-\lambda \delta}.
\]

It follows that for \( \delta \) sufficiently large an actual inverse may be obtained from the Taylor series formula

\[
D_{u^\phi_{\text{pre}}}^{-1} = (Q^\delta D_{u^\phi_{\text{pre}}})^{-1} Q = \sum_{k \geq 0} (I - Q^\delta D_{u^\phi_{\text{pre}}})^k Q.
\]
A uniform quadratic estimate for $\mathcal{F}_\delta^D$ may be obtained from the Sobolev embedding theorem: After redefining $C > 0$ we have for all $\xi_1, \xi_2$

\begin{equation}
\|D_i \mathcal{F}_\delta^D(\xi_1) - D_{u^\pre}_\delta \xi_1\| \leq C \|\xi_1\|_{1,p,\lambda} \|\xi_2\|_{1,p,\lambda}.
\end{equation}

To prove this we require some estimates on parallel transport. Let

\begin{equation}
T_\delta^x(m, \xi) : \Lambda^{0,1} T^*_x C_\delta \otimes T_x X \to \Lambda^{0,1}_j(m) T^*_x C_\delta \otimes T_{\exp_\delta(\xi)} X
\end{equation}

denote pointwise parallel transport. Consider its derivative

\begin{equation}
DT_\delta^x(m, \xi, m_1, \xi_1; \eta) = \nabla_t |_{t=0} T_\delta^x (m + t m_1, \xi + t \xi_1) \eta.
\end{equation}

For a map $u : C \to x$ we denote by $DT_u$ the corresponding map on sections. By Sobolev multiplication, there exists a constant $c$ such that

\begin{equation}
\|DT_u^\delta(m, \xi, m_1, \xi_1; \eta)\|_{0,p,\lambda} \leq c \|(m, \xi)\|_{1,p,\lambda} \|(m_1, \xi_1)\|_{1,p,\lambda} \|\eta\|_{0,p,\lambda}.
\end{equation}

Differentiate the equation

\begin{equation}
T_u^\delta(x)(m, \xi) \mathcal{F}_\delta^D(m, \xi) = \nabla \partial_{\delta}(m) (\exp_{u^\pre}(\xi))
\end{equation}

with respect to $(m_1, \xi_1)$ to obtain

\begin{equation}
DT_u^\delta(m, \xi, m_1, \xi_1, \mathcal{F}_\delta^D(m, \xi)) + T_u^\delta(m, \xi)(D\mathcal{F}_\delta^D(m, \xi, m_1, \xi_1)) = (D\nabla_{j}(m)\exp_\delta(\xi))(Dj^\delta(m_1), D\exp_\delta(\xi, \xi_1)).
\end{equation}

Using the pointwise inequality

\begin{equation}
|\mathcal{F}_\delta^D(m, \xi)| < c|\exp_{u^\pre}(\xi)| < c(|du^\pre| + |\nabla\xi|)
\end{equation}

for $m, \xi$ sufficiently small, the estimate (52) yields a pointwise estimate

\begin{equation}
|T_u^\delta(\xi)^{-1} DT_u^\delta(m, \xi, m_1, \xi_1, \mathcal{F}_\delta^D(m, \xi))| \leq c(|du^\pre| + |\nabla\xi|) \|(m, \xi)\| \|(\xi_1, m_1)\|.
\end{equation}

Hence

\begin{equation}
\|T_u^\delta(\xi)^{-1} DT_u^\delta(m, \xi, m_1, \xi_1, \mathcal{F}_\delta^D(m, \xi))\|_{0,p,\lambda} \leq c(1 + |du^\delta|_{0,p,\lambda} + |\nabla\xi|_{0,p,\lambda}) \|(m, \xi)\|_{L^\infty} \|(\xi_1, m_1)\|_{L^\infty}.
\end{equation}

It follows that

\begin{equation}
\|T_u^\delta(\xi)^{-1} DT_u^\delta(m, \xi, m_1, \xi_1, \mathcal{F}_\delta^D(m, \xi))\|_{0,p,\lambda} \leq c\|(m, \xi)\|_{1,p,\lambda} \|(m_1, \xi_1)\|_{1,p,\lambda}
\end{equation}

since the $W^{1,p}$ norm controls the $L^\infty$ norm by the uniform Sobolev estimates. Then, as in McDuff-Salamon [78, Chapter 10], Abouzaid [3] there exists a constant $c > 0$ such that for all $\delta$ sufficiently small, after another redefinition of $C$ we have

\begin{equation}
\|T_u^\delta(\xi)^{-1} D\exp_{u^\pre}(\xi)(Dm^\delta(m_1), D\exp_{u^\pre}(\xi)\xi_1)) - D_{u^\pre}(m_1, \xi_1)\|_{0,p,\lambda} \leq C\|(m, \xi)\|_{1,p,\lambda} \|(m_1, \xi_1)\|_{1,p,\lambda}.
\end{equation}

Combining these estimates and integrating completes the proof of claim (51). Applying the implicit function theorem using the estimates (48), (50), (51) produces a unique solution $m(\delta), \xi(\delta)$ to the equation $\mathcal{F}_\delta^D(m(\delta), \xi(\delta)) = 0$ for each $\delta$, such that the maps $u(\delta) := \exp_{u^\pre}(\xi(\delta))$ depend smoothly on $\delta$. However, the implicit
function theorem by itself does not give that the maps \( u_\delta \) are distinct, since each \( u_\delta \) is the result of applying the contraction mapping principle in a different Sobolev space.

5. The break-up process

In this section we construct a neck-stretching limit of the Fukaya algebra, and show that this Fukaya algebra has unobstructed and non-vanishing Floer cohomology. By stretching a tubular neighborhood of the separating hypersurface one obtains a family of almost complex structures and an associated degeneration of the moduli space of (weighted treed) disks, discussed in the case of spheres in Ionel-Parker [52] and Li-Ruan [74]. This degeneration produces holomorphic curves which “match”, in a certain sense, at the resulting hypersurface. A second degeneration, discussed in Bourgeois [20] and Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [19], involves choosing a Hamiltonian perturbation on the neck and letting the perturbation tend to zero as the neck lengthens. This degeneration produces holomorphic curves together with Morse flow lines and the result is the broken Fukaya category of the previous section. We show that, even in the absence of a strong symplectic gluing theorem, the unobstructedness of the broken Fukaya algebra implies unobstructedness of the original Fukaya algebra.

5.1. Breaking a symplectic manifold. First we recall some terminology from Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder [19].

