ABSTRACT. We use the technique of stabilizing divisors introduced by Cieliebak-Mohnke [9] to construct finite dimensional, strictly unital Fukaya algebras of compact, oriented, relatively spin Lagrangians in compact symplectic manifolds with rational symplectic classes. The homotopy type of the algebra and the moduli space of solutions to the weak Maurer-Cartan equation are shown to be independent of the choice of perturbation data. The Floer cohomology is the cohomology associated to a complex of bundles over the space of solutions to the weak Maurer-Cartan equation and is shown to be independent of the choice of perturbation data up to gauge equivalence.

1. Introduction

The *Fukaya algebra* of a Lagrangian submanifold of a symplectic manifold was introduced by Fukaya in [16] to solve the problem that the Floer cohomology of a

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Lagrangian submanifold is not always defined. When defined, the Fukaya algebra is a homotopy-associative algebra whose higher composition maps are counts of configurations involving perturbed pseudoholomorphic disks with boundary in the Lagrangian as in Figure 4. Stasheff’s homotopy-associativity equation follows from studying the boundary strata in the moduli space of treed disks as in Figure 1.

Because the moduli spaces of disks involved in the construction are usually singular, there are technical issues involved in its construction similar to those involved in the construction of virtual fundamental classes for moduli spaces of pseudoholomorphic curves. Fukaya and collaborators [17] introduced a method of solving these issues using Kuranishi structures in which one first constructs local thickenings of the moduli spaces and then introduces perturbations constructed locally. The details involved are formidable and lengthy, which makes generalizations to other theories involving pseudoholomorphic curves challenging.

In this paper we construct Fukaya algebras of Lagrangians in a compact rational symplectic manifold using a perturbation scheme that we find particularly convenient for various computations: the stabilizing divisors scheme introduced by Cieliebak-Mohnke [9]. We also incorporate Morse gradient trees introduced by Fukaya [16] and Cornea-Lalonde [12]. This construction allows us to take our Floer cochain spaces $CF(L)$ to be finite-dimensional spaces of Morse cochains over a Novikov field $\Lambda$. Similar constructions appear in Seidel [32] and Charest [7] in the exact and monotone cases respectively. An approach using polyfolds is under development by J. Li and K. Wehrheim. The structure constants for the Fukaya algebras in the stabilizing divisors approach count holomorphic disks with Lagrangian boundary conditions and Morse gradient trajectories on the Lagrangians with domain-dependent almost complex structures and Morse functions depending on the position of additional markings mapping to a stabilizing divisor. Because the additional marked points must be ordered in order to obtain a domain without automorphisms, this scheme gives a multi-valued perturbation and the resulting structure maps $\mu^n : CF(L)^{\otimes n} \to CF(L), \quad n \geq 0$

for the Fukaya algebra are defined only using rational coefficients.

For later applications, it is important that our Fukaya algebras have strict units so that disk potentials are defined. To achieve this we incorporate a slight enhancement, similar to that of homotopy units, in which perturbation systems compatible with breakings are homotoped to perturbation systems that admit forgetful maps. Then the maximum of the Morse function, if it is unique, defines a strict unit $e_L \in CF(L)$. 

![Figure 1. Moduli space of stable treed disks](image-url)
We also prove that the Fukaya algebras so constructed satisfy a natural convergence property so that the Maurer-Cartan map
\[
\mu : CF(L) \to CF(L), \quad b \mapsto \sum_{n \geq 0} \mu^n(b, \ldots, b)
\]
is well-defined. Denote the space of solutions to the weak Maurer-Cartan equation.
\[
\widetilde{MC}(L) := \mu^{-1}(\Lambda e_L) \subset CF(L)
\]
where \( e_L \in CF(L) \) is the strict unit. The Floer cohomology of a Lagrangian brane is the complex of vector bundles \((CF(L)|\widetilde{MC}(L), \partial)\). That is,
\[
HF(L) = \bigcup_{b \in \widetilde{MC}(L)} HF(L)_b, \quad HF(L)_b := \ker(\partial_b) / \text{im}(\partial_b), \quad b \in \widetilde{MC}(L)
\]
is the collection of cohomologies of the fiber. The Floer cohomology \( HF(L) \) is said to be non-vanishing if the fiber \( HF(L)_b \) is non-vanishing for some \( b \in \widetilde{MC}(L) \). The main result is the following:

**Theorem 1.1.** Let \((M, \omega)\) be a compact symplectic manifold with rational symplectic class \([\omega] \in H^2(M, \mathbb{Q})\) and \(L \subset M\) a compact embedded Lagrangian submanifold admitting a relative Spin structure. For a comeager subset of perturbation data, counting weighted treed holomorphic disks defines an convergent \(A_\infty\) structure with strict unit
\[
\mu^n : CF(L)^{\otimes n} \to CF(L), \quad n \geq 0 \quad e_L \in CF(L)
\]
that is independent of all choices up to convergent strictly-unital \(A_\infty\) homotopy. Furthermore, \( HF(L) \) is independent of all choices up to gauge equivalence (to be explained below).

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### 2. Fukaya algebras

The vector space underlying the Fukaya algebra of a Lagrangian is meant to be some version of cochains on the Lagrangian.\(^1\) Here the underlying vector space will be that of Morse cochains. Counting holomorphic treed disks will define a sequence of higher composition maps satisfying a homotopy-associativity relations introduced by Stasheff. In order to obtain strict units, a weighted treed disks extension of the usual construction of Floer cohomology is required that incorporates a homotopy of the perturbation data to one for which certain forgetful maps exist.

\(^1\)Or perhaps, cochains on the loop space of the Lagrangian.
2.1. $A_\infty$ algebras. The theory of homotopy-associative algebras was introduced by Stasheff [33] in order to capture algebraic structures on the space of cochains on loop spaces. Let $g > 0$ be an even integer. A $\mathbb{Z}_g$-graded $A_\infty$ algebra consists of a $\mathbb{Z}_g$-graded vector space $A$ together with for each $d \geq 0$ a multilinear composition map

$$\mu^d : A^\otimes d \rightarrow A[2 - d]$$

satisfying the $A_\infty$-associativity equations

$$0 = \sum_{n,m \geq 0 \atop n+m \leq d} (-1)^{n+m+\sum_{i=1}^n |a_i|} \mu^{d-m+1}(a_1, \ldots, a_n,$$

$$\mu^{m}(a_{n+1}, \ldots, a_{n+m}), a_{n+m+1}, \ldots, a_d)$$

for any tuple of homogeneous elements $a_1, \ldots, a_d$ with degrees $|a_1|, \ldots, |a_d| \in \mathbb{Z}_g$. The signs are the shifted Koszul signs, that is, the Koszul signs for the shifted grading in which the structure maps have degree one as in Kontsevich-Soibelman [23]. The notation $[2 - d]$ denotes a degree shift by $2 - d$, so that $\mu^1$ has degree 1, $\mu^2$ has degree 0 etc. The element $\mu^0(1) \in A$ (where 1 $\in \Lambda$ is the unit) is called the curvature of the algebra. The $A_\infty$ algebra $A$ is flat if the curvature vanishes. A strict unit for $A$ is an element $e_A \in A$ such that

$$\mu^2(e_A, a) = a = (-1)^{|a|}\mu^2(a, e_A), \quad \mu^n(\ldots, e_A, \ldots) = 0, \forall n \neq 2.$$ 

A strictly unital $A_\infty$ algebra is an $A_\infty$ algebra equipped with a strict unit. The cohomology of a flat $A_\infty$ algebra $A$ is defined by

$$H(\mu^1) = \frac{\ker(\mu^1)}{\text{im}(\mu^1)}.$$ 

The algebra structure on $H(\mu^1)$ is given by

$$[a_1a_2] = (-1)^{|a_1|}\mu^2(a_1, a_2).$$

If $e_A$ is a strict unit for $A$ then $[e_A]$ is a unit for $H(\mu^1)$. For certain curved $A_\infty$ algebras we give a construction of a cohomology complex over the space of solutions to the weak Maurer-Cartan in Section 2.7 below.

Let $A$ be an $A_\infty$ algebra without units over a field $\Lambda$ and let $e_A, \hat{e}_A$ be formal symbols. By a homotopy unit we mean an $A_\infty$ structure on the direct sum

$$\hat{A} = A \oplus \Lambda e_A \oplus \Lambda \hat{e}_A$$

extending the given structure on $A$ so that $e_A$ is a strict unit, and $e_A \in \mu^1(\hat{e}_A) + A$. That is, up to cohomology $e_A$ lies in the subalgebra $A$.

2.2. Associahedra. The combinatorics of the definition above is closely related to a sequence of spaces introduced by Stasheff under the name associahedra. We will use several related constructions of these spaces. The first of these is the moduli spaces of metric trees. A tree is a connected, cycle-free planar graph

$$T = (\text{Edge}(T), \text{Vert}(T))$$
where Vert($T$) is the set of vertices and Edge($T$) is the set of edges. We introduce the following notation for the edges:

(a) if Vert($T$) is non-empty, then the set of edges Edge($T$) consists of
   (i) combinatorially finite edges Edge$_f(T)$ connecting two vertices and
   (ii) semi-infinite edges Edge$_i(T)$ with a single endpoint, or
(b) if Vert($T$) is empty, then $T$ has one infinite edge and we denote by Edge$_i(T)$ its two ends.

From Edge$_i(T)$, an open endpoint is distinguished as the root of the tree, the others being referred to as its leaves.

A moduli space of trees is obtained by allowing the finite edges to acquire lengths. A metric tree is a pair $T = (T, \ell)$ consisting of a tree $T$ equipped with a metric $\ell$. By definition a metric is labelling

$$\ell : \text{Edge}_f(T) \to [0, \infty]$$

of its combinatorially finite edges by elements of $[0, \infty]$ called lengths. An equivalence relation on metric trees is defined by collapsing edges of length zero: Given a tree with an edge of length zero, removing the edge and identifying its head and tail gives an equivalent metric tree.

The metric trees with edges of infinite length are often thought of as broken metric trees. More precisely, a broken metric tree is obtained from a finite collection of metric trees by gluing roots to leaves. Given two metric trees $T_1, T_2$ and semi-infinite root edge $e_2 \in \text{Edge}_i(T_2)$ and leaf edge $e_1 \in \text{Edge}_i(T_2)$, let $T_1$ resp. $T_2$ denote the space obtained by adding a point $\infty_2$ resp. $\infty_1$ at the open end of $e_2$ resp. $e_1$. The space

$$T := T_1 \cup_{\infty_1 \sim \infty_2} T_2$$

is a broken metric tree, the point $\infty_1 \sim \infty_2$ being called a breaking.

![Figure 2. Creating a broken tree](image)

In general, broken metric trees are obtained by repeating this process in such a way that the resulting space is connected and has no non-contractible cycles. An equivalence relation on broken metric trees is defined by adding a breaking on edges of infinite length and the latter description will then be preferred. If a combinatorially finite edge has infinite length then one attaches an additional positive integer to that edge indicating its number of breakings, see [7].
In order to obtain a Hausdorff moduli space of broken trees a stability condition is imposed. A broken metric tree is stable if it has no infinite edge. The moduli space of stable metric trees is a finite cell complex studied in, for example, Boardman-Vogt [5]; this is the first realization of Stasheff’s associahedron as a moduli space of geometric objects. However, the natural cell structure on this moduli space is a refinement of the canonical cell structure on the associahedra.

A realization of the associahedron that reproduces the canonical cell structure involves nodal disks with boundary markings. A nodal disk with a single boundary node is a topological space $S$ obtained from a disjoint union of holomorphic disks $S_1, S_2$ by identifying pairs of boundary points $w_{12} \in S_1, w_{21} \in S_2$ on the boundary of each component so that

$$S = S_1 \cup_{w_{12} \sim w_{21}} S_2.$$  

Figure 3. Creating a nodal disk

A nodal disk with a single interior node is defined similarly, except in this case $S_2$ is a holomorphic sphere and $w_{12}, w_{21}$ are points in the boundary. The image of $w_{12}, w_{21}$ in the space $S$ is the nodal point. A nodal disk with multiple nodes is obtained by repeating this process with $S_1$ a nodal disk with fewer nodes. For an integer $n \geq 0$ a nodal disk with $n + 1$ boundary markings is a nodal disk $S$ equipped with a finite ordered collection of points $x = (x_0, \ldots, x_n)$ on the boundary, disjoint from the nodes, in counterclockwise cyclic order around the boundary. An $(n + 1)$-marked nodal disk is stable if each component has at least three special (nodal or marked) points. The moduli space of $(n + 1)$-marked stable disks forms a compact cell complex, which is yet another geometric realization of the associahedron. More generally we allow interior markings in the definition of marked nodal disks, and so that converging interior markings bubble off onto sphere components. A nodal disk with interior markings arises from a nodal genus zero curve equipped with an anti-holomorphic involution with non-empty fixed point set by taking the quotient by the involution (see [7]); the components meeting the fixed point set give rise to disk components while the components disjoint from the fixed point set give rise to sphere components in the quotient. Similarly, markings in the fixed point set give rise to boundary markings while markings disjoint from the fixed point set give rise to interior markings.

A combination of the above constructions involves both trees and disks, as in Oh [28], Cornea-Lalonde [12], Biran-Cornea [4], and Seidel [32].