Definition 5.1. (a) (Neck-stretched manifold) Let \( X \) be a closed almost complex manifold, and \( Z \subset X \) a separating real hypersurface. Let \( X^\circ \) denote the manifold with boundary obtained by cutting open \( X \) along \( Z \). Let \( Z', Z'' \) denote the resulting copies of \( Z \). For any \( \tau > 0 \) let

\[
X^\tau = X^\circ \bigcup_{Z'' = \{-\tau\} \times Z, \{\tau\} \times Z = Z'} \left[ -\tau, \tau \right] \times Z
\]

obtained by gluing together the ends of \( X^\circ \) using a neck of length \([ -\tau, \tau ]\).

(b) (Neck-stretched form with perturbation) For \( \epsilon \) small we consider the following perturbations corresponding to a Morse function \( H : Y \to \mathbb{R} \). We denote by the same notation the pull-back of \( H \) to \( Z \). Choose a cutoff function \( \beta \in X^\tau \) equal to 1 on \([-\tau + 1, \tau - 1]\) \times Z, and with support in \([-\tau, \tau] \times Z \). Let \( \alpha \in \Omega^1(Z) \) be a connection one-form, and let

\[
\omega_\epsilon = \pi_Y^* \omega_Y + \epsilon \text{d}(\beta H \alpha) \in \Omega^2(X^\tau).
\]

Define the perturbed Reeb vector field by

\[
v_\epsilon \in \text{Vect}(X^\tau), \quad \iota(v_\epsilon) \omega_\epsilon = 0, \quad \alpha(v_\epsilon) = 1.
\]

Differentiating (58) with respect to \( \epsilon \) at \( \epsilon = 0 \) we obtain

\[
\ell \left( \frac{d}{d\epsilon} v_\epsilon |_{\epsilon=0} \right) \pi_Y^* \omega_Y = -\ell(v) \text{d}(\beta H \alpha), \quad \alpha \left( \frac{d}{d\epsilon} v_\epsilon |_{\epsilon=0} \right) = 0.
\]

This implies that on the neck region, \( \frac{d}{d\epsilon} v_\epsilon |_{\epsilon=0} \) is the horizontal lift of minus the Hamiltonian vector field of \( H \).
Definition 5.2. (Almost complex structures on neck-stretched manifolds) Let $\pi_Y : Z \to Y$ be a circle bundle over a symplectic manifold $Y$ and $X = \mathbb{R} \times Z$. Let $\pi_Z : X \to \mathbb{R}, \pi_Z : X \to Z$ the projections onto factors and $\ker(\pi_Z)$ the vertical subspace. Let $\omega_Z$ denote the pullback of the symplectic form $\omega_Y$ to $Z$. Then

\begin{equation}
V = \ker(D\pi_Y) \oplus \ker D(\pi_Z) \subset TX
\end{equation}

is a rank two subbundle complementary to $H = \ker(\alpha) \subset \pi^*_Z TZ \subset TX$.

The $\mathbb{R}$ action by translation and $U(1)$ action on $Z$ combine to a smooth $\mathbb{C}^\times \cong \mathbb{R} \times U(1)$ action on $X$. We say that an almost complex structure on $X$ is cylindrical iff $J = v$ in (60) and is invariant under the $C^\times$-action. Any cylindrical almost complex structure induces an almost complex structure $J_Y$ on $Y$ by projection, and we have an isomorphism of complex vector bundles

\begin{equation}
TX \cong H \oplus V, \quad H \cong \pi^*_Z TY, \quad V \cong X \times \mathbb{C}.
\end{equation}

Given a cylindrical almost complex structure and a Hamiltonian perturbation as in (58) define the perturbed almost complex structure by requiring that $J_{\epsilon} \in J(X^\tau)$ induce the same almost complex structure on $Y$ but

\begin{equation}
J_{\epsilon} \frac{\partial}{\partial \tau} = v_{\epsilon}.
\end{equation}

5.2. Breaking perturbation data. The next lemmas allow us to develop a perturbation scheme for broken pseudoholomorphic maps which will be compatible with degeneration. Suppose that $J_t$ is a family of almost complex structures on $X$ obtained by stretching the neck at $Z$, using the Morse function $H$.

Lemma 5.3. (Symplectic sums for pairs) Suppose that $X_{\pm}$ are symplectic manifolds both containing a symplectic hypersurface $Y \subset X_{\pm}$ such that the normal bundles $N_Y^\pm$ of $Y$ in $X_{\pm}$ are inverse. Suppose furthermore that $D_{\pm} \subset X_{\pm}$ are codimension two symplectic submanifolds such that $D_{\pm} \cap Y = D_Y$ for some $D_Y \subset Y$. Then the symplectic sum $D = D_- \# D_Y D_+$ is naturally a symplectic submanifold of $X = X_- \# Y X_+$ preserved by the family of almost complex structures $J_t$.

Proof. This is an application of symplectic local models. Choose a metric on $X_{\pm}$ near $Y$ so that $D_{\pm}$ is totally geodesic in $X_{\pm}$, as in [73, Lemma 6.8]; this can be done in stages so that the metrics on $Y$ (considered as submanifolds of $X_{\pm}$) agree. The geodesic exponential map $N_Y^\pm \to X_{\pm}$ identifies $N|D_Y$ with $D$ locally. The constant rank embedding theorem in Marle [69] identifies a neighborhood of $D_Y$ in $D$ with $N|D_Y$ symplectically. Now consider the identification of $N$ with a neighborhood of $Y$ on $X_{\pm}$ which maps $N|D_Y$ to $D$. Let $\omega_Y$ denote the pull-back of the symplectic form on $X_{\pm}$ and $\omega_Y$ the symplectic form induced from a connection on $N$.

We have $\omega_1 - \omega_0|(D \cup Y) = 0$. Now $D \cap Y$ is by assumption transverse, hence there exists a deformation retract of a neighborhood $U$ of $Y$ to $(D \cup Y) \cap U$. The
standard homotopy formula products a one-form $\alpha$ such that $d\alpha = \omega_1 - \omega_0$ and $\alpha$ vanishes on $(D \cup Y) \cap U$. Then $\omega_t = t\omega_1 + (1 - t)\omega_0$ satisfies $\frac{d}{dt} \omega_t = d\alpha$. Define a vector field $v_t$ by $\iota(v_t)\omega_t = \alpha$. Then $v_t$ vanishes on $(D \cup Y) \cap U$ and defines a symplectomorphism from $\omega_0$ to $\omega_1$ in a neighborhood of $Y$ equal to the identity on $D$. Once local models have been chosen compatibly, the construction is the same as in Lerman [66] and Gompf [43].

**Lemma 5.4.** Suppose that $L \subset X_-$ is a rational Lagrangian. There exists a family of divisors $D_t$ on $X$ so that the image of $L$ in $X$ is exact in the complement of $D_t$, and $D_t$ degenerates as $t \to 0$ to a broken symplectic manifold $\mathcal{D}$ where $D_\pm$ are divisors in $X_\pm$ so that $L$ is exact in the complement of $D_-$. 

**Proof.** Let $(D_+, D_-)$ be a broken divisor. Since the normal bundles $N_Y^\pm$ of $Y$ in $X_\pm$ are inverses, the restrictions $N_Y^\pm|_{DY}$ are also inverses. Then $(Y, DY)$ has a symplectic tubular neighborhood of the form $(N_Y^+, N_Y^+|_{DY})$, where the latter has symplectic structure induced by the choice of connection as in the Lemma 5.3. Furthermore, $D$ is almost complex with respect to every element of the family $J$, by construction. □

We now construct a system of perturbations for the breaking process. We say that a **breaking disk** is a disk with a **breaking parameter** $\rho \in [0, \infty]$ such that if $\rho < \infty$ resp. $\rho = \infty$ then the disk is unbroken resp. broken. The compactified moduli space of breaking disks of type $\Gamma$ is denoted $\overline{\mathcal{M}}_\Gamma$ with a universal curve $\overline{U}_\Gamma$ that has surface part $\overline{S}_\Gamma$ and tree part $\overline{T}_\Gamma$.

**Definition 5.5.** (Perturbation data) A **perturbation datum** for breaking curves of type $\Gamma$ is a datum

$$P_\Gamma = (J_\Gamma, F_\Gamma, G_\Gamma, H_\Gamma),$$

$$J_\Gamma : \overline{S}_\Gamma \to \mathcal{J}(X), \quad F_\Gamma : \overline{T}_\Gamma^b \to C^\infty(L), \quad G_\Gamma : \overline{T}_\Gamma \to \mathcal{G}(L), \quad H_\Gamma : \overline{T}_\Gamma \to C^\infty(Y)$$

such that $J_\Gamma$ takes values in the space of almost complex structures adapted to the neck in a neighborhood of the curves with parameter $\epsilon = 0$, that is, are obtained by gluing from a map to $\mathcal{J}(X_0^\circ) \times_{\mathcal{J}(\mathbb{R} \times \mathbb{Z})} \mathcal{J}(X_0^{\circ+})$ via the map (61).

Associated to any morphism of combinatorial types is a morphism of perturbation data, as before. However in this case there is a new kind of morphism of combinatorial types corresponding to **gluing of cylindrical end symplectic manifolds**. The corresponding morphism of perturbation data takes the almost complex structure on the cylindrical parts of the corresponding curve by pull-back as in (57), (61). We say that a collection of perturbations are coherent if, near the boundary of the moduli space of domains, the perturbations are given by pulling back under the morphisms induced by morphisms of graphs of type (Breaking an edge of infinite length), (Product axiom), (Collapsing edges), (Making edges or weights finite or non-zero). Given a domain $C$, we obtain a perturbation datum on $C$ by pulling back the perturbation data on $\mathcal{U}_{f(\Gamma)}$, where $f(\Gamma)$ is the type of the stabilization of $C$. 
Definition 5.6. Given a perturbation datum an adapted breaking stable map $u$ of type $\Gamma$ is a breaking disk $C$ with breaking parameter $\rho \in [0, \infty)$ together with a map $C \to X$ that is $J_\Gamma$-holomorphic for $\rho > 0$, or a map $C \to X$ for $\rho = 0$ that is $J_\Gamma^0$ resp. $J_\Gamma^{+, -}$-holomorphic on the pieces $C_0$ resp. $C_1, \ldots, C_k$ resp. $C_{k+1}$.

The following compactness result is a Lagrangian version of a result of Bourgeois [20], which technically seems to be new but is a combination of already established cases. (Bourgeois et al. [20] does not treat curves with boundary or treed disks, while Abbas [1] treats, to some extent, curves with boundary but not the Morse-Bott limit.)

Theorem 5.7. Let $P = (P_t)$ be a coherent collection of perturbation data for breaking disks. Let $\tau_\nu \to \infty$ be a sequence of neck lengths with $0 < \epsilon_0 \leq \epsilon_\nu \to 0$ a sequence of perturbation parameters such that $\tau_\nu \epsilon_\nu \to \infty$. Any sequence of adapted stable maps $u_\nu : C_\nu \to X^\tau_\nu$ holomorphic with respect to $J_{\nu}$ with bounded energy converges, after passing to a subsequence, to a stable adapted broken map $u : C \to X$ with the same index. The proof of the compactness result Theorem 5.7 is similar to standard Gromov compactness, with the exception that because the Hamiltonian perturbation is vanishing in the limit, the annulus lemma fails so that the bubbles appearing in the limit need not connect. Instead, the bubbles are connected by Morse gradient segments. We first prove the following lemma.

Lemma 5.8. (Convergence on cylinders to Morse trajectories) Let $J$ be an almost complex structure on $X$, $H$ a smooth function on $X$, $C_\nu := [-s_\nu, s_\nu] \times S^1$ a sequence of cylinders with $s_\nu \to \infty$, $\rho_\nu \to 0$ a sequence of positive real numbers, and $u_\nu : C_\nu \to Z \times \mathbb{R}$ a sequence of $J_\nu, \pi_Y H$-holomorphic maps (see (58)) such that $\rho_\nu \to 0$ has sup $|\pi_Y du_\nu|/\rho_\nu$ bounded with respect to the standard metric on the domain $C_\nu$. Then there exists a cylinder $C := [-s_0, s_0] \times S^1$ and a smooth map $u_Y : C \to Y$ such $\partial_s u = 0$, $\partial_t u = -\text{grad}_H(u)$ and $u_\nu$ converges to a lift of $u_Y$ uniformly in all derivatives on compact subsets of the interior.