Definition 2.1. A treed disk $C$ is a triple $(T, S, o)$ consisting of
(a) a broken metric tree
\[ T = (T, \ell); \]
(b) a collection
\[ S = (S_v, \Xi_v, \Xi_v)_{v \in \text{Vert}(T)} \]
of (boundary and interior) marked nodal disks for each vertex \( v \) of \( T \), with
number of boundary markings equal to the valence of \( v \), and
(c) an ordering \( o \) of the set of interior markings
\[ \Xi = \bigcup_v \Xi_v \in \text{int}(S). \]

Denote the interior markings \( z_1, \ldots, z_m \) using this ordering.

The \textit{topological realization} of a treed disk \( C \) is obtained by removing the vertices
from \( T \) and gluing in the nodal disks by attaching the boundary markings \( \Xi_v \) to the
edges of the tree meeting \( v \). A treed disk \( C = (T, S, o) \) is \textit{stable} iff

(a) the tree \( T \) is stable; that is, each valence is at least three;
(b) each (nodal) disk \( S_v, v \in \text{Vert}(T) \) is stable; that is, each disk contains at least
three special boundary points or one marking in the interior and at least one
special boundary point.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{treed_disk.png}
\caption{A treed disk with three disk components and one sphere component}
\end{figure}

See Figure 4. An \textit{isomorphism} of treed disks is an equivalence of broken metric
trees together with an isomorphism of collections of nodal disks preserving the or-
dering of the interior markings. In particular, a treed disk with a boundary node
will be equivalent to the same treed disk where an edge of length zero is substituted
to the latter.

In order to obtain Fukaya algebras with strict units, we attach additional parameters
to certain of the semi-infinite edges called \textit{weightings}. A \textit{weighting} of a (broken)
treed disk \( C = (T, S, o) \) consists of

(a) (Weighted, forgettable, and unforgettable edges) A partition of the semi-
infinite edges
\[ \text{Edge}^\bullet(T) \sqcup \text{Edge}^\circ(T) \sqcup \text{Edge}^*(T) = \text{Edge}_i(T) \]
into \textit{weighted} resp. \textit{forgettable} resp. \textit{unforgettable} edges, and
(b) (Weighting) a map

$$\rho : \text{Edge}_i(T) \to [0, \infty]$$

satisfying the property: each of the semi-infinite $e$ edges is assigned a weight $\rho(e)$ such that

$$\rho(e) \in \begin{cases} 
\{0\} & e \in \text{Edge}^i(T) \\
[0, \infty] & e \in \text{Edge}^s(T) \\
\{\infty\} & e \in \text{Edge}^\infty(T)
\end{cases}.$$ 

The type of semi-infinite edges (weighted, forgettable or unforgettable) and the weights, if any, are required to agree on the semi-infinite edges adjacent to a breaking.

An isomorphism of weighted treed disks is an isomorphism of treed disks that preserves the weightings.

The combinatorial type of any weighted treed disk is the corresponding tree with additional data recording which lengths resp. weights are zero or infinite. Namely if $C = (T, S, o)$ is a weighted treed disk then its combinatorial type is the graph $\Gamma = \Gamma(C)$ obtained by gluing together the combinatorial graphs $\Gamma(S_v)$ of the disks $S_v$ along the edges corresponding to the edges of $T$; and equipped with the additional data of

(a) the subsets

$$\text{Edge}^s(T) \text{ resp. } \text{Edge}^\infty(T) \text{ resp. } \text{Edge}^i(T) \subset \text{Edge}_i(T)$$

of weighted, resp. forgettable, resp. unforgettable semi-infinite edges;

(b) the subsets

$$\text{Edge}^\infty_f(T) \text{ resp. } \text{Edge}^0_f(T) \text{ resp. } \text{Edge}^{0,\infty}_f(T) \subset \text{Edge}_f(T)$$

of combinatorially finite edges of infinite resp. zero length resp. non-zero finite length;

(c) the subset

$$\text{Edge}^{\infty,\infty}(T) \text{ resp. } \text{Edge}^{0,0}(T) \subset \text{Edge}^s(T)$$

of weighted edges with infinite resp. zero weighting.

The set of vertices admits a partition

$$\text{Vert}(\Gamma) = \text{Vert}_s(\Gamma) \cup \text{Vert}_d(\Gamma)$$

corresponding to spheres or disks. The length function on edges of $T$ extends to the edges of $\Gamma$ corresponding to disk and spherical nodes, by setting the lengths of those edges to be zero. We will from now on refer to the edges of $T$ as being edges of $\Gamma$.

A weighted treed disk is stable if the underlying treed disk is stable or if there are no vertices, a single weighted leaf and an unweighted root.

The moduli spaces of stable weighted treed disks are naturally cell complexes with multiple cells of top dimension. For integers $n, m \geq 0$ denote by $\overline{M}_{n,m}$ the moduli
space of isomorphism classes of stable treed disks with \( n \) leaves and \( m \) interior markings. For each combinatorial type \( \Gamma \) denote by \( \mathcal{M}_\Gamma \subset \mathcal{M}_{n,m} \) the set of isomorphism classes of weighted stable treed disks of type \( \Gamma \) so that the moduli spaces decomposes into strata of fixed type
\[
\mathcal{M}_{n,m} = \bigcup_{\Gamma} \mathcal{M}_\Gamma.
\]
The dimension of \( \mathcal{M}_\Gamma \) is equal to the \( n + 2m - 2 \) plus the number of weighted leaves. In particular, if \( \Gamma \) has no vertices then the dimension is zero. In Figure 5 a subset of the moduli space with one interior marking is shown, where the interior marking is constrained to lie on the line half-way between the special points on the boundary.

**Remark 2.2.** (a) The moduli spaces of weighted treed disks are, in most cases, related to unweighted moduli spaces by taking products with intervals: If \( \Gamma \) has at least one vertex and \( \Gamma' \) denotes the combinatorial type of \( \Gamma \) obtained by forgetting the weightings then
\[
\mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma'} \times [0, \infty)^{\text{Edge}} \times [0, \infty]^{(\Gamma)}.
\]
(b) The exceptional case is that of a type with no vertices. If \( \Gamma \) is a type with a single weighted leaf and no vertices, the outgoing edge may be unforgettable or forgettable, assigning a weighting on the leaf will be irrelevant and then \( \mathcal{M}_\Gamma \) will be a point.

The moduli spaces admit universal curves, which admit partitions into one and two-dimensional parts. For any combinatorial type \( \Gamma \) let \( \mathcal{U}_\Gamma \) denote universal treed disk consisting of isomorphism classes of pairs \((C, z)\) where \( C \) is a treed disk of type \( \Gamma \) and \( z \) is a point in \( C \), possibly on a disk component, sphere component, or one of the edges of the tree. The map
\[
\mathcal{U}_\Gamma \twoheadrightarrow \mathcal{M}_\Gamma, \quad [C, z] \rightarrow [C]
\]
is the universal projection. Because of the stability condition, there is a natural bijection
\[
\mathcal{U}_\Gamma = \bigcup_{[C] \in \mathcal{M}_\Gamma} C.
\]
We denote by
\[
\mathcal{S}_\Gamma = \{[C = S \cup T, z] \in \mathcal{U}_\Gamma \mid z \in C\}
\]
the locus where \( z \) lies on a disk or sphere of \( C \). Denote by
\[
\mathcal{T}_\Gamma = \{[C = S \cup T, z] \in \mathcal{U}_\Gamma \mid z \in T\}
\]
the locus where \( z \) lies on an edge of \( C \). Hence
\[
\mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma
\]
and \( \mathcal{S}_\Gamma \cap \mathcal{T}_\Gamma \) is the set of points on the boundary of the disks meeting the edges of the tree. In case \( \Gamma \) has no vertices we define \( \mathcal{U}_\Gamma \) to be the real line, considered as a fiber bundle over the point \( \mathcal{M}_\Gamma \).
Later we will need local trivializations of the universal treed disk and the associated families of complex structures and metrics on the domains. For a stable combinatorial type $\Gamma$ let

$$U_i^\Gamma \to \mathcal{M}_i^\Gamma \times C, i = 1, \ldots, l$$

be a collection of local trivializations of the universal treed disk, identifying each nearby fiber with $(C, z, w)$ such that each point in the universal treed disk is contained in one of these local trivializations. The complex structures on the fibers induce a family

$$\mathcal{M}_i^\Gamma \to \mathcal{J}(S), \quad m \mapsto j(m)$$

of complex structures on the two-dimensional locus $S \subset C$.

The following operations on treed disks will be referred to in the coherence conditions on perturbation data.

**Definition 2.3.** (Morphisms of graphs) A morphism of graphs $\Upsilon : \Gamma' \to \Gamma$ is a surjective morphism of the set of vertices $\text{Vert}(\Gamma') \to \text{Vert}(\Gamma)$ obtained by combining the following elementary morphisms:

![Collapsing an edge](image)

**Figure 6.** Collapsing an edge

(a) (Cutting edges) $\Upsilon$ cuts an edge with infinite length if there exists $e \in \text{Edge}_f(\Gamma'), \ell(e) = \infty$ so that the map $\text{Vert}(\Gamma) \to \text{Vert}(\Gamma')$ on vertices is a bijection, but $\text{Edge}(\Gamma) \cong \text{Edge}(\Gamma') - \{e\} + \{e_+, e_-\}$ where $e_\pm \in \text{Edge}_i(\Gamma)$ are attached to the vertices contained in $e$. The edges $e_\pm$ are taken to be of the same type (forgettable, weighted, or unforgettable).
as $e$ and with the same weight (if weighted) resp. $e_\pm \in \text{Edge}_{i,s}(\Gamma)$. Since our curves have genus zero, $\Gamma$ is disconnected with pieces $\Gamma_-, \Gamma_+$. The edge corresponds to a broken segment and $\Gamma_-, \Gamma_+$ are types of stable treed disks. The ordering on $\text{Edge}_{i,s}(\Gamma)$ induces one on $\text{Edge}_{i,s}(\Gamma_\pm)$ by viewing the latter as a subset of the former.

(b) (Collapsing edges) $\Upsilon$ collapses an edge if the map on vertices is a bijection except for a single vertex $v' \in \text{Vert}(\Gamma)$

$$\text{Vert}(\Upsilon) : \text{Vert}(\Gamma') \to \text{Vert}(\Gamma), \quad \text{Vert}(\Upsilon)^{-1}(v') = \{v_-, v_+\}$$

that has two pre-images $v_\pm \in \text{Vert}(\Gamma')$. The vertices $v_-, v_+$ are connected by an edge $e \in \text{Edge}(\Gamma')$ so that

$$\text{Edge}(\Gamma) \cong \text{Edge}(\Gamma') - \{e\}.$$  

(c) (Making an edge length finite or non-zero) $\Upsilon$ makes an edge finite resp. non-zero if $\Gamma'$ is the same graph as $\Gamma$ and the lengths of the edges are the same except for a single edge $e$:

$$\ell|_{\text{Edge}(\Gamma') - \{e\}} = \ell'|_{\text{Edge}(\Gamma') - \{e\}}.$$

For the edge $e$ we require

$$\ell'(e) = \infty \text{ resp. } 0, \quad \ell(e) \in (0, \infty).$$

(d) (Forgetting tails) $\Upsilon : \Gamma' \to \Gamma$ forgets a tail (semi-infinite edge) and collapses edges to make the resulting combinatorial type stable. The ordering on $\text{Edge}_{i,s}(\Gamma')$ then naturally defines one on $\text{Edge}_{i,s}(\Gamma)$ viewing the latter as a subset.

(e) (Making an edge weight finite or non-zero) makes a weight finite or non-zero if $\Gamma'$ is the same graph as $\Gamma$ and the weights of the edges $\rho(e), e \in \text{Edge}(\Gamma)$ are the same except for a single edge $e$,

$$\rho|_{\text{Edge}(\Gamma') - \{e\}} = \rho'|_{\text{Edge}(\Gamma') - \{e\}}.$$

For the edge $e$ we have

$$\rho'(e) = \infty \text{ resp. } 0, \quad \rho(e) \in (0, \infty).$$

Each of the above operations on graphs corresponds to a map of moduli spaces of stable marked treed disks.

**Definition 2.4.** (Morphisms of moduli spaces)
(a) (Cutting edges) Suppose that $\Gamma$ is obtained from $\Gamma'$ by cutting an edge $e$. There are diffeomorphisms
\[ \overline{M}_\Gamma \to \overline{M}_{\Gamma'}, \quad [C] \to [C'] \]
obtained as follows. Given a treed disk $C$ of type $\Gamma'$, let $z_+, z_-$ denote the endpoints at infinity of the edge corresponding to $e$. Form a treed disk $C'$ by identifying $z_+ \sim z_-$ and choosing the ordering of the interior markings to be that of $\Gamma'$.

(b) (Collapsing edges) Suppose that $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge. There is an embedding $\iota_{\Gamma'} : M_{\Gamma'} \to M_{\Gamma}$. In the case of an edge of Edge$_0(\Gamma')$, the image of $\iota_{\Gamma'}^* M_{\Gamma'}$ is a 1-codimensional corner of $\overline{M}_\Gamma$. In the case of an edge of Edge$_{i,s}(\Gamma')$ the image $\iota_{\Gamma'}^* M_{\Gamma'}$ is a 2-codimensional submanifold of $\overline{M}_\Gamma$.

(c) (Making an edge or weight finite resp. non-zero) If $\Gamma$ is obtained from $\Gamma'$ by making an edge finite resp. non-zero then $M_{\Gamma'}$ also embeds in $\overline{M}_\Gamma$ as the 1-codimensional corner. The image is the set of configurations where the edge $e$ reaches infinite resp. zero length $\ell(e)$ or weight $\rho(e)$.