Proof. We write $u_\nu = (\phi_\nu, v_\nu)$ in terms of the components $X = \mathbb{R} \times Z$. Because the domain is a sequence of annuli converging to a trivial cylinder, the map $\phi_\nu$ converges to $\phi(s) = \mu s$ and $\phi(t)$ converges to the Reeb orbit $\gamma(t)$ with multiplicity $\mu$. In cylindrical coordinates the map $u_\nu$ is given on $A_{w, \nu}$ satisfies

$$\partial_s u_\nu + J_{\rho_\nu H} \partial_t u_\nu = 0,$$

hence $\partial_s \pi_Y u_\nu + J_Y (\partial_t \pi_Y u_\nu - \alpha(\partial_t u) \rho_\nu H(\pi_Y u)) = 0$

where $v_H$ is the perturbed Reeb vector field of (58). After rescaling we obtain a sequence of cylinders $\rho_\nu C_\nu := [-\rho_\nu s_\nu, \rho_\nu s_\nu] \times \rho_\nu S^1$ and $(J_Y, H)$-holomorphic maps $\pi_Y u_\nu : \rho_\nu C_\nu \to Y$. After passing to the universal cover we obtain $(J_Y, H)$-holomorphic maps $\tilde{u}_\nu : [-\rho_\nu s_\nu, \rho_\nu s_\nu] \times \mathbb{R} \to Y$ with bounded derivative of $\pi_Y \tilde{u}_\nu$, which must converge to a $(J_Y, H)$-holomorphic map $\tilde{u}_Y : [-s_0, s_0] \times \mathbb{R} \to Y$ in the limit, uniformly in all derivatives on compact subsets of the interior. Since $\tilde{u}_\nu$ is $\rho_\nu$ periodic in $t$, the limit $\tilde{u}_Y := \pi_Y \tilde{u}$ must be constant in $s$ and $\tilde{u}$ is a lift of $\tilde{u}_Y$. ☐

Proof of Theorem. We begin with some historical remarks. In the case of separating hypersurfaces of contact type, almost complex structures preserving the horizontal
and vertical subspaces, and no Lagrangian boundary conditions, a proof is sketched in Bourgeois [20]. Recall that the sft compactness result (again for separating hypersurfaces of contact type) is extended to the case of Lagrangian boundary conditions in Abbas [1].

We first construct the domain of the limiting map. As in the references above and Theorem 4.5 we equip the surface parts $S_\nu$ with hyperbolic metrics with injectivity radii $r_\nu : S_\nu \to \mathbb{R}_{>0}$, and add marked points until there exist bounds on the first derivatives $\sup |du_\nu|/r_\nu$. The argument of (4.5), [1] shows that there exists a stable limit $S$ of the curve parts $S_\nu$ such that $u_\nu|S_\nu$ converges uniformly in all derivatives on compact subsets of complement of the the nodes to some smooth map $u : S^\times \to X$, where $S^\times$ is the complement of the nodes. However, the bubbles may not “connect” to a continuous map because of the failure of the annulus lemma. In particular, the domain captures all bubbling sequences $z_\nu \in S_\nu$ such that for some sequence local coordinate near $z_\nu$ with radius of definition bounded from below, $|d\pi_y u_\nu(z_\nu)|$ is bounded below.

We claim that the various surface components of the limit are connected by gradient trajectories, as in Bourgeois [20]. Choose identifications of $S^\times$ with the complements $S^\times_\nu$ of a finite set of circles in $S_\nu$ corresponding to the nodes; such an identification may be defined by considering $S_\nu$ as fibers of a Lefschetz fibration with central fiber $S$ and choosing a connection on the complement of the singular set. Choose small disks $B^\pm_w$ on either side of the node and consider the cylinder $S^\times_\nu \setminus B_w$ between $B^\pm_w$. The union of $B^\pm_w$ with the circle corresponding to the node is a finite cylinder (annulus) denoted $A_{w,\nu} \subset S_\nu$. By the collar lemma (see e.g. [1, Theorem 1.78]) the annulus may be realized as a hyperbolic surface as a subset of the upper half plane $\mathbb{H}/(z \sim \rho_0z)$ satisfying an inequality of the form $\theta^- \leq \arg(z)/\pi \leq \theta^+$ where $\theta^- \to 0, \theta^+ \to 1$, and the convergence rate is sufficiently slow compared to the convergence of $\rho_0$ to zero so that the lengths of the paths $\tau \exp(i\theta^-\tau), \tau \in [1, \rho_0]$ converges to zero. In the cylindrical metric on the finite cylinders, the bound on $c_\nu = \sup |du_\nu|/r_\nu$ still holds, since the conformal rescaling factor $\|\theta^-\|_{\mathrm{cyl}}/\|\theta^-\|_{\mathrm{hyp}}$ between the two metrics (subscripts denoting cylindrical and hyperbolic metrics respectively) lies in $[\tau_\nu/2\rho_0, 2\tau_\nu/\rho_0]$, see the discussion in [1, p. 230] for comparison of the metric and [1, Lemma 1.83] for a formula for the injectivity radius. In the cylindrical metric the injectivity radius $r_\nu = \rho_0$ is a constant.

We distinguish the following cases depending on how the blow-up rate of the first derivative compares to the rate at which the radii of the cylinders converges to zero. If $c_\nu r_\nu$ converges to infinity (after passing to a subsequence) then since $du_\nu$ is bounded, the cylinders must converge to a critical point of $H$, and so the maps $u_\nu$ necessarily converge to a constant. Choose a $\zeta$-neighborhood $\mathrm{crit}(H)_\zeta$ of $\mathrm{crit}(H)$ with $\zeta$ sufficiently small so that the balls around each critical point are disjoint, let $A'_{w,\nu,\zeta} = S^1 u^{-1}_\nu(\mathrm{crit}(H)_\zeta)$ denote the $t$-flow-out of the inverse image of the complement, and $A''_{w,\nu,\zeta} = A_{w,\nu} - A'_{w,\nu,\zeta}$. Hence the decomposition $A_{w,\nu} = A'_{w,\nu,\zeta} \cup A''_{w,\nu,\zeta}$ is a decomposition of the neck region $A_{w,\nu}$ corresponding to the node $w$ into further finite cylinders $A'_{w,\nu,\zeta,k}, A''_{w,\nu,\zeta,l}$ (possibly infinite in number) so
that $A'_{w,\nu,\zeta,k}$ map to neighborhoods of the critical locus $\text{crit}(H)$ and on $A''_{w,\nu,\zeta,l}$ the derivative $dH$ is bounded from below.

![Figure 13. Limit of cylinders with small derivative](image)

Hence if $c_\nu$ denotes the supremum of the first derivative on $A''_{w,\nu,\zeta,l}$, then $c_\nu \epsilon_\nu$ is bounded. We rescale by $\epsilon_\nu$ so that the first derivative of the rescaled map $u_\nu(\epsilon_\nu s, \epsilon_\nu t)$ is bounded (possibly converging to zero). By Lemma 5.8, in the limit $\nu \to \infty$ one obtains for each sequence $A'_{w,\nu,\zeta,l}$ a subsequence that converges to a piece $u : [-s_l, s_l] \times S^1 \to Y$ constant in $t$ and a gradient trajectory of $H$, possibly of finite or infinite length. Any trajectory connecting distinct components of $\text{crit}(H)_\zeta$ must decrease the value of $H$, and it follows that the numbers of pieces $A''_{w,\nu,\zeta,l}$ with non-trivial limits (and so connecting different components of $\text{crit}(H)_\zeta$) must be finite. The same argument with $\zeta$ replaced with $\zeta/2^m$ further gives limiting Morse trajectories which connect components of $\text{crit}(H)_{\zeta/2^m}$. A standard diagonal argument gives a sequence which connects $\text{crit}(H)_{\zeta/2^m}$ for any $m$, hence a broken Morse trajectory. See Figure 13 where the regions $\text{crit}(H)_\zeta$ are the shaded balls and $A'_{w,\nu,\zeta,l}$ are the shaded regions on the cylinder.