(d) (Forgetting tails) Suppose that $\Gamma$ is obtained from $\Gamma'$ by forgetting the $i$-th tail (either in Edge$_{i,s}(\Gamma')$ or Edge$_i(\Gamma')$). Forgetting the $i$-th marking and collapsing the unstable components and their distance to the stable components (if any) defines a map $\overline{M}_{\Gamma'} \to \overline{M}_\Gamma$. Each weighted semi-infinite edge for $\Gamma'$ defines a weighted semi-infinite edge for $\Gamma$ with the same weight.

Each of the maps involved in the operations (Collapsing edges/Making edges/weights finite or non-zero), (Forgetting tails), (Cutting edges) extends to a smooth map of universal treed disks. In the case that the type is disconnected we have
\[ \overline{M}_\Gamma = \overline{M}_{\Gamma_1} \sqcup \overline{M}_{\Gamma_2} \]
In this case the universal disk $\overline{U}_\Gamma$ is the disjoint union of the pullbacks of the universal disks $\overline{U}_{\Gamma_1}$ and $\overline{U}_{\Gamma_2}$: If $\pi_1, \pi_2$ are the projections on the factors above then
\[ \overline{U}_\Gamma = \pi_1^* \overline{U}_{\Gamma_1} \sqcup \pi_2^* \overline{U}_{\Gamma_2}. \]
Orientations on the main strata (i.e. of maximal dimension) of the moduli space of (non-weighted) treed disks may be constructed as follows.

**Definition 2.5.** (Orientations on moduli of treed disks)

(a) (A single disk) On the strata made of treed disks having a single disk, choosing an orientation amounts to choosing orientations on the spaces of stable marked disks. One can identify a smooth disk $D$ with $n+1$ boundary markings $x_0, \ldots, x_n$ and $m \geq 1$ interior markings $z_1, \ldots, z_m$ with the positive half-space $\mathbb{H} \subset \mathbb{C}$ by a map
\[ \phi : D - \{x_0\} \to \mathbb{H}, \quad z_i \mapsto i \]
so that the boundary markings $x_i, i \geq 1$ map to an ordered tuple in $\mathbb{R} \subset \mathbb{C}$. For $m = 0$ interior markings, then there are $n + 1 \geq 3$ boundary markings
and we can identify
\[ \phi : D - \{ x_0 \} \rightarrow \mathbb{H}, \quad x_1 \mapsto 0, \quad x_2 \mapsto 1 \]
so that the remaining boundary markings \( x_i, i \geq 3 \) map to an ordered tuple of \( \mathbb{H} \subset \mathbb{R} \subset \mathbb{C} \). The moduli space \( \mathcal{M}_\Gamma \) of disks of this type then inherits an orientation from the canonical orientation on \( \mathbb{R}^{n-2} \times \mathbb{C}^m \).

(b) (Multiple disks) One can extend these orientations to the top dimensional strata of treed disks having more than a single disk as follows. The closures of the main strata are attached to strata with fewer edges with finite non-zero lengths via isomorphisms of treed disks identifying a boundary node with an edge of length zero. The addition of an edge of finite non-zero length corresponds to identifying the closures of two main strata on a 1-codimensional corner strata. For \( n,m \geq 0 \) we choose orientations \( O_{n,m} \) on the main strata \( M_\Gamma \) of \( \mathcal{M}_{n,m} \) so that they induce opposite orientations, on the latter 1-codimensional strata. The closures of the top-dimensional strata \( \mathcal{M}_\Gamma \) then fit together to an oriented manifold with corners (see [7]).

2.3. Treed holomorphic disks. The composition maps in the Fukaya algebra will be obtained by counting treed holomorphic disks, which we now define.

**Definition 2.6.**

(a) (Gradient flow lines) Let \( L \) be a compact connected smooth manifold. We denote by \( \mathcal{G}(L) \) the space of smooth Riemannian metrics on \( L \). Fix a metric \( G \in \mathcal{G}(L) \) and a Morse function \( F : L \rightarrow \mathbb{R} \) having a unique maximum \( x_M \in L \). Let \( I \subset \mathbb{R} \) be a connected subset containing at least two elements, that is, an open or closed interval. The *gradient vector field* of \( F \) is defined by
\[
\text{grad}_F : L \rightarrow TL, \quad G(\text{grad}_F, \cdot) = dF \in \Omega^1(L).
\]
A gradient flow line for \( F \) is a map
\[
u : I \rightarrow L, \quad \frac{d}{ds} \nu = -\text{grad}_F(\nu)
\]
where \( s \) is a unit velocity coordinate on \( I \). Given a time \( s \in \mathbb{R} \) let
\[
\phi_s : X \rightarrow X, \quad \frac{d}{ds} \phi_s(x) = \text{grad}_F(\phi_s(x)), \quad \forall x \in X
\]
denote the time \( s \) gradient flow of \( F \).

(b) (Stable and unstable manifolds) Denote by
\[
\mathcal{I}(L) : = \mathcal{I}(L, F) := \text{crit}(F) \subset X
\]
the space of critical points of \( F \). Taking the limit of the gradient flow determines a discontinuous map
\[
X \rightarrow \text{crit}(F), \quad y \mapsto \lim_{s \rightarrow \pm \infty} \phi_t(y).
\]
By the stable manifold theorem each \( x \in \mathcal{I}(L) \) determines stable and unstable manifolds
\[
W_x^\pm : = \left\{ y \in L \mid \lim_{s \rightarrow \pm \infty} \phi_s(y) = x \right\} \subset L
\]
consisting of points whose downward resp. upwards gradient flow converges to \( x \). The pair \((F, G)\) is Morse-Smale if the intersections
\[ W^+_x \cap W^-_x \subset L \]
are transverse for \( x, x' \in \mathcal{I}(L) \).

(c) (Almost complex structures) Let \((X, \omega)\) be a symplectic manifold. An almost complex structures on \((X, \omega)\) given by
\[ J : \mathcal{T}X \to \mathcal{T}X, \quad J^2 = -I \]
is tamed iff \( \omega(\cdot, J\cdot) \) is a positive definite and compatible if it is in addition symmetric, hence a Riemannian metric on \( X \). We denote by \( \mathcal{J}_\tau(X) \) the space of tamed almost complex structures. The space \( \mathcal{J}_\tau(X) \) has a natural manifold structure locally isomorphic to the space of sections
\[ \delta J : X \to \text{End}(\mathcal{T}X), \quad J(\delta J) = -(\delta J)J. \]

In order to obtain the necessary transversality our Morse functions and almost complex structures must be allowed to depend on a point in the domain. Fix a compact subset
\[ \mathcal{T}^{cp}_\Gamma \subset \mathcal{T}_\Gamma \]
containing, in its interior, at least one point on each edge. Also fix a compact subset
\[ \mathcal{S}^{cp}_\Gamma \subset \mathcal{S}_\Gamma - \{ w_e \in \mathcal{S}_\Gamma, \ e \in \text{Edge}_{\phi,\psi}(\Gamma) \} - \{ z_e \in \mathcal{S}_\Gamma, \ e \in \text{Edge}_{\psi,\phi}(\Gamma) \}\]
disjoint from the boundary, the nodes and interior markings, containing in its interior at least one point on each sphere and disk component. Thus the complement
\[ \mathcal{T}_\Gamma - \mathcal{T}^{cp}_\Gamma \subset \mathcal{T}_\Gamma \]
is a neighborhood of infinity on each edge and the complement
\[ \mathcal{S}_\Gamma - \mathcal{S}^{cp}_\Gamma \subset \mathcal{S}_\Gamma \]
is a neighborhood of the boundary, nodes, and interior markings.

**Definition 2.7.**
(a) (Domain-dependent Morse functions) Suppose that \( \Gamma \) is a type of stable treed disk, and \( \mathcal{T}_\Gamma \subset \mathcal{U}_\Gamma \) is the tree part of the universal treed disk. Let \((F, G)\) be a Morse-Smale pair. For an integer \( l \geq 0 \) a domain-dependent perturbation of \( F \) of class \( C^l \) is a \( C^l \) map
\[ F_\Gamma : \mathcal{T}_\Gamma \times L \to \mathbb{R} \]
equal to the given function \( F \) away from the compact part:
\[ F_\Gamma|((\mathcal{T}_\Gamma - \mathcal{T}^{cp}_\Gamma) = \pi_2^*F \]
where \( \pi_2 \) is the projection on the second factor in (5).

(b) (Domain-dependent almost complex structure) Let \( J \in \mathcal{J}_\tau(X) \) be a tamed almost complex structure. Let \( l \geq 0 \) be an integer. A domain-dependent almost complex structure of class \( C^l \) for treed disks of type \( \Gamma \) is a map from the two-dimensional part \( \mathcal{S}_\Gamma \) of the universal curve \( \mathcal{U}_\Gamma \) to \( \mathcal{J}_\tau(X) \) given by a \( C^l \) map
\[ J_\Gamma : \mathcal{S}_\Gamma \times X \to \text{End}(\mathcal{T}X) \]
equal to the given $J$ away from the compact part:

$$J_\Gamma|((\mathcal{S}_\Gamma - \mathcal{S}_\Gamma^0) = \pi_2^* J$$

where $\pi_2$ is the projection on the second factor in (5).

Let $(X, \omega)$ be a compact symplectic manifold and $L \subset X$ a Lagrangian submanifold. Let $J \in \mathcal{J}_\omega(X)$ be a base almost complex structure and $G \in \mathcal{G}(L)$ a base metric so that $(F, G)$ is Morse-Smale.

**Definition 2.8.** (Perturbation data) A perturbation datum is pair $P_\Gamma = (F_\Gamma, J_\Gamma)$ consisting of a domain-dependent Morse function $F_\Gamma$ and a domain-dependent almost complex structure $J_\Gamma$.

The following are three operations on perturbation data:

**Definition 2.9.**
(a) (Cutting edges) Suppose that $\Gamma$ is a combinatorial type obtained by cutting an edge of $\Gamma'$. A perturbation datum for $\Gamma$ gives rise to a perturbation datum for $\Gamma'$ by pushing forward $P_\Gamma$ under the map $\pi_{\Gamma'}: \mathcal{U}_{\Gamma'} \to \mathcal{U}_{\Gamma}$. That is, define

$$J_{\Gamma'}(z', x) = J_\Gamma(z, x), \quad \forall z \in (\pi_{\Gamma'}^{-1}(z)).$$

This is well-defined by the (Constant near the nodes and markings) axiom. The definition for $F_{\Gamma'}$ is similar.

(b) (Collapsing edges/making an edge/weight finite/non-zero) Suppose that $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making an edge/weight finite/non-zero. Any perturbation datum $P_\Gamma$ for $\Gamma$ induces a datum for $\Gamma'$ by pullback of $P_\Gamma$ under $\iota_{\Gamma'}: \mathcal{U}_{\Gamma'} \to \mathcal{U}_{\Gamma}$.

(c) (Forgetting tails) Suppose that $\Gamma$ is a combinatorial type of stable treed disk is obtained from $\Gamma'$ by forgetting a marking. In this case there is a map of universal disks $f_{\Gamma'}: \mathcal{U}_{\Gamma'} \to \mathcal{U}_{\Gamma}$ given by forgetting the marking and stabilizing. Any perturbation datum $P_\Gamma$ induces a datum $P_{\Gamma'}$ by pullback of $P_\Gamma$.

We are now ready to define coherent collections of perturbation data. These are data that behave well with each type of operation in Definition 2.9.

**Definition 2.10.** (Coherent families of perturbation data) A collection of perturbation data $\mathcal{P} = (P_\Gamma)$ is coherent if it is compatible with the morphisms of moduli spaces of different types in the sense that

(a) (Cutting edges) if $\Gamma$ is obtained from $\Gamma'$ by cutting an edge of infinite length, then $P_{\Gamma'}$ is the pushforward of $P_\Gamma$;

(b) (Collapsing edges/making an edge/weight finite/non-zero) if $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge or making an edge/weight finite/non-zero, then $P_{\Gamma'}$ is the pullback of $P_\Gamma$;
(c) (Products) if $\Gamma$ is the union of types $\Gamma_1, \Gamma_2$ obtained by cutting an edge of $\Gamma'$, then $P_{\Gamma}$ is obtained from $P_{\Gamma_1}$ and $P_{\Gamma_2}$ as follows: Let

$$\pi_k : \mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2} \to \mathcal{M}_{\Gamma_k}$$

denote the projection on the $k$th factor. Then $\overline{U}_\Gamma$ is the union of $\pi_1^*\overline{U}_{\Gamma_1}$ and $\pi_2^*\overline{U}_{\Gamma_2}$. Then we require that $P_{\Gamma}$ is equal to the pullback of $P_{\Gamma_k}$ on $\pi_k^*\overline{U}_{\Gamma_k}$:

$$P_{\Gamma}|_{\overline{U}_{\Gamma_k}} = \pi_k^*P_{\Gamma_k}.$$  

(6)

There is one important clarification corresponding to the case that the configuration is constant on one of the pieces: suppose that $\Gamma_1$ corresponds to a configuration with a single unmarked disk and two incoming leaves, one of which is weighted as in the middle configuration on the bottom row in Figure 10. Then we declare the corresponding incoming leaf of $\Gamma_2$ to be weighted, with the same weight as the leaf of $\Gamma_1$, and require (6).

(d) (Infinite weights) If a weight parameter $\rho(e_i)$ is equal to infinity, then the $P_{\Gamma'}$ is pulled back under the forgetful map forgetting the $e_i$ semi-infinite edge and stabilizing from the perturbation datum $P_{\Gamma}$ given by (Forgetting tails). The last sentence of the previous item guarantees that this condition is compatible with the product axiom, in the case that forgetting an edge with infinite weight leads to a collapse of a disk component (so that the stabilization of $\Gamma$ is $\Gamma_2$, in the notation of the previous paragraph.)

Let $C$ be a possibly unstable treed disk of type $\Gamma$. The stabilization of $C$ is the stable treed disk of some type $s(\Gamma)$ obtained by collapsing unstable surface and tree components. Thus the stabilization of any treed disk is the fiber of a universal treed disk $U_s(\Gamma)$. Given perturbation datum for the stabilization of the type $\Gamma$, we obtain a domain-dependent almost complex structure and Morse function for $C$, still denoted $J_{\Gamma}, F_{\Gamma}$, by pull-back under the map $C \to U_{\Gamma}$.

**Definition 2.11.** (Perturbed holomorphic treed disks) Given perturbation datum $P_{\Gamma}$, a holomorphic treed disk in $X$ with boundary in $L$ consists of a treed disk $C = S \cup T$ and a continuous map $u : C \to X$ such that

(a) (Boundary condition) $u(\partial S \cup T) \subset L$;

(b) (Surface equation) on the surface part $S$ of $C$ the map $u$ is $J$-holomorphic for the given domain-dependent almost complex structure: if $j$ denotes the complex structure on $S$ then

$$J_{\Gamma,u(z),z} \, du|_S = du|_S \, j;$$

(c) (Tree equation) on the tree part $T \subset C$ the map $u$ is a collection of gradient trajectories:

$$\frac{d}{ds} u|_T = - \text{grad}_{F_{\Gamma,(s,u(s))}}(u|_T)$$

where $s$ is a local coordinate with unit speed. Thus for each edge $e \in \text{Edge}_f(\Gamma)$ the length of the trajectory is given by the length $u|_{e \subset T}$ is equal to $\ell(e)$. 

The stability condition for weighted treed disks is the following.

**Definition 2.12.** A weighted treed disk \( u : C \to X \) with markings \((z_1, \ldots, z_k)\) and interior and boundary nodes \( w_1, \ldots, w_m \) is **stable** if

- (a) each disk component on which \( u \) is constant has at least three special boundary points or one boundary node and one interior special point
  \[
  \text{du}(C_i) = 0, \quad C_i \text{ disk } \implies 2\#\{z_k, w_k \in \text{int}(C_i)\} + \#\{w_k \in \partial C_i\} \geq 3;
  \]
- (b) each sphere \( C_i \subset C \) component on which \( u \) is constant has at least three special points:
  \[
  \text{du}(C_i) = 0, \quad C_i \text{ sphere } \implies \#\{z_k, w_k \in C_i\} \geq 3
  \]

  and

- (c) each infinite line on which \( u \) is constant has a weighted leaf and an unforgettable or forgettable root:
  \[
  \text{du}(C_i) = 0, \quad C_i \text{ line } \implies e_0 \in \text{Edge}^0(\Gamma) \text{ and } e_1 \in \text{Edge}^*\cap(\Gamma) \cup \text{Edge}^*(\Gamma).
  \]

The stability condition in Definition 2.12 is not quite the same as having no automorphisms because of the third item. The latter is an exceptional case in which the trajectory does admit automorphisms, corresponding to translations of the line. However, the moduli space is still of expected dimension because of the condition on the weighting.

Equivalence of weighted treed disks is defined as follows. Given a non-constant holomorphic treed disk \( u : C \to X \) with leaf \( e_i \) for which the weighting \( \rho(e_i) = 0 \) resp. \( \infty \), we declare \( u \) to be **equivalent** to the holomorphic treed disk \( u' : C \to X \) obtained by adding a constant trajectory with weighted incoming and forgettable resp. unforgettable outgoing edge.

![Figure 8. Equivalent weighted treed disks](image)

Removing a constant segment and relabelling gives equivalent holomorphic treed disks. Also, any two configurations with an outgoing weighted edge with the same underlying metric tree are considered equivalent. See Figure 9.
We introduce notation for various moduli spaces of equivalence classes of weighted treed disks. For integers \( n, m \) denote by \( \overline{\mathcal{M}}_{n,m}(L) \) the moduli space of equivalence classes of connected treed holomorphic disks with \( n \) leaves and \( m \) interior markings. For any connected combinatorial type \( \Gamma \) of treed holomorphic disk, denote by \( \mathcal{M}(\Gamma) \) the subset of type \( \Gamma \). Define

\[
\hat{I}(L) = I(L) - \{x_M\} \cup \{x_M^\circ, x_M^*, x_M^\bullet\}.
\]

Thus \( I(L) \) is the union of the critical points of \( F \), with the maximum \( x_M \) replaced by three copies \( x_M^\circ, x_M^*, x_M^\bullet \). We extend the index map on \( I(L) \) to \( \hat{I}(L) \) by

\[
i(x_M^\circ) = i(x_M^*) = 0, \quad i(x_M^\bullet) = -1.
\]

We define

\[
\hat{I}_d(L) = \{x \in \hat{I}(L) \mid i(x) = d\}.
\]

The moduli spaces above break into components depending on the limits along the semi-infinite edges. An admissible labelling of a (non-broken) weighted treed disk \( C \) with leaves \( e_1, \ldots, e_n \) and outgoing edge \( e_0 \) is a sequence \( x = (x_0, \ldots, x_n) \in \hat{I}(L) \) satisfying:

(a) (Label axiom) If \( x_i = x_M^\circ \) resp. \( x_M^\circ \) resp. \( x_M^\bullet \) then the corresponding semi-infinite edge is required to be weighted resp. forgettable resp. unforgettable, that is,

\[
x_i = x_M^\circ \text{ resp. } x_M^\circ \text{ resp. } x_M^\bullet \implies e_i \in \text{Edge}^\circ(\Gamma) \text{ resp. } e_i \in \text{Edge}^\circ(\Gamma) \text{ resp. } e_i \in \text{Edge}^\ast(\Gamma).
\]

Furthermore, in this case the limit along the \( i \)-th semi-infinite edge is required to be \( x_M^\circ \):

\[
\lim_{s \to \infty} u(\varphi_e(s)) = x_M^\circ.
\]

In every other case the semi-infinite edge is required to be unforgettable:

\[
x_i \notin \{x_M^\circ, x_M^\circ, x_M^\bullet\} \implies e_i \in \text{Edge}^\bullet(\Gamma).
\]

(b) (Outgoing edge axiom)
(i) The outgoing edge $e_0$ is weighted only if there are two incoming leaves, exactly one of which is weighted with the same weight, and the other is forgettable and equal to infinity, and there is a single disk with no markings.

(ii) The outgoing edge $e_0$ can only be forgettable if either

- there are two forgettable incoming leaves, or
- there is a single leaf that is weighted and the configuration has no interior marking.

See Figure 10.

The (Outgoing edge axiom) treats the case of constant holomorphic treed disks. As is typical in Floer theory, constant configurations must be treated with great care. Denote by $M_\Gamma(L, \underline{x}) \subset M_\Gamma(L)$ the moduli space of isomorphism classes of stable holomorphic treed disks with boundary in $L$ and admissible labelling $\underline{x} = (x_0, \ldots, x_n)$.

The expected dimension of the moduli space is given by a formula involving the Maslov indices of the disks and the indices and number of semi-infinite edges. The expected dimension of the moduli space is given by the usual index formula involving the Maslov indices of the disk and sphere parts. If $u_i = u|C_i, i = 1, \ldots, k$ of $u$, denote by the Maslov index $I(u_i)$. The expected dimension of $M_\Gamma(L, \underline{x})$ at $[u : C \to X]$ is given by

\begin{equation}
i(\Gamma, \underline{x}) := i(x_0) - \sum_{i=1}^{n} i(x_i) + \sum_{i=1}^{k} I(u_i) + n - 2 - |\text{Edge}_{\Gamma}^{0}\text{d}(\Gamma)| - |\text{Edge}_{\Gamma}^{s}(\Gamma) - (n + 1)|/2
- 2|\text{Edge}_{\infty, s}(\Gamma)| - \sum_{e \in \text{Edge}_{\infty, s}(\Gamma)} m(e) - \sum_{e \in \text{Edge}_{<\infty, s}(\Gamma)} m(e).
\end{equation}

where $|\text{Edge}_{\Gamma}(\Gamma) - (n + 1)|/2$ corresponds to the number of breakings of $\Gamma$.

Remark 2.13. (Constant trajectories) If $x_1 = x^s_M$ and $x_0 = x^s_M$ resp. $x_0 = x^s_M$ then the moduli space $M(L, x_0, x_1)$ contains a configuration with no disks and single edge on which $u$ is constant, corresponding to a weighted leaf and an outgoing end that is unforgettable resp. forgettable. These trajectories are pictured in Figure 10.

2.4. Transversality. In this section we use domain-dependent almost complex structures and metrics to regularize the moduli space of holomorphic disks with boundary in a Lagrangian submanifold. In order to use domain-dependent almost complex structures, the surface part of the domains must be stable. We begin by recalling the existence of stabilizing divisors for Lagrangian manifolds in Charest-Woodward [8], see also Cieliebak-Mohnke [10].

Definition 2.14. (Rational Lagrangians)
A symplectic manifold $X$ with two-form $\omega \in \Omega^2(X)$ is rational if the class $[\omega] \in H^2(X, \mathbb{R})$ is rational, that is, in the image of $H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{R})$. Equivalently, $(X, \omega)$ is rational if there exists a linearization of $X$: a line bundle $\tilde{X} \to X$ with a connection whose curvature is $(2\pi k/i)\omega$ for some integer $k > 0$.

(b) A Lagrangian $L \subset X$ of a rational symplectic manifold $X$ with linearization $\tilde{X}$ is strongly rational if $\tilde{X}|L$ has a covariant constant section $L \to \tilde{X}|L$.

(c) Let $h_2 : \pi_2(X, L) \to H_2(X, L)$ be the degree two relative Hurewicz morphism. On the other hand, we denote by $[\omega]^\vee : H_2(X, L) \to \mathbb{R}$ the map induced by pairing with $[\omega] \in H^2(X, L)$. An immersed Lagrangian submanifold $L \subset X$ is rational if

$$[\omega]^\vee \circ h_2(\pi_2(X, L)) = \mathbb{Z} \cdot e \subset \mathbb{R}$$

for some $e > 0$.

The rationality assumptions guarantee that we can find stabilizing divisors in the following sense.

**Definition 2.15.** (Stabilizing divisors)

(a) A divisor in $X$ is a closed codimension two symplectic submanifold $D \subset X$. An almost complex structure $J : TX \to TX$ is adapted to a divisor $D$ if $D$ is an almost complex submanifold of $(X, J)$.

(b) A divisor $D \subset X$ is stabilizing for a Lagrangian submanifold $L \subset X$ iff

(i) it is disjoint from $L$, that is, $D \subset X - L$, and

(ii) any disk $u : (C, \partial C) \to (X, L)$ with non-zero area $\omega([u]) > 0$ intersects $D$ in at least one point.
A divisor $D \subset X$ is weakly stabilizing for a Lagrangian submanifold $L \subset X$ iff

(a) it is disjoint from $L$, that is, $D \subset X - L$, and

(b) there exists an almost-complex structure $J_D \in J(X, \omega)$ adapted to $D$ such that any non-trivial $J_D$-holomorphic disk $u : (\mathbb{C}, \partial \mathbb{C}) \to (X, L)$ intersects $D$ in at least one point.

Recall from Charest-Woodward [8] the existence result for stabilizing divisors:

**Theorem 2.16.** [8, Section 4] There exists a divisor $D \subset X - L$ that is weakly stabilizing for $L$. Moreover, if $L$ is rational resp. strongly rational then there exists a divisor $D \subset X - L$ that is stabilizing for $L$ resp. stabilizing for $L$ and such that $L$ is exact in $(X - D, \omega|_{X - D})$.

The existence of the divisors in the theory is an application of the theory of Donaldson-Auroux-Gayet-Mohsen stabilizing divisors [13], [3]. In the case that $X$ is a smooth projective algebraic variety, stabilizing divisors may be obtained using a result of Borthwick-Paul-Uribe [6]. Roughly speaking one chooses an approximately holomorphic section concentrated on the Lagrangian; then a generic perturbation defines the desired divisor. For example, in the case $L$ is a circle in the symplectic two-sphere $X$ then a stabilizing divisor $D$ is given by choosing a point in each component of the complement of $L$. There is also a time-dependent version of this result which will be used later to prove independence of the homotopy type of the Fukaya algebra from the choice of base almost complex structure: If $J^t, t \in [0, 1]$ is a path of almost complex structures then (see [8, Lemma 4.20] ) there exists a path of $J^t$-stabilizing divisors $D^t, t \in [0, 1]$ connecting $D^0, D^1$.

**Definition 2.17.** (Adapted stable treed disks) Let $(X, \omega)$ be a symplectic manifold with Lagrangian $L$ and a codimension two submanifold $D$ disjoint from $L$. A stable treed disk $u : C \to X$ with boundary in $L$ is adapted to $D$ iff

(a) (Stable domain) The surface part of $C$ is stable (that is, each disk component $C_i \subset C$ has either one interior special point (a marking $z_i \in C$ or node $w_i \in C$) and one special point on the boundary $w_j \in \partial C_i$ or three special points on the boundary $w_i, w_j, w_k \in \partial C_i$; and each sphere component has at least three special points;

(b) (Non-constant spheres) Each component $C_i$ of $C$ that maps to $D$ is constant: $u(C_i) \subset D \implies du|_{C_i} = 0$.

That is, there are no non-constant sphere components of $C$ mapping entirely to $D$.