We claim that the limit is adapted and has uncrowded type. By assumption, the Morse trajectories are disjoint from the stabilizing divisor, so there are no ghost components containing interior markings attached to Morse trajectories. Thus any maximal ghost component with at least two interior markings is connected to at least one non-constant component. Either it is connected to a single component, then the adjacent non-constant component has intersection multiplicity two at the node, by conservation of intersection multiplicity under degeneration. Since this is a codimension two condition, it cannot occur. Similarly if the ghost bubble is adjacent to two non-constant components then by removing the ghost bubble we obtain a configuration with a node mapping to the divisor, which is again a codimension two condition and so does not occur.

\[ \square \]

**Theorem 5.9.** Suppose that $u : C \to X$ is a regular broken map with $k$ levels. Then there exists $\delta_0 > 0$ such that for each gluing parameter $\delta \in (0, \delta_0)$ there exists an unbroken map $u_\delta : C_\delta \to X$, with the property that $u_\delta$ depends smoothly on $\delta$ and $\lim_{\delta \to 0}[u_\delta] = [u]$.

We prove Theorem 5.9 in Section 5.3. In fact, we only use the following weaker version for Theorem 5.9 whose proof is easier:
**Theorem 5.10.** Suppose that a sequence of holomorphic weighted treed disks \( u_\nu : C_\nu \to X \) converges to a regular broken weighted treed disk \( u : C \to X \) as \( \nu \to \infty \). Then

(a) the indices of \( u_\nu \) and \( u \) matching for \( \nu \) sufficiently large;

(b) any sequence of elements in the kernel resp. cokernel of \( D u_\nu \) converges with norm one (after passing to a subsequence) to a non-zero element in the kernel resp. cokernel of \( D u \).

**Proof.** Equality of indices follows from the index formula in Proposition 4.17. Identification of the kernels and cokernels is a standard argument using elliptic regularity and a gluing theorem for elements of the kernel and cokernel, see for example [103, Theorem 2.4.5].

**Corollary 5.11.** Suppose that a collection \( \mathcal{P} \) of regular perturbations for broken maps have been chosen. For each \( E > 0 \), there exists \( \rho_0 \) such that if \( \rho > \rho_0 \) then \( \mathcal{M}^{<E}(X, L, \mathcal{P}_\rho) \) is independent of \( \rho \) and every element is regular.

**Proof.** By Theorem 5.7, any sequence in \( \mathcal{M}(X_\nu, L, x) \) with bounded energy converges to an element of \( \mathcal{M}(X_0, L, x) \). If \( \xi_\nu \) is a sequence of elements in the cokernel of \( D u_\nu \) with norm one then after passing to a subsequence one obtains an element in the cokernel of the limiting linearized operator \( D u \) by the second part of Theorem 5.10. Since \( D u \) is surjective by assumption, \( D u_\nu \) is surjective for sufficiently large \( \nu \) as well.

**Corollary 5.12.** The higher composition maps \( \mu^{n,t} \) of the \( A_\infty \) algebra \( \widehat{CF}(X, L, \mathcal{P}_\rho) \) have a limit as \( t \to \infty \):

\[
\mu^{n,\infty} := \lim_{t \to \infty} \mu^{n,t}.
\]

The limit \( \widehat{CF}(X, L, \mathcal{P}_\infty) \) is convergent-\( A_\infty \)-homotopy equivalent to \( \widehat{CF}(X, L, \mathcal{P}_\rho) \) for any finite \( t \).

**Proof.** Since for any energy bound \( E \), the terms in \( \mu^{n,\rho} \) of order at most \( q^E \) are independent of \( t \) for \( \rho > \rho_0 \) by Lemma 5.11. The \( A_\infty \) axiom follows from the \( A_\infty \) axiom for \( \mu^{n,\rho} \) for each bound \( E \). Convergent homotopy equivalence follows from the fact that for any energy \( E \) and any path \( J^t \) there are at most finitely many homotopy classes of holomorphic disks and spheres with energy at most \( E \); the same holds for sufficiently small \( C^2 \) perturbations.

**Corollary 5.13.** For any energy bound \( E > 0 \), there exists \( \rho_0 > 0 \) such that if \( \rho > \rho_0 \) then there is bijection between \( \mathcal{M}^{<E}(X, L, D, \mathcal{P}_\rho, l, x) \) and \( \mathcal{M}^{<E}(X, L, D, l, x) \).

**Proof.** By Theorem 5.9 and the compactness result Theorem 5.7.

**Theorem 5.14.** Suppose that \( \mathcal{P}_\rho \) converges to a regular coherent stabilizing perturbation system \( \mathcal{P} \) for broken treed disks as above. Then the limit \( \lim_{\rho \to \infty} \widehat{CF}(X, L, \mathcal{P}_\rho) \) is equal to the broken Fukaya algebra \( \widehat{CF}(X, L) \).
Proof. By Theorem 5.13, there is a bijection between the moduli spaces defining the structure coefficients of the Fukaya algebras $\widehat{CF}(X, L, P_{\rho})$ for $\rho$ sufficiently large and $\widehat{CF}(X, L)$. The bijection preserves the area $A([u])$ of the map as well as the homology class $[\partial u] \in H_1(L)$ of the restriction of the map to the boundary, hence preserves the holonomies of the flat connection on the brane around the boundary of the disk. It follows from the gluing results in e.g. [103] that the bijection is orientation preserving. (One may treat the Morse trajectory as a cylinder of Chern number zero as in (65).)

We now specialize to the case that $X_-$ is the toric piece near the exceptional locus of a reverse flip as in Theorem 4.24.

**Proposition 5.15.** Let $X$ be obtained by a reverse flip or blow-up, $L \subset X$ a regular Lagrangian near the exceptional locus. Let $P_{\rho}$ be as in Theorem 5.14. The limit $\lim_{\rho \to \infty} \widehat{CF}(X, L, P_{\rho})$, hence $\widehat{CF}(X, L, P_{\rho})$ for any finite $\rho$, is weakly unobstructed.