(c) (Markings) Each interior marking $z_i$ maps to $D$ and each connected component of $u^{-1}(D)$ contains an interior marking:

$u(z_i) \in D, \quad C_i \subset u^{-1}(D)$ connected component $\implies \exists z_k, z_k \in C_i$.

Thus $u : C \to X$ consists of components on which $u$ is non-constant which meet $D$ in finitely many points and components on which $u$ is constant. Note that any ghost
component (component on which \( u \) is constant) containing an interior marking is required to map to \( D \).

The moduli space of adapted treed holomorphic disks is stratified by combinatorial type as follows. We denote by \( \Pi(X) \) resp. \( \Pi(X, L) \) the space of homotopy classes of maps from the two sphere resp. disk with boundary in \( L \). For any interior marking \( z_e \) near which \( u \) is non-constant, we denote by \( m(z_e) \in \mathbb{Z}_{\geq 0} \) the multiplicity of the intersection with the divisor \( D \) at \( z_e \). This is the winding number of small loop around \( z_e \) in the complement of the zero section in a tubular neighborhood of \( D \) in \( X \). The combinatorial type of a holomorphic treed disk \( u : C \to X \) adapted to \( D \) consists of

- (a) the combinatorial type \( \Gamma = (\text{Vert}(\Gamma), \text{Edge}(\Gamma)) \) of its domain \( C \) together with
- (b) the labelling
  \[ d : \text{Vert}(\Gamma) \to \Pi(X) \cup \Pi(X, L) \]
  of each vertex \( v \) of \( \Gamma \) corresponding to a disk or sphere component with the corresponding homotopy class and
- (c) the labelling
  \[ m : \text{Edge}_{i,s}(\Gamma) \cup \text{Edge}_{f,s}(\Gamma) \to \mathbb{Z}_{\geq 0} \]
  recording the order of tangency of the map \( u \) to the divisor \( D \) at each of the interior markings \( z_e, e \in \text{Edge}_{i,s}(\Gamma) \) and spherical nodes \( w_e, e \in \text{Edge}_{\infty,s}(\Gamma) \).

Note that at any spherical node mapping to the divisor, at least one component on one side of the node is constant mapping to the divisor by the (Markings) axiom. We take the order of tangency at this node to be the order of tangency on this constant component, or zero if the components on both sides map to the divisor. Let \( \mathcal{M}(L, D) \) the moduli space of stable treed marked disks in \( X \) with boundary in \( L \) adapted to \( D \) and \( \mathcal{M}_\Gamma(L, D) \) the locus of combinatorial type \( \Gamma \). For \( x \in \hat{I}(L)^n \) let
\[
\mathcal{M}_\Gamma(L, D, x) \subset \mathcal{M}_\Gamma(L, D)
\]
denote the adapted subset made of holomorphic treed disks of type \( \Gamma \) adapted to \( D \) with limits \( x = (x_0, \ldots, x_n) \in \hat{I}(L) \) along the root and leaves.

In order to obtain transversality we begin by fixing an open subset of the universal curve on which the perturbations will vanish. Let
\[
\overline{U}_\Gamma^{\text{cp}} \subset \overline{U}_\Gamma
\]
be a compact subset disjoint from the nodes and attaching points of the edges such that the interior of \( \overline{U}_\Gamma^{\text{cp}} \) in each two and one-dimensional component is open. Suppose that perturbation data \( P_t \) for all boundary types \( \mathcal{U}_t \subset \overline{U}_\Gamma \) have been chosen. Let
\[
\mathcal{P}_\Gamma^t(X, D) = \{ P_t = (F_t, J_t) \}
\]
denote the space of perturbation data \( P_t = (F_t, J_t) \) of class \( C^4 \) that are

- equal to the given pair \( (F, J) \) on \( \overline{U}_\Gamma - \overline{U}_\Gamma^{\text{cp}} \), and such that
• the restriction of $P_{\Gamma}$ to $U_{\Gamma'}$ is equal to $P_{\Gamma'}$, for each boundary type $\Gamma'$, that is, type of lower-dimensional stratum $\mathcal{M}_{\Gamma'} \subset \mathcal{M}_{\Gamma}$.

The second condition will guarantee that the resulting collection satisfies the (Collapsing edges/Making edges/weights finite or non-zero) axiom of the coherence condition Definition 2.10. Let $P_{\Gamma}(X, D)$ denote the intersection of the spaces $P_{\Gamma}(X, D)$ for $l \geq 0$.

One cannot expect, using stabilizing divisors, to obtain transversality for all combinatorial types. The reason is a rather trivial analog of the multiple cover problem: once one has a ghost bubble mapping to the divisor and containing a marking then one has configurations with arbitrary number of markings on that component, whose expected dimension goes to minus infinity but which are all non-empty. A type $\Gamma$ will be called *uncrowded* if each maximal ghost component contains at most one marking. Write $\Gamma' \leq \Gamma$ iff $\Gamma$ is obtained from $\Gamma'$ by (Collapsing edges/making edge lengths or weights finite/non-zero). or by gluing $\Gamma'$ at a breaking.

**Theorem 2.18.** (Transversality) Suppose that $\Gamma$ is an uncrowded type of stable treed marked disk of expected dimension $i(\Gamma, x) \leq 1$, see (8). Suppose regular coherent perturbation data for types of stable treed marked disk $\Gamma'$ with $\Gamma' \leq \Gamma$ are given. Then there exists a comeager subset $P_{\Gamma}^{\text{reg}}(X, D) \subset P_{\Gamma}(X, D)$ of regular perturbation data for type $\Gamma$ compatible with the previously chosen perturbation data such that if $P_{\Gamma} \in P_{\Gamma}^{\text{reg}}(X, D)$ then

(a) (Smoothness of each stratum) the stratum $\mathcal{M}_{\Gamma}(L, D)$ is a smooth manifold of expected dimension;

(b) (Tubular neighborhoods) if $\Gamma$ is obtained from $\Gamma'$ by collapsing an edge of $\text{Edge}_{<\infty, d}(\Gamma')$ or making an edge or weight finite/non-zero or by or by gluing $\Gamma'$ at a breaking then the stratum $\mathcal{M}_{\Gamma'}(L, D)$ has a tubular neighborhood in $\mathcal{M}_{\Gamma}(L, D)$; and

(c) (Orientations) there exist orientations on $\mathcal{M}_{\Gamma}(L, D)$ compatible with the morphisms (Cutting an edge) and (Collapsing an edge/Making an edge/weight finite/non-zero) in the following sense:

(i) If $\Gamma$ is obtained from $\Gamma'$ by (Cutting an edge) then the isomorphism $\mathcal{M}_{\Gamma'}(L, D) \to \mathcal{M}_{\Gamma}(L, D)$ is orientation preserving.

(ii) If $\Gamma$ is obtained from $\Gamma'$ by (Collapsing an edge) or (Making an edge/weight finite/non-zero) then the inclusion $\mathcal{M}_{\Gamma'}(L, D) \to \mathcal{M}_{\Gamma}(L, D)$ has orientation (using the decomposition

$$T\mathcal{M}_{\Gamma}(L, D)|_{\mathcal{M}_{\Gamma'}(L, D)} \cong \mathbb{R} \oplus T\mathcal{M}_{\Gamma'}(L, D)$$

and the outward normal orientation on the first factor) given by a universal sign depending only on $\Gamma, \Gamma'$. 

*Proof.* The proof is the similar to the corresponding proof in [8], with the added variation of the Morse functions on the tree part of the universal treed disk. For a
combinatorial type $\Gamma$ represented by a nodal disk $C$, integers $p \geq 2$ and $k$ sufficiently large we denote by $\text{Map}^{k,p}(C, L, X)$ the space of continuous maps $u$ from $C$ to $X$ of Sobolev class $W^{k,p}$ on each disk, sphere and edge component such that the boundaries $\partial C := \partial S \cup T$ of the disks and edges mapping to $L$. In each local chart for each component of $C$ and $X$ the map $u$ is given by a collection of continuous functions with $k$ partial derivatives of class $L^p$. The space $\text{Map}^{k,p}(C, L, X)$ has the structure of a Banach manifold, with a local chart at $u \in \text{Map}^{k,p}(C, L, X)$ given by the geodesic exponential map

$$W^{k,p}(C, u^*TX, (u|_{\partial C})^*TL) \to \text{Map}^{k,p}(C, L, X), \quad \xi \mapsto \exp_u(\xi)$$

where we assume that the metric on $X$ is chosen so that $L$ is totally geodesic, that is, preserved by geodesic flow. Denote by

$$\text{Map}^{k,p}_\Gamma(C, L, X, D) \subset \text{Map}^{k,p}(C, L, X)$$

the subset of maps such that $u$ has the prescribed homology class on each component, and the prescribed order of vanishing at markings and nodes; this is a $C^q$ Banach submanifold where $q < k - n/p - \max_e m(e)$ where $m(e)$ is the order of tangency.

To construct the local universal moduli spaces we take the approach of Dragnev [14]. For each local trivialization of the universal tree disk as in (4) we consider the ambient moduli space defined as follows. Let $\text{Map}^{k,p}_\Gamma(C, L, D)$ denote the space of maps of Sobolev class $k \geq 1, p > 2$ mapping the boundary of $C$ into $L$, the interior markings into $D$, and constant on each disk with no interior marking, and with the prescribed order of vanishing at the intersection points with $D$. Let $l \gg k$ be an integer and

$$\mathcal{B}^{i}_{k,p,l,\Gamma} := \mathcal{M}^i_\Gamma \times \text{Map}^{k,p}_\Gamma(C, L, D) \times \mathcal{P}^i_\Gamma(X, D).$$

Consider the map given by the local trivialization

$$\mathcal{M}^{\text{univ},i}_\Gamma \to \mathcal{J}(S), \quad m \mapsto j(m).$$

Consider the fiber bundle $\mathcal{E}^i = \mathcal{E}^i_{k,p,l,\Gamma}$ over $\mathcal{B}^{i}_{k,p,l,\Gamma}$ given by

$$(\mathcal{E}^i_{k,p,l,\Gamma})_{m,u,J} \subset \Omega^{0,1}_{j,J,\Gamma}(S, (u|_{S})^{*}TX)_{k-1,p} \oplus \Omega^{1}(T, (u|_{T})^{*}TL)_{k-1,p}$$

the space of 0,1-forms with respect to $j(m)$, $J$ that vanish to order $m(e) - 1$ at the node or marking corresponding to each contact edge $e$. The Cauchy-Riemann and shifted gradient operators applied to the restrictions $u_S$ resp. $u_T$ of $u$ to the two resp. one dimensional parts of $C = S \cup T$ define a $C^q$ section

$$\overline{\partial}_\Gamma : \mathcal{B}^{i}_{k,p,l,\Gamma} \to \mathcal{E}^i_{k,p,l,\Gamma}, \quad (m, u, J, F) \mapsto \left(\overline{\partial}_{j(m),J} u, (\frac{d}{ds} - \text{grad}_F)u_T\right)$$

where

$$\overline{\partial}_{j(m),J} u := \frac{1}{2}(Jdu_S - du_S j(m)),$$

and $s$ is a local coordinate with unit speed. The local universal moduli space is

$$\mathcal{M}^{\text{univ},i}_\Gamma(L, D) = \overline{\partial}^{-1} \mathcal{B}^{i}_{k,p,l,\Gamma}$$
where $B^i_{k,p,l,T}$ is embedded as the zero section.

We claim that the local universal moduli space is cut out transversally. To see this, suppose that
\[ \eta = (\eta_2, \eta_1) \in \Omega^0(S, (u|_S)^*TX)_{k-1,p} \oplus \Omega^1(T, (u|_T)^*TL)_{k-1,p} \]
is in the cokernel of derivative of (9) with 2-dimensional part $\eta_2$ and one-dimensional part $\eta_1$. Variation of (9) with respect to the section $\xi_1$ on the one-dimensional part $T$ gives
\[ 0 = \int_T (Du_1 \xi_1, \eta_1) ds = \int_T (\xi_1, Du_1^* \eta_1) ds, \forall \xi \in \Omega^0(u_1^*TX) \]
hence
\[ (11) \quad \nabla_\xi \ast \eta_1 = 0. \]

On the other hand, the linearization of (9) with respect to the Morse function $F$ $F \mapsto - \text{grad}_F u_T$ is pointwise surjective:
\[ \{- \text{grad}_F, u_T(s) \mid F \in C^\delta_c(\mathcal{T}_\Gamma \times L)\} = T_{u_T(s)}X. \]
This implies that
\[ (12) \quad \eta_1(u_T(s)) = 0, \quad \forall s \in \overline{U}_\Gamma - \overline{U}_{\text{thin}}. \]
Combining (11) and (12) implies that $\eta_1 = 0$. By similar arguments the two-dimensional part $\eta_2$ satisfies
\[ (13) \quad (Du_2 \xi_2, \eta_2) = 0, \quad \int_S (Y \circ du \circ j) \wedge \eta_2 = 0 \]
for every $\xi_2 \in \Omega^0(S, u^*TX)$ with given orders of vanishing at the intersection points with $D$ and variation of almost complex structure $Y \in \Omega^0(S, u^* \text{End}(TX))$ as in [27, Chapter 3]. Hence in particular $Du_2^* \eta_2 = 0$ away from the intersection points with $D$. In particular if $du$ is somewhere non-vanishing at a point $z \in S$ then $\eta_2$ is vanishing on that component in a neighborhood of $z$. It follows that $\eta_2$ vanishes everywhere on the component of $S$ containing $z$ by unique continuation except possibly at the intersection points with $D$. At these points $\eta_2$ could in theory be a sum of derivatives of delta functions; That $\eta_2$ also vanishes at these intersection points follows from [9, Lemma 6.5, Proposition 6.10].

It remains to consider components of the two-dimensional part on which the map is constant. If $u : C \to X$ is a map that is constant on $S' \subset C$ the linearized operator is constant on each disk component of $S'$ and surjective by a doubling trick. However, we also must check that the matching conditions at the nodes are cut out transversally. For this it suffices to consider a “maximal ghost component” $S'' \subset S$ consisting of a union of disks on which $u$ is constant, attached by nodes. let $S''$ denote the normalization of $S'$, obtained by replacing each nodal point in $S'$ with a pair of points in $S''$. Since the combinatorial type of the component is a subgraph of a tree, the combinatorial type must itself be a tree. We denote by $T_u L$
the tangent space at the constant value of $u$ on $S'$. Taking the differences of the maps at the nodes defines a map

$$
\delta : \ker(D_u|S') \cong T_u L^k \to T_u m, \quad \xi \mapsto (\xi(w^+_i) - \xi(w^-_i))_{i=1}^m.
$$

An explicit inverse to $\delta$ is given by defining recursively as follows. Consider the orientation on the combinatorial type $\Gamma'' \subset \Gamma$ induced by the choice of outgoing semi-infinite edge of $\Gamma$. For $\eta \in T_u m$ define an element $\xi \in T_u L^k$ by

$$
\xi(t(e)) = \xi(h(e)) + \eta(e)
$$

whenever $t(e), h(e)$ are the head and tail of an edge $e$ corresponding to a node. This element may be defined recursively starting with the edge $e''_0$ of $\Gamma''$ closest to the outgoing edge of $\Gamma$ and taking $\xi(v) = 0$ for $v$ the vertex corresponding to the disk component closest to outgoing edge. The matching conditions at the nodes connecting $S'$ with the complement $S - S'$ are also cut out transversally. Indeed on the adjacent components the linearized operator restricted to sections vanishing at the node is already surjective as in [9, Lemma 6.5].

Using the surjectivity of the parametrized linearized operator in the previous paragraphs we apply the implicit function theorem to conclude that the local universal moduli space is a Banach manifold. More precisely, $\mathcal{M}^{\text{univ},i}(L,D)$ is a Banach manifold of class $C^q$, and the forgetful morphism

$$
\varphi_i : \mathcal{M}^{\text{univ},i}(L,D)_{k,p,l} \to \mathcal{P}_i(L,D)_l
$$

is a $C^q$ Fredholm map. Let

$$
\mathcal{M}^{\text{univ},i}(L,D)_{d} \subset \mathcal{M}^{\text{univ},i}(L,D)
$$

denote the component on which $\varphi_i$ has Fredholm index $d$. By the Sard-Smale theorem, for $k, l$ sufficiently large the set of regular values $\mathcal{P}^{\text{reg}}_i(X)_l$ of $\varphi_i$ on $\mathcal{M}^{\text{univ},i}(L,D)_d$ in $\mathcal{P}_i(X)_l$ is comeager. Let

$$
\mathcal{P}^{\text{reg}}_i(X)_l = \cap_i \mathcal{P}^{\text{reg},i}(X)_l.
$$

A standard argument shows that the set of smooth domain-dependent $\mathcal{P}^{\text{reg}}_i(L,D)$ is also comeager. Fix $(J_\Gamma, F_\Gamma) \in \mathcal{P}^{\text{reg}}_i(L,D)$. By elliptic regularity, every element of $\mathcal{M}_i^\Gamma(L,D)$ is smooth. The transition maps for the local trivializations of the universal bundle define smooth maps

$$
\mathcal{M}^\Gamma_i(L,D)|_{\mathcal{M}_i^\Gamma \cap \mathcal{M}_i^{\Gamma'}} \to \mathcal{M}^\Gamma_i(L,D)|_{\mathcal{M}_i^\Gamma \cap \mathcal{M}_i^{\Gamma'}}.
$$

This construction equips the space

$$
\mathcal{M}^\Gamma_i(L,D) = \bigcup_i \mathcal{M}_i^\Gamma(L,D)
$$

with a smooth atlas. Since $\mathcal{M}^\Gamma_i$ is Hausdorff and second-countable, so is $\mathcal{M}^\Gamma_i(L,D)$ and it follows that $\mathcal{M}^\Gamma_i(L,D)$ has the structure of a smooth manifold.

The construction of tubular neighborhoods and orientations is similar to the case treated in [8], with the added ingredient of estimates for gluing Morse trajectories in the case of a broken Morse trajectory from, e.g., Schwarz [29].
Remark 2.19. The strata $\mathcal{M}_\Gamma(L,D)$ of expected dimension zero are of two possible types.

(a) The first possibility is that $\Gamma$ has a broken edge, see Figure 11.

![Figure 11. Treed disk with a broken edge](image1)

In the case that no component with no vertex is involved, the normal bundle to the stratum has fiber canonically isomorphic to $\mathbb{R}_{\geq 0}$, corresponding to deformations that make the length finite. Note that the broken edge may be a leaf and that that leaf may have label $x_M^*$ and weight 0 or $\infty$ as in Remark 2.13. In this case any configuration $u$ of type $\Gamma$ is constant on the first segment in the leaf and the normal bundle is again isomorphic to $\mathbb{R}_{\geq 0}$, corresponding to deformations that replace that segment with an unbroken segment labelled $x_M^*$ and deform the weight away from 0 resp. $\infty$.

(b) The second possibility is that $\Gamma$ corresponds to a stratum with a boundary node: either $T$ has an edge of length zero or equivalently $S$ has a disk with a boundary node. See Figure 12.

![Figure 12. Treed disk with a zero-length edge](image2)
We call $\mathcal{M}_Γ(L, D)$ in the first resp. second case a true resp. fake boundary stratum. A fake boundary stratum is not a part of the boundary in the sense that the space is not locally homeomorphic to a manifold with boundary near such a stratum since any such treed disk may be deformed either by deforming the disk node, or deforming the length of the edge corresponding to the node to a positive real number.

We introduce orientations on the moduli spaces of treed holomorphic disks as follows. Let $[u : C \to X] \in \mathcal{M}_Γ(L, D)$ be an equivalence class of stable adapted holomorphic treed disks of combinatorial type $Γ$ of expected dimension 0. Assuming that regularity has been achieved, the linearization $D\bar{∂}_Γ|_{P_Γ}$ of the section $\bar{∂}_Γ$ of (9) restricted to the perturbation $P_Γ$ is an isomorphism. Choose orientations on the stable and unstable manifolds of the Morse function $W^±(x_i)$ for $x_i \in \mathcal{I}(L)$ so that the map

$$T_{x_i}W^-(x_i) \oplus T_{x_i}W^+(x_i) \to T_{x_i}L$$

induces an isomorphism of determinant lines

$$\det(T_{x_i}W^-(x_i) \oplus T_{x_i}W^+(x_i)) \to \det(T_{x_i}L).$$

In case $x_i = x^*_M$ we define

$$W^±(x^*_M) = W^±(x_M) \times \mathbb{R}$$

and choose orientations similarly. One naturally obtains an orientation of the determinant line of the linearized operator from the isomorphism

$$\det(D\bar{∂}_Γ|_{P_Γ}) \to \det(T\mathcal{M}_Γ) \otimes \det(TL) \otimes \det(TW^+(x_0)) \otimes \bigotimes_{i=1}^n \det(TW^-(x_i)),$$

the orientations on the stable and unstable manifolds and the orientation on the underlying moduli space of treed disks. See [7], [34] for similar discussions. The case of the type $Γ$ of a trivial trajectory with no disks is treated separately: In the case of a trajectory connecting $x^*_M$ with $x^*_M$ resp. $x^*_M$, the moduli space is a point and we define the orientation to agree resp. disagree with the standard orientation.

2.5. Compactness. In this section we show that the subset of the moduli space satisfying an energy bound is compact for suitable perturbation data. For arbitrary combinatorial types, compactness of the spaces of adapted treed disks (so that they have stable surface part domains) can fail since bubbles mapping entirely to the stabilizing divisor can develop.

**Definition 2.20.** For $E > 0$, an almost complex structure $J_D \in J_Γ(X, D)$ is $E$-stabilized by a divisor $D$ iff

- (Non-constant spheres) $D$ contains no non-constant $J_D$-holomorphic spheres of energy less than $E$; and
- (Sufficient intersections) each non-constant $J_D$-holomorphic sphere $u : C \to X$ resp. $J_D$-holomorphic disk $u : (C, ∂C) \to (X, L)$ with energy less than $E$
has at least three resp. one intersection points resp. point with the divisor $D$:

$$E(u) < E \implies \#u^{-1}(D) \geq 1 + 2(\chi(C) - 1)$$

where $\chi(C)$ is the Euler characteristic.

Denote by $H_2(X, \mathbb{Z})_{\text{sph}} \subset H_2(X, \mathbb{Z})$ the set of classes representing non-constant $J_D$-holomorphic spheres, and $H_2(X, \mathbb{Z})_{\text{disk}} \subset H_2(X, \mathbb{Z})$ the set of classes representing non-constant $J_D$-holomorphic disks with boundary in $L$. A divisor $D$ with Poincaré dual $[D]^\vee = k[\omega]$ for some $k \in \mathbb{N}$ has sufficiently large degree for an almost complex structure $J_D$ iff

$$
\begin{align*}
(\{D\}^\vee, \alpha) & \geq 2(c_1(X), \alpha) + \dim(X) + 1 \quad \forall \alpha \in H_2(X, \mathbb{Z})_{\text{sph}} \\
(\{D\}^\vee, \beta) & \geq 1 \quad \forall \beta \in H_2(X, L, \mathbb{Z})_{\text{disk}}.
\end{align*}
$$

We introduce the following notation for spaces of almost complex structures sufficiently close to the given one. Given $J \in \mathcal{J}_\tau(X, \omega)$ denote by

$$
\mathcal{J}_\tau(X, D, J, \theta) = \{J_D \in \mathcal{J}_\tau(X, \omega) \parallel J_D - J \parallel < \theta, \quad J_D(TD) = TD\}
$$

the space of tamed almost complex structures close to $J$ in the sense of [9, p. 335] and preserving $TD$.

**Lemma 2.21.** For $\theta$ sufficiently small, suppose that $D$ has sufficiently large degree for an almost complex structure $\theta$-close to $J$. For each energy $E > 0$, there exists an open and dense subset $\mathcal{J}^*(X, D, J, \theta, E)$ in $\mathcal{J}_\tau(X, D, J, \theta)$ such that if $J_D \in \mathcal{J}^*(X, D, J, \theta, E)$, then $J_D$ is $E$-stabilized by $D$. Similarly, if $D = (D^t)$ is a family of divisors for $J^t$, then for each energy $E > 0$, there exists a dense and open subset $\mathcal{J}^*(X, D^t, J^t, \theta, E)$ in the space of time-dependent tamed almost complex structures $\mathcal{J}_\tau(X, D^t, J^t, \theta)$ such that if $J_{D^t} \in \mathcal{J}^*(X, D^t, J^t, \theta, E)$, then $J_{D^t}$ is $E$-stabilized for all $t$.

We restrict to perturbation data taking values in $\mathcal{J}^*(X, D, J, \theta, E)$ for a (weakly or strictly) stabilizing divisor $D$ having sufficiently large degree for an almost-complex structure $\theta$-close to $J$. Let $J_D \in \mathcal{J}_\tau(X, D, J, \theta)$ be an almost complex structure that is stabilized for all energies, for example, in the intersection of $\mathcal{J}^*(X, D, J, \theta, E)$ for all $E$. For each energy $E$, there is a contractible open neighborhood $\mathcal{J}^{**}(X, D, J_D, \theta, E)$ of $J_D$ in $\mathcal{J}^*(X, D, J, \theta, E)$ that is $E$-stabilized.

The construction of perturbation data satisfying compactness depends crucially on the following relationship between the energy and the number of interior markings. Let $U$ be a type of stable treed disk. Disconnecting the components that are connected by boundary nodes with positive length one obtains types $\Gamma_1, \ldots, \Gamma_l$, and a decomposition of the universal curve $U_U$ into components $\overline{U}_{\Gamma_1}, \ldots, \overline{U}_{\Gamma_l}$. Let $n(\Gamma_i)$ denote the number of markings on $\overline{U}_{\Gamma_i}$, and $C(k)$ is the increasing linear function of $k$ arising in the construction of $D$ in Section 4 of [8].
Proposition 2.22. Any stable treed disk \( u : C \to X \) with domain of type \( \Gamma \) and only transverse intersections with the divisor has energy at most
\[
E(u|_{C_i}) \leq n(\Gamma, k) := \frac{n(\Gamma_k)}{C(k)}
\]
on the component \( C_i \subset C \) contained in \( \mathcal{U}_{\Gamma_i} \).

This follows immediately from the relationship \( PD[D] = k[\omega] \).

Definition 2.23. A perturbation datum \( P = (P_\Gamma, J_\Gamma) \) for a type of stable treed disk \( \Gamma \) is stabilized by \( D \) if \( J_\Gamma \) takes values in \( J^*(X, D, J, \theta, n(\Gamma_i, k)) \) on \( \mathcal{U}_{\Gamma_i} \) (in particular, if \( J_\Gamma \) takes values in \( J^{**}(X, D, J, \theta, n(\Gamma_i, k)) \)).

In other words, perturbations sufficiently close to the almost complex structure \( J_D \) have the property that any component of an adapted disk or sphere in the domain is stabilized by its intersections with the stabilizing divisor.