**Proof.** By Theorems 4.24 and 5.14.

**Corollary 5.16.** Let $X$ be obtained by a reverse flip or blow-up, $L \subset X$ a regular Lagrangian near the exceptional locus. For any perturbation system $P$, there exists a weakly bounding cochain and local system so that $\widehat{CF}(X, L, P)$ is weakly unobstructed and has non-trivial cohomology.

**Proof.** Let $X_-$ be the toric piece from Theorem 4.24. By Theorem 4.24, $L$ considered as a Lagrangian in the broken symplectic manifold $(X_-, X_+)$ is weakly unobstructed with non-trivial Floer cohomology. By Proposition 5.15, the same holds for $L$ considered as a Lagrangian submanifold of $X$.

**Remark 5.17.** (Standard Lagrangians in Darboux charts are weakly unobstructed) If $x$ is an arbitrary rational compact symplectic manifold, and $(q_1, p_1, \ldots, q_n, p_n)$ are Darboux coordinates near a point $x \in X$, the same argument shows that torus orbits $L = \{(p_{2j}^2 + q_j^2) = c_j, j = 1, \ldots, n\}$ for some small constants $c_1, \ldots, c_n$, have unobstructed but trivial Floer cohomology.

### 5.3. Getting back together, redux.

The existence of a tubular neighborhood for broken strata requires a gluing theorem generalizing that in Section 5.3, which is similar but slightly different from the gluing theorem for Morse trajectories and holomorphic curves in Bourgeois-Oancea [18]. Given a broken curve $C$ with, for simplicity, levels $C_+, C_-$ and no intermediate Morse trajectory, a standard gluing procedure creates, for any small **gluing parameter** $\delta \in \mathbb{C}$, an unbroken curve $C_\delta$ obtained by removing small disks around the attaching points of the trajectory and gluing in using a map given in local coordinates by $z \mapsto \delta/z$.

**Theorem 5.18.** Suppose that $u : C \to X$ is a regular adapted broken map of type $\Gamma$ with a single gradient trajectory for $F_\Gamma$. Let $\tilde{\Gamma}$ denote the combinatorial type of unbroken map obtained by replacing the gradient segment with a cylinder. For any function $\epsilon(\tau)$ with $\epsilon(\tau) \to 0$ as $\tau \to \infty$, let $J_{\Gamma}$ denote a Hamiltonian perturbation on the neck as in (61). There exists an $\tau_0 > 0$ and for $\tau > \tau_0$ a family of unbroken
curves $C_\tau$ and unbroken maps $u_x : C_\tau \to X, \tau > \tau_0$ of type $\tilde{\Gamma}$ that are $(J_{\tilde{\Gamma}}, F_{\Gamma})$-holomorphic and satisfying $\lim_{\delta \to 0} [u_\delta] = [u]$.

**Proof.** The proof is similar to the previous result Theorem 4.25; however, now the equation to be solved has a Hamiltonian perturbation on the neck region. The fundamental new issue is gluing a Morse trajectory to a pseudoholomorphic map. To illustrate the new issue, we assume that $X = \mathbb{R} \times Z$ is cylindrical, the broken map has a single level and two sublevels joined by a Morse trajectory of length one.

First we lift the Morse trajectory to the cylinder. We suppose the broken Morse segment with the Morse cylinder produces a map whose domain $\tilde{\Gamma}$ and unbroken maps $\tilde{\Gamma}$ is (66) $\tilde{\Gamma}$

Denote by Lemma 5.19. Suppose that $u_T : [-1, 1] \to Y$, for simplicity. Choose a map $\phi_e : \mathbb{R} \to [-1, 1]$ satisfying

\begin{equation}
\phi_e(s) = \begin{cases} 
1 & s \geq 1/\epsilon \\
\epsilon s & -1/\epsilon + 1 \leq s \leq 1/\epsilon - 1 \\
-1 & s \leq -1/\epsilon 
\end{cases}
\end{equation}

(63)

Given a time-dependent Morse function on $[-1, 1] \times X$, the pull-back of the gradient flow equation to $\mathbb{R}$ is

\begin{equation}
\frac{d}{ds} u(s) = (\phi'_e) \text{grad} H(\phi_e(s), u(s)).
\end{equation}

(64)

Define the Morse cylinder

\begin{equation}
\tilde{u}_T(s, t) = u_T(\phi_e(s)), \quad (s, t) \in \mathbb{R} \times S^1.
\end{equation}

(65)

The Morse cylinder satisfies the equation

\begin{equation} 
\partial_s \tilde{u}_T(s, t) + J_0(\tilde{u}_T(s, t) - (\phi'_e)F_{\Gamma}^\#) = 0.
\end{equation}

(67)

The map $\tilde{u}_T$ is $(J, (\phi'_e)F_{\Gamma}^\#)$-holomorphic and equal to $u_T(\epsilon s)$ for $s \in (-1/\epsilon + 1, 1/\epsilon - 1)$.

Replace the Morse segment with the Morse cylinder produces a map whose domain is the disjoint union of three curves (the first with boundary and segments attached.) Let $\tilde{u}$ denote the map with disjoint domain

\begin{equation}
\tilde{\Gamma} = C_- \sqcup (\mathbb{R} \times S^1) \sqcup C_+.
\end{equation}

(66)

Let $\Omega(\tilde{\Gamma}, \tilde{u}^*TX)_{1,p,\lambda}$ denote the space of maps of Sobolev class $1, p, \lambda$ in the sense of (44), satisfying matching condition at the ends. There is a space of Sobolev class $W_{1,p}$ maps near $\tilde{u}$ that are $J$-holomorphic on $C_{\pm}$ and $(J, (\phi_k^0)H_{\Gamma}^\#)$-holomorphic on $\mathbb{R} \times S^1$, described as the zeros of a map

(67)

$$
F_\delta^D : \mathcal{M}_1^\delta \times \Omega^0(\tilde{\Gamma}, \tilde{u}^*TX)_{1,p} \to \Omega^{0,1}(\tilde{\Gamma}, \tilde{u}^*TX)_{1,p,\lambda} \oplus \bigoplus_{i=1}^n T_{u_{\lambda(z_i)}} X / T_{\alpha(z_i)} D
$$

(67)

where in the last factor $\pi_D$ is projection onto $D$ using a local coordinate chart. We denote by $D_{\tilde{u}}$ the associated linearized operator.