Theorem 2.24. (Compactness for fixed type) For any collection \( P = (P_\Gamma) \) of coherent, regular, stabilized perturbation data and any uncrowded type \( \Gamma \) of expected dimension at most one, the moduli space \( \mathcal{M}_{\Gamma}(L, D) \) of adapted stable treed marked disks of type \( \Gamma \) is compact and the closure of \( \mathcal{M}_{\Gamma}(L, D) \) contains only configurations with disk bubbling.

Proof. The proof is essentially the same as that in [8]. Because of the existence of local distance functions, similar to [27, Section 5.6], it suffices to check sequential compactness. Let \( u_\nu : C_\nu \to X \) be a sequence of stable treed disks of type \( \Gamma \), necessarily of fixed energy \( E(\Gamma) \). The sequence of stable disks \( [C_\nu] \) converges to a limiting stable disk \( [C] \) in \( \overline{\mathcal{M}}_{\Gamma} \). Then \( u_\nu : C_\nu \to X \) has a stable Gromov-Floer limit \( u : \hat{C} \to X \), where \( \hat{C} \) is a possibly unstable sphere or disk with stabilization \( C \). We show that \( u \) is adapted. From the (Compatible with the divisor) axiom, we have
\[
(J_\Gamma|\overline{\mathcal{U}}_{\Gamma})|D = J_D|D.
\]
Now \( J_D \in J^*(X, D, J, \theta, n(\Gamma_i, k)) \) was chosen so that \( D \) contains no \( J_D \)-holomorphic spheres. Hence the limit \( u \) satisfies the (Non-constant spheres) property.

To show the (Markings) property, note that each connected component \( C_i \) of \( u^{-1}(D) \) is either a point \( C_i = \{z_i\} \) or a union of sphere and disk components. In the first case, the intersection multiplicity \( m(z_i) \) with the divisor at the point \( z_i \) is positive while in the second, the intersection multiplicity at each node connecting the component \( C_i \) with the rest of the domain \( C \) is positive. In the first case, there necessarily exists a sequence \( z_\nu \) of points in \( u_\nu^{-1}(D) \) with positive intersection multiplicity converging to \( z_i \). By the (Markings) property of \( u_\nu \), the points \( z_\nu \) must be markings. Hence \( z_i \) is a marking as well. In the second case, the positivity of the intersection multiplicity implies that there exists a sequence of points \( z_{\nu,i} \in u_\nu^{-1}(D) \subset C_\nu \) converging to \( C_i \). Either \( z_{\nu,i} \) are isolated, or lie on some sequence of connected components \( C_{\nu,i} \) of \( u_\nu^{-1}(D) \subset C_\nu \) on which \( u_\nu \) is constant in which case any
limit point of $C_{\nu,i}$ lies in $C_i$. Since each $C_{\nu,i}$ contains a marking by the (Markings) property, so does $C_i$. Note that if $u_{\nu}(z_{i,\nu}) \in D$ then $u(z_i) \in D$, by convergence on compact subsets of complements of the nodes. This shows the (Markings) property.

To show the (Stable domains) property note that since $D$ is stabilizing for $\mathcal{L}$, any disk component of $C$ must have at least one interior intersection point with $D$. Since this point lies in the interior, and the boundary has at least one special point any such component is stable. Since $J_0$ is regular, the trajectories $u_{\nu}$ have only transverse intersections with $D$. Any unstable spherical resp. disk component $\hat{C}_i$ of $\hat{C}$ attached to a component of $C$ in $\mathcal{U}_{\Gamma_i}$ has energy at most $n(\Gamma_i,k)$. Suppose that $u$ is non-constant on $\hat{C}_i$. Then since $J_0$ is constant and equal to an element of $\mathcal{J}^*(X,D,J,\theta,n(\Gamma_i,k))$ on $\hat{C}_i$, the restriction of $u$ to $\hat{C}_i$ has at least three intersection points resp. one intersection point with $D$. By definition these points must be markings, which contradicts the instability of $\hat{C}_i$. Hence the stable map $u$ must be constant on $\hat{C}_i$, and thus $\hat{C}_i$ must be stable. This shows that $\hat{C}$ is equal to $C$ so that we get the (Stable domain) property.

It remains to rule out sphere bubbling. Suppose $C$ has a spherical component on which $u$ is non-constant. After forgetting all but one marking on maximal ghost components we obtain a configuration in an uncrowded stratum $\mathcal{M}_{\Gamma_i}'(\mathcal{L},D)$ of negative expected dimension, a contradiction. Hence all spherical components of $C$ are ghost components. Suppose that $\hat{C}_\nu$ is a maximal ghost component of $C$. Necessarily, $\hat{C}_\nu$ contains at least two markings that are limits of markings on $C_{\nu,i}$. By invariance of local intersection degree, the node at which $\hat{C}_\nu$ attaches a disk component has intersection degree at least two with the divisor. Such a configuration lies in a stratum with negative expected dimension, a contradiction. \[\square\]

2.6. Composition maps. In this section we use holomorphic treed disks to define the structure coefficients of the Fukaya algebra. Let $q$ be a formal variable and $\Lambda$ the universal Novikov field of formal sums with rational coefficients

\begin{equation}
\Lambda = \left\{ \sum_i c_i q^{\rho_i} \mid c_i \in \mathbb{C}, \rho_i \in \mathbb{R}, \rho_i \to \infty \right\}
\end{equation}

We denote by $\Lambda_{\geq 0}$ resp. $\Lambda_{>0}$ the subalgebra with only non-negative resp. positive powers.

We define the space of Floer cochains for Lagrangians with additional data called brane structures. Let $X$ be a compact symplectic manifold and let $\text{Lag}(X)$ denote the fiber bundle over $X$ whose fiber $\text{Lag}(X)_x$ at $x$ consists of Lagrangian subspaces of $T_xX$. Let $g$ be an even integer. A Maslov cover is an $g$-fold cover $\text{Lag}^g(X) \to \text{Lag}(X)$ such that the induced two-fold cover

\[ \text{Lag}^2(X) := \text{Lag}^g(X)/\mathbb{Z}_{g/2} \to \text{Lag}(X) \]

is the oriented double cover. A Lagrangian submanifold $L$ is admissible if $L$ is compact and oriented; we also assume for simplicity that $L$ is connected. A grading
on $L$ is a lift of the canonical map
\[ L \to \text{Lag}(X), \quad l \mapsto T_l L \]
to $\text{Lag}^g(X)$. A relative spin structure for an oriented Lagrangian $L \subset X$ is a lift of the Čech class of its tangent bundle to relative non-abelian cohomology with values in $\text{Spin}$ relative to the map $i : L \to X$; equivalently, a lift of the transition maps $\psi_{\alpha \beta}$ for $TL$ to $\text{Spin}$ satisfying the cocycle condition
\[ \psi_{\alpha \beta} \psi_{\alpha \gamma}^{-1} = i^* \epsilon_{\alpha \beta \gamma} \]
where $\epsilon_{\alpha \beta \gamma}$ is a 2-cycle on $X$. We denote by
\[ \Lambda^\times = \{ c_0 + \sum_{i > 0} c_i q^i \subset \Lambda_{\geq 0} \mid c_0 \neq 0 \} \]
the subgroup of formal power series with invertible leading coefficient. A rank one local system with values in $\Lambda^\times$ is a representation $\pi_1(L) \to \Lambda^\times$. A brane structure on $L$ is a relative spin structure with the given background class, a grading for the given Maslov cover, and a rank one local system with values in $\Lambda^\times$. An admissible Lagrangian brane is an admissible Lagrangian submanifold equipped with a brane structure. For $L$ an admissible Lagrangian brane define the space of Floer cochains
\[ \widehat{CF}(L) = \bigoplus_{d \in \mathbb{Z}_q} \widehat{CF}_d(L), \quad \widehat{CF}_d(L) = \bigoplus_{x \in \hat{I}_d(L)} \Lambda <x> \]
where $\hat{I}_d(L)$ are as in (7). Let $CF(L) \subset \widehat{CF}(L)$ the subspace generated by $x \in \hat{I}(L)$.

The composition maps are defined in the following. Given a Lagrangian brane $L$, we denote by $\text{Hol}_L(u) \in \mathbb{C}^\times$ the evaluation of the local system on the homotopy class of loops defined by going around the boundary of the treed disk once. We denote by $\sigma([u])$ the number of interior markings of $[u] \in \bar{M}_\Gamma(L, D, x)$.  

**Definition 2.25.** (Higher composition maps) For regular stabilizing coherent perturbation data $(P_\Gamma)$ define
\[ \mu^n : \widehat{CF}(L)^{\otimes n} \to \widehat{CF}(L) \]
on generators by
\[ \mu^n(x_1, \ldots, x_n) = \sum_{x_0, [u] \in \bar{M}_\Gamma(L, D, x_0)} (-1)^{\bigtriangledown} (\sigma([u])!)^{-1} \text{Hol}_L(u) q^{E([u])} \epsilon([u]) x_0 \]
where $\bigtriangledown = \sum_{i=1}^n i |x_i|$.

**Remark 2.26.** (Zero-th composition map is a quantum correction) Any configuration with a only single outgoing edge must have at least one non-constant disk. Hence $\mu^0(1)$ has no term with coefficient $q^0$.

**Remark 2.27.** (Leading order term in the first composition map) As in Remark 2.2 (??) and 2.13, the constant trajectories at the maximum $x_M$ with weighted leaf and outgoing unweighted root are part of $\bar{M}_\Gamma(L, D, x_0, x_M^*)_0$. The orientations on these
trajectories are determined by the orientation on \( \mathcal{M}_\Gamma \) which by the discussion after (15) is negative resp. positive for \( x_0 = x_M^\bullet \) resp. \( x_0 = x_M^\circ \). Hence

\[
\mu^1(x_M^\circ) = x_M^\circ - x_M^\bullet + \sum_{x_0, [u] \in \overline{\mathcal{M}}_\Gamma(L, D, x_M^\circ, x_0) \atop [u] \in \mathcal{M}_\Gamma(L, D, x_M^\circ, x_0) \atop E([u]) > 0} (-1)^{\sigma([u])!} q^{-E([u])} \epsilon([u]) \text{Hol}_L(u)x_0.
\]

This formula is similar to that in Fukaya-Oh-Ohta-Ono [17, (3.3.5.2)]. Presumably the discussion here is a version of their treatment of homotopy units.

\[ \text{Theorem 2.28. (A}_\infty \text{ algebra for a Lagrangian) For any coherent regular stabilizing perturbation system } \mathcal{P} = (\mathcal{P}_\Gamma) \text{ the maps } (\mu^n)_{n \geq 0} \text{ satisfy the axioms of a (possibly curved) } A_\infty \text{ algebra } \hat{CF}(L) \text{ with strict unit. The subspace } CF(L) \text{ is a subalgebra without unit.} \]

\[ \text{Proof. Since the one-dimensional component of the moduli space with bounded energy is a finite union of compact oriented one manifolds, the boundary points of the moduli space come in pairs. This gives the identity} \]

\[
0 = \sum_{\Gamma \in \mathcal{I}_{n,m}} \sum_{[u] \in \partial \mathcal{M}_{\Gamma}(L, D, x_0)} (-1)^{\sigma([u])!} q^{-E([u])} \epsilon([u]) \text{Hol}_L(u).
\]

In case the moduli spaces do not involve weightings then each combinatorial type \( \Gamma \) with a single interior edge of infinite length is obtained by gluing together graphs \( \Gamma_1, \Gamma_2 \) with \( n - n_2 + 1 \) and \( n_2 \) leaves along a single leaf, say with \( m_1 \) resp. \( m_2 \) interior markings, and by the (Products) axiom we have an isomorphism

\[
\overline{\mathcal{M}}_{\Gamma}(L, D, x_0) = \cup_{y, \Gamma_1, \Gamma_2} \overline{\mathcal{M}}_{\Gamma_1}(L, D, x_0, x_1, \ldots, x_{i-1}, y, x_{i+n_2+1}, \ldots, x_n) \times \overline{\mathcal{M}}_{\Gamma_2}(L, D, y, x_i, \ldots, x_{i+n_2-1}).
\]

Say \( \sigma([u]) = m \). Since there are \( m \) choose \( m_1, m_2 \) ways of distributing the interior markings to the two components graphs,

\[
0 = \sum_{i, m_1 + m_2 = m} (m!)^{-1} \binom{m}{m_1} q^{-E([u_1]) - E([u_2])} \epsilon([u_1]) \epsilon([u_2]) \text{Hol}_L(u_1) \text{Hol}(u_2)
\]

which is the \( A_\infty \) axiom up to signs. In the case of a weighted marking one has additional boundary components corresponding to when the weighting becomes zero or infinity, but those configurations correspond to splitting off a constant Morse trajectory with weighted leaf and outgoing forgettable or unforgettable root, that is, the terms \( x_M^\circ \) and \( x_M^\bullet \) in \( \mu^1(x_M^\bullet) \).

Note that the above one dimensional moduli spaces actually have more boundary components than there are terms in the \( A_\infty \) relations. Indeed, a treed disk with a weighting \( \rho \in [0, \infty] \) on a branch asymptotic to the maximum can develop a breaking
on the end near that leaf. The resulting broken configuration is made of a flow line from the maximum to a point critical $x$ of index 1 and a composed configuration satisfying a perturbation with a $|0, \infty|$ weight on the branch with limit $x$. The latter is not a configuration represented by a term in the $A_\infty$ relations, which at first makes it appear that the $A_\infty$ relations do not hold. However, since the Morse differential of the maximum vanishes, there exists another similar configuration broken at $x$ with the opposite sign that will cancel it.