**Lemma 5.19.** Suppose that $D_{\tilde{u}}$ is surjective. There exists $\delta_0 > 0$ such that for $\delta < \delta_0$, the operator $D_{\tilde{u}}$ is surjective.
Proof. Our argument is similar to but slightly different from that in Bourgeois-Oancea [18, Propositions 4.8, 4.9]. First we apply a Fourier analysis on the neck piece. On the neck piece \( \mathbb{R} \times S^1 \) in \( \tilde{C} \) from (66), the operator \( D_{\tilde{u}} \) is \( S^1 \)-invariant and the kernel and cokernel break into weight spaces for the \( S^1 \)-action:

\[
\text{coker}(D_{\tilde{u}}) = \bigoplus_{m \in \mathbb{Z}} \text{coker}(D_{\tilde{u}})_m
\]

where \( \text{coker}(D_{\tilde{u}})_m \) is spanned by elements of the form \( \eta(s) \exp(2\pi imt)(ds + i dt) \).

If \( \xi(\delta_{\nu}) \) is a sequence of sections in \( \text{coker}(D_{\tilde{u}})_m \) with \( \delta_{\nu} \to 0 \) then, via elliptic regularity, a subsequence of \( \xi(\delta_{\nu}) \) converges to a non-zero element of \( \text{coker}(\nabla_s + J\nabla_t)_m \). The latter is empty, hence no such sequence exists. It follows that for \( \delta \) sufficiently small, \( \text{coker}(D_{\tilde{u}})_m = 0 \) unless \( m = 0 \).

By the previous paragraph, to show surjectivity of the linearized operator it suffices to consider a triple \( \eta = (\eta_-, \eta_0, \eta_+) \) of one-forms on \( C_-, \mathbb{R} \times S^1, C_+ \) respectively with the property that \( \eta_0 \) is \( S^1 \)-invariant. We wish to construct \( \xi \) such that \( D_{\tilde{u}}\xi = \eta \). By integrating we may find \( \xi_0^\pm \in \Omega^0(\mathbb{R} \times S^1, \tilde{u}^*TX) \) such that

\[
\lim_{s \to \pm \infty} \xi_0^\pm(s, t) = 0, \quad D_{\tilde{u}}\xi_0^\pm|_{U} = \eta_0|_{U}
\]

for some open neighborhood \( U \) of \( [\pm \infty, \pm(|\ln(\delta)| - 1)] \). Subtracting \( D_{\tilde{u}}\xi_0^\pm \) from \( \eta_0 \) we may assume that \( \eta_0 \) is supported in the region on which \( \phi_\epsilon \) is a diffeomorphism. Then pushing-forward \( \eta \) to \((-1, 1) \) we obtain a one-form on \( C \) with values in \( u^*TX \). By surjectivity of \( D_{\tilde{u}} \), there exists a triple \( \xi = (\xi_-, \xi_0, \xi_+) \) such that \( D_{\tilde{u}}\xi = \eta \). Pulling back the one-dimensional piece to \( \mathbb{R} \times S^1 \) under \( \phi_\epsilon \) we obtain a triple \( \xi = (\xi_-, \tilde{\xi}_0, \xi_+) \) with \( D_{\tilde{u}}\xi = \eta \), as desired. \( \square \)

Once the gradient segment has been replaced with an infinite cylinder, the construction of a family of solutions converging to the broken solution is much the same as the previous gluing result in Section 4.6. An approximate solution is given by gluing together \( u_\pm \) and \( \tilde{u}^T \) using a cutoff function. We assume that \( \beta \in C^\infty(\mathbb{R}) \) has the property that \( \beta(s) = 0, s \leq 0 \) and \( \beta(s) = 1, s \geq 1 \) as in (26). Let \( y \in Y \) be the evaluation of \( u_\pm \) at the node and \( \gamma(t) \) the Reeb orbit in the fiber over \( y \) in \( Z \). We form a domain curve \( C_\gamma \) by gluing together \( C_-^\circ, C_+^\circ, \mathbb{R} \times S^1 \) so that the cylindrical coordinates \( s, t \) on the gluing region are such that so that \( u_\pm \) considered locally as maps to \( X^0 \) are asymptotic to \( \gamma(t) \). We write

\[
u_\pm(s, t) = \exp_{\mu, \gamma_\pm(t)}(\xi_\pm(s, t)), \quad \tilde{u}^T = \exp_{\mu, \gamma_\pm(t)}(\tilde{\xi}^T(s, t)).
\]

Recall that \( \epsilon(\delta) \) is a number much smaller than \( \delta \). We assume that

\[
\lim_{\delta \to 0} \epsilon(\delta)|\ln(\delta)| = 0
\]
so that the intervals \([1/\epsilon - |\ln(\delta)|/2, 1/\epsilon + |\ln(\delta)|/2]\) are disjoint and far away from 0.

\[
(69) 
\begin{align*}
    u^\text{pre}_\delta(s, t) = \\
    \begin{cases}
        u_-(s + |\ln(\delta)|/2 + 1/\epsilon, t) & s \leq -|\ln(\delta)|/2 - 1/\epsilon \\
        \tilde{u}^T(s, t) & |\ln(\delta)|/2 - 1/\epsilon \leq s \leq -|\ln(\delta)|/2 + 1/\epsilon \\
        u_+(s - |\ln(\delta)|/2 - 1/\epsilon, t) & s \geq |\ln(\delta)|/2 + 1/\epsilon.
    \end{cases}
\end{align*}
\]

There are two gluing regions missing from the description above, corresponding to the values \(|s| \in [1/\epsilon - |\ln(\delta)|/2, 1/\epsilon + |\ln(\delta)|/2]\). In these regions, one interpolates between the two maps on either side using the cutoff function and geodesic exponentiation. On the two regions corresponding to gluing a holomorphic map to a Morse trajectory define

\[
(70) 
\begin{align*}
    u^\text{pre}_\delta(s, t) = \exp(\mu s, \gamma(t)) (\beta(-s + |\ln(\delta)|/2 + 1/\epsilon) \zeta_-(s + |\ln(\delta)| + 1/\epsilon, z) \\
    + (s - |\ln(\delta)|/2 - 1/\epsilon) \zeta^T(s, t)).
\end{align*}
\]

Thus on the first of gluing region while on the second

\[
(71) 
\begin{align*}
    u^\text{pre}_\delta(s, t) = \exp(\mu s, \gamma(t)) (\beta(-s - |\ln(\delta)|/2) \zeta_+(s - |\ln(\delta)| - 1/\epsilon, z) \\
    + (s - |\ln(\delta)|/2 - 1/\epsilon) \zeta^T(s, t)).
\end{align*}
\]

Thus on each gluing region, \(u^\text{pre}_\delta\) is approximately equal to a \(\mu\)-fold cover of the orbit corresponding to the matching condition \(u^T(\pm1)\).

We solve for an unbroken map near the approximate solution above by the contraction mapping principle in a suitable Sobolev space of maps. Define Sobolev spaces with exponential weight on the gluing regions as follows. Define a Sobolev weight function

\[
\kappa_\lambda : C_\delta \to [0, \infty), \quad (s, t) \mapsto \beta(|\ln(\delta)|/2 + 1/\epsilon - |s|) \beta(|\ln(\delta)|/2 - 1/\epsilon + |s|) p\lambda(1 + |\ln(\delta)|/2 - |s - 1/\epsilon|).
\]

The function \(\kappa_\lambda\) takes values \(p\lambda|\ln(\delta)|/e\) in the middle of the neck region \(s = 1/\epsilon\), and 0 away from the neck regions. Given a smooth map \(u : C_\delta \to X\), element \(m \in \mathcal{M}^1_{\text{loc}}\) and a section \(\xi : C_\delta \to u^*TX\) define

\[
(72) 
\|(m, \xi)\|_{1,\mu, \lambda} := \left(\|m\|^p + \int_{C_\delta} (\|\nabla \xi\|^p + \|\xi\|^p) \exp(\kappa_\lambda) d\text{Vol}_{C_\delta}\right)^{1/p}.
\]

Let \(\Omega^0(C_\delta, u^*TX)_{1,\mu, \delta}\) be the space of \(W^{1,p}_{\text{loc}}\) sections with finite norm (72). Similarly, for a 0, 1-form \(\eta \in \Omega^{0,1}(C_\delta, u^*TX)\) define

\[
\|\eta\|_{0,\mu, \lambda} = \left(\int_{C_\delta} \|\eta\|^p \exp(\kappa_\lambda) d\text{Vol}_{C_\delta}\right)^{1/p}.
\]
As before let \( \exp_{u^\text{pre}} \) denote pointwise geodesic exponentiation and \( T_{u^\text{pre}}(\xi) \) parallel transport along \( \exp(t\xi) \). Define

\[
\mathcal{F}_\delta^D : \mathcal{M}_1^* \times \Omega^0(C_\delta, (u^\text{pre}_\delta)^*TX)_{1,p} \to \Omega^{0,1}(C_\delta, (u^\text{pre}_\delta)^*TX)_{0,p} \oplus \bigoplus_{i=1}^n T_{u(z_i)}X/T_{u(z_i)}D
\]

\[
\xi \mapsto (T_{u^\text{pre}}(\xi|S))^{-1}\mathcal{F}_{\Lambda, j(m)} \exp_{u^\text{pre}}(\xi|S),
\]

\[
\frac{d}{dt} - \text{grad}(F_T)(\xi_U), \pi_D(\exp(\xi(z_i)))_{i=1}^n
\]

Treed pseudoholomorphic maps close to \( u^\text{pre}_\delta \) correspond to zeroes of \( \mathcal{F}_\delta \). In addition, because we are working in the adapted setting, our curves \( C_\delta \) have a collection of markings \( z_1, \ldots, z_n \) and we work subject to the constraint

\[
(\exp_{u^\text{pre}}(\xi))(z_i) \in D, \quad i = 1, \ldots, n.
\]

First, as in the case of gluing without a Morse trajectory one has the failure of the approximate solution to be an exact solution:

\[
\|\mathcal{F}_\delta^D(0)\|_{0,p,\lambda} < C(\exp(-|\ln(\delta)|)(1-\lambda)/2) = C\delta^{1-\lambda}
\]

which is proved in the same way as (48), using that the cylinder \( \tilde{u}^T \) is an exact solution to the perturbed Cauchy-Riemann equation. We remark that in the more complicated case of a surface component mapping to a neck piece \( X[t]_k = \mathbb{R} \times Z \), there is an additional contribution to (74) arising from the Hamiltonian perturbation away from the gluing regions. However, the assumed estimate (68) implies that this error estimate is also exponentially small in \( |\ln(\delta)| \).

The analogs of estimates (48), (50), and (51) are then proved as in the case of gluing holomorphic curves. Given a one-form \( \eta \in \Omega^{0,1}(C_\delta, (u^\text{pre}_\delta)^*TX) \), we obtain one-forms \( \eta_-, \eta_0, \eta_+ \) using the cutoff function:

\[
\eta_\pm = \beta(\pm(s - |\ln(\delta)|/2 - 1/\epsilon))\eta, \quad \eta_0(s,t) = \eta - \eta_- - \eta_+.
\]

By assumption the linearized operator \( D_u \) for the limit \( u \) is surjective; this means that there exists a tuple \( \xi = (\xi_-, \xi_0, \xi_+) \) with

\[
D_u(\xi_-, \xi_0, \xi_+) = (\eta_-, \eta_0, \eta_+)
\]

and \( \xi \) satisfying matching conditions at the joining points of the Morse trajectory with the surface part of the domain. Now we define an element \( \xi \in \Omega^0(C_\delta, (u^\text{pre}_\delta)^*TX) \) by interpolating between the maps \( (\xi_-, \xi_0, \xi_+) \):

\[
\xi(s,t) = \beta(s - |\ln(\delta)|/2 + 1/\epsilon)\xi_-(s + |\ln(\delta)|/2 + 1/\epsilon, t)
\]

\[
+ (1 - \beta(s - |\ln(\delta)|/2 - 1/\epsilon))(1 - \beta(s - |\ln(\delta)|/2 + 1/\epsilon)\xi_0(s)
\]

\[
+ \beta(s - 1/\epsilon + |\ln(\delta)|/2)\xi_+(s - |\ln(\delta)|/2 - 1/\epsilon, t
\]

Then \( D_{u^\text{pre}}\xi - \eta \) is the sum of terms arising from differentiating the cutoff functions, each of which contributes a term of size bounded by \( C\delta^{p-\lambda} \). The uniform quadratic estimate is similar to that in (51). Applying the implicit function theorem produces
a unique solution $\xi(\delta)$ to the equation $F^P_D(m, \xi(\delta)) = 0$ for each $\delta$ with $\xi(\delta)$ in the image of the uniformly bounded right inverse, giving rise to a smooth family $\exp_{\nu_3}^{\text{pre}}(\xi(\delta))$.

The case that $u$ is a map with two levels (rather than two sublevels) is similar, but now involves three gluing regions: two where the (necessarily broken) Morse trajectory meets the surface part of the domain, and one gluing region where the Morse trajectories are glued together. The latter is standard while in the former regions the gluing estimates are identical to those above. □

References

[38] K. Fukaya. Floer homology for 3-manifolds with boundary I, 1999. unpublished manuscript.


MATHEMATICS-HILL CENTER, RUTGERS UNIVERSITY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019, U.S.A.
E-mail address: ctw@math.rutgers.edu