It remains to justify the signs given by orientations on the moduli spaces of treed disks. First consider the moduli space $\overline{D}_{n,m}$ of stable disks which were already discussed in Section 2.1. Any stratum $\mathcal{D}_\Gamma \subset \overline{D}_{n,m}$ corresponding to disks with $\nu$ nodes has a neighborhood homeomorphic to a neighborhood of $\mathcal{D}_\Gamma$ in $\mathcal{D}_\Gamma \times [0, \infty)^\nu$. Suppose that $\nu = 1$ so that

$$\mathcal{D}_\Gamma \cong \mathcal{D}_{n,1} \times \mathcal{D}_{i,2} \cong \mathcal{D}_{n-i+1, m_1} \times \mathcal{D}_{i, m_2}$$

for some $i \geq 2$ or $m_2 \geq 1$. The homeomorphism has inverse given by a gluing map of the form

$$\begin{align*}
(0, \epsilon) \times \mathcal{D}_{n-i+1, m_1} \times \mathcal{D}_{i, m_2} \to \mathcal{D}_{n, m} \\
(\delta, (x_1, \ldots, x_{n-i+1}, z_2, \ldots, z_{m_2}),(x_1', \ldots, x_i', z_2', \ldots, z_{m_1}')) \mapsto \\
(x_1, \ldots, x_j, x_{j+1} + \delta^{-1}x_1', \ldots, x_{j+1} + \delta^{-1}x_i', x_{j+2}, \ldots, x_{n-i+1}, \\
x_{j+1} + \delta^{-1}x_j, x_{j+1} + \delta^{-1}z_2', \ldots, x_{j+1} + \delta^{-1}z_{m_1}', z_2, \ldots, z_{m_2}).
\end{align*}$$

The inclusion $\mathcal{D}_{n-i+1, m_1} \times \mathcal{D}_{i, m_2} \to \mathcal{D}_{n, m}$ then has orientation sign $(-1)^{(n-i-j)+j}$.

By construction, the inclusion of $\mathcal{M}_{n-i+1, m_1} \times \mathcal{M}_{i, m_2}$ in $\mathcal{M}_{n, m}$ has the same sign as that of the inclusion of $\mathcal{D}_{n-i+1, m_1} \times \mathcal{D}_{i, m_2}$ in $\mathcal{D}_{n, m}$. The case involving a constant trajectory will be treated separately below.

The signs in the $A_\infty$ axiom are derived as follows, using the orientations constructed in (15). We omit interior markings to simplify notation, since the deformations associated to these are even dimension and so do not affect the computation. If $i \geq 2$ or $m_2 \geq 1$ then the gluing construction for treed holomorphic disks gives rise to a gluing map

$$\begin{align*}
(0, \epsilon) \times \mathcal{M}_{n-i+1} (x_0, x_1, \ldots, x_j, y, x_{j+1}, \ldots, x_n)_0 \times \mathcal{M}_i (y, x_{j+1}, \ldots, x_{j+i})_0 \\
\to \mathcal{M}_n (x_0, \ldots, x_n)_1.
\end{align*}$$

In the special case $i = 1$ and $m_2 = 0$ where the breaking is that of a trivial trajectory, the gluing map is given by replacing the $i$-th weight with the gluing parameter resp. its reciprocal if $y = x_M$ resp. $x_M^\ast$. The orientation on the determinant line of $\mathcal{M}_i (y, x_{j+1}, \ldots, x_{j+i})$ is induced from an isomorphism

$$\begin{align*}
\det(T\mathcal{M}_i (y, x_{j+1}, \ldots, x_{j+i})) \to \\
\det(T\mathcal{M}_i \oplus TL \oplus T_j W^+ \oplus T_{x_{j+1}} W^- \oplus \cdots \oplus T_{x_{j+i}} W^-).
\end{align*}$$

Here $T_x W^-$ is the negative part of the tangent space $T_x L$ with respect to the Hessian of $F$, with an additional factor of $\mathbb{R}$ if $x_i = x_M^\ast$. Similarly the orientation on the
determinant line of $TM_{n-i+1}(x_0, x_1, \ldots, y, \ldots, x_n)$ is induced from an isomorphism

$\det(TM_{n-i+1}(x_0, x_1, \ldots, y, \ldots, x_n)) =
\det(TM_{n-i+1, m_1} \oplus TL \oplus TW^+_0 \oplus TW^-_{x_1} \oplus \ldots \oplus TW^-_{x_n}).$

Transposing $TM_i$ with $TL \oplus T_{x_0} W^+ \oplus T_{x_1} W^- \oplus \ldots \oplus TW^-_y \oplus \ldots \oplus T_{x_n} W^-$ yields a sign of $(-1)^i$ to the power $i(n - i + 1)$. Transposing $TL \oplus T_{y} W^+ \oplus T_{x_{j+1}} W^- \oplus \ldots \oplus T_{x_{j+i+1}} W^-$ with $TL \oplus TW^+_0 \oplus TW^-_{x_{j+i+1}} \oplus \ldots \oplus TW^-_{x_n}$ yields a sign $(-1)^i$ to the power $i\left(\sum_{k=j+i+1}^n |x_{j+1}|\right)$. The gluing map $M_i \times M_{n-i+1} \to M_n$ has sign $(-1)^i$ to the power $i(n - i - j) + j$, see (23). Comparing the contributions from $(-1)^i$ gives a sign of $(-1)^i$ to the power

$$
(27) \quad \sum_{k=1}^n k|x_k| + \sum_{k=1}^i k|x_{j+k}| + \sum_{k=1}^j k|x_{i+k}| + \sum_{k=1}^{n-i-j} (j + k + 1)|x_{j+i+k}| + (j + 1)|y| \equiv 2j(|y| + i) + (i - 1)(\sum_{k=1}^{n-i-j} |x_{j+i+k}|) + (j + 1)|y| \equiv 2|y| + ji + (i - 1)(\sum_{k=1}^{n-i-j} |x_{j+i+k}|)
$$

while the sign in the $A_\infty$ axiom contributes $\sum_{k=1}^j (|x_k| - 1)$. Combining the signs one obtains in total

$$
(28) \quad i\left(\sum_{k=i+j+1}^n |x_k|\right) + i(n - i + 1) + i(n - i - j) + j + ji + (i - 1)\left(\sum_{k=i+j+1}^n |x_k|\right) + |y| + \sum_{k=1}^j (|x_k| - 1) = \sum_{k=1}^j (|x_k| - 1) + |y| + i \sum_{k=i+j+1}^n |x_k| \equiv 2\sum_{k=1}^n |x_k|
$$

which is independent of $i, j$. A similar computation holds in the case of splitting off a trivial trajectory: in that case, $i = 1$ and $m_2 = 0$ then transposing the gluing parameter with the determinant lines of $TM_n$, $TL$, $T_{x_0}^+ W$, and $T_{x_k}^- W$, $1 \leq k \leq j$ yields a sign $(-1)^{n+\sum_{k \leq j} |x_k|}$, while the signs $\forall$ for the various moduli spaces combine to $(-1)^j$. Combining with the sign in the $A_\infty$ axiom gives $(-1)^{n-j-|x_0|+j} = (-1)^{\sum_{k=1}^n |x_k|}$ as in the previous case. The $A_\infty$ axiom follows.

Finally we prove the assertion on the strict units. We claim that a strict unit is given by the element $e_L = x_M^t \in \text{OF}(L)$. The (Infinite weights) axiom implies that the perturbation data $P_T$ for the moduli spaces $\overline{M}(L, D, \ldots, x_{i-1}, x_i = x_M^t, x_{i+1}, \ldots)$ is pulled back from data $P_{T'}$ for $\overline{M}(L, D, \ldots, x_{i-1}, x_{i+1}, \ldots)$ for the type $\Gamma'$ obtained from $\Gamma$ by forgetting the corresponding incoming leaf, as long
as the type $\Gamma$ has a stabilization. It follows that for dimension reasons, the compositions $\mu^n(x_1, \ldots, x_M)$ vanish except for the case that the resulting type has no stabilization: That is, $n = 2$ and the underlying configuration has no non-constant disks and the map is constant on the domain. One obtains from the configuration with constant values $x_i$ (or $x_M$ if $x_i \in \{x_1^\circ, x_M^\circ, x_M^\bullet\}$)

$$\mu^2(x_1, x_M^\circ) = (-1)^{\deg(x_1)} \mu^2(x_M^\circ, x_1) = x_1, \quad \forall x_1 \in \hat{I}(L)$$

as in Figure 10. \qed

2.7. Maurer-Cartan moduli space. In this section we define a cohomology complex associated to an $A_\infty$ algebra satisfying suitable convergence conditions which is a complex of vector bundles over a space of solutions to a weak Maurer-Cartan equation. Let $A$ be a $A_\infty$ algebra defined over $\Lambda_{\geq 0}$ with strict unit $e_A$. Suppose that the vector space underlying $A$ is a finite rank and free $\Lambda_{\geq 0}$-module and furthermore admits a $\mathbb{Z}$-grading; we write

$$A = \bigoplus_{d \in [d_-, d_+]} A^d, \quad A_{\leq 0} = \bigoplus_{d \geq 0} A^d, \quad A_{< 0} = \bigoplus_{d < 0} A^d$$

for some integers $d_- \leq d_+$; define $A_{\geq 0}$ and $A_{> 0}$ similarly. Consider the decomposition

$$\text{Hom}(A^{\otimes n}, A) = \bigoplus_m \text{Hom}(A^{\otimes n}, A)_m$$

where elements of $\text{Hom}(A^{\otimes n}, A)_m$ have degree $m$. Write the map $\mu^n$ in terms of its components,

$$\mu^n = \sum_m \mu^{n,m}, \quad \mu^{n,m} \in \text{Hom}(A^{\otimes n}, A)_{2-n+m}$$

so that

$$\mu^{n,m}(A^{d_1} \otimes \ldots A^{d_n}) \subset A^{d_1 + \ldots + d_n + 2-n+m}.$$ We say that an $A_\infty$ algebra $A$ as above is convergent iff there exists a sequence $E_m \to \infty$ such that

$$(29) \quad \mu^{n,m}(A) \subset q^{E_m} A, \quad \forall n \geq 0.$$ We introduce a suitable domain for the weak Maurer-Cartan equation as follows. Given a convergent $A_\infty$ algebra, define

$$(30) \quad A^+ := A_{\leq 0} + \Lambda_{>0} A_{>0}, \quad A^{++} := A_{<0} + \Lambda_{>0} A_{\geq 0}.$$ In other words, in positive degree (resp. non-negative degree) the elements of $A^+$ resp. $A^{++}$ must have positive exponents of the formal variable $q.$

Lemma 2.29. For $b \in A^+$ the sum

$$(31) \quad \mu^0_b(1) := \mu^0(1) + \mu^1(b) + \mu^2(b, b) + \ldots$$

is well-defined.
Proof. For \( b \in A^+ \) write
\[
b = b_{\leq 0} + q^E b_{> 0}, \quad b_{\leq 0} \in A_{\leq 0}, \quad b_{> 0} \in A_{> 0}\]
for some \( E > 0 \). Consider the set \( S_{n_0} \) of tuples \((a_1, \ldots, a_n)\) where at most \( n_0 \) of the elements are equal to \( b_{> 0} \) and the remaining elements are equal to \( b_{\leq 0} \). Since the terms not in \( S_{n_0} \) have \( \mu^n(a_1, \ldots, a_n) \in q^{E n_0} A \), it suffices to show that for any \( n_0 \), the sum
\[
(32) \quad \sum_{n_0 \leq n, (a_1, \ldots, a_n) \in S_{n_0}} \mu^n(a_1, \ldots, a_n)
\]
converges in \( A \). Since the elements \( b_{> 0} \) are of degree at most \( d_+ \) and the output must be of degree at least \( d_- \),
\[
\mu^n(a_1, \ldots, a_n) = \sum_{m \geq n - 2 - d_+ n_0 - d_-} \mu^{n,m}(a_1, \ldots, a_n).
\]
By the convergent condition (2.33), we have
\[
\sum_{n \geq n_0, (a_1, \ldots, a_n) \in S_{n_0}} \mu^n(a_1, \ldots, a_n) \in q^{E n_1 - 2 + d_+ n_0 - d_-} A
\]
so the sum (32) converges. \( \square \)

More generally the same argument implies convergence of the deformed composition map
\[
\mu^n_b(a_1, \ldots, a_n) = \sum_{i_1, \ldots, i_{n+1}} \mu^{n+i_1+\ldots+i_{n+1}}(b, \ldots, b, a_1, b, \ldots, b, a_2, b, \ldots, b, a_n, b, \ldots, b)
\]
over all possible combinations of insertions of the element \( b \in A^+ \) between (and before and after) the elements \( a_1, \ldots, a_n \), is convergent for similar reasons. The maps \( \mu^n_b \) define an \( A_\infty \) structure on \( A \). In particular
\[
(\mu^1_b)^2(a_1) = \mu^2_b(\mu^1_b(1), a_1) - \mu^2_b(a_1, \mu^1_b(1)).
\]
The weak Maurer-Cartan equation for \( b \in A^+ \) is
\[
(33) \quad \mu^0_b(1) = \mu^0(1) + \mu^1(b) + \mu^2(b, b) + \ldots \in \Lambda e_A.
\]
Denote by \( \widetilde{MC}(A) \) the space of solutions to the weak Maurer-Cartan equation (33). Any solution to the weak Maurer-Cartan equation defines an \( A_\infty \) algebra such that \( (\mu^1_b)^2 = 0 \) and so has a well-defined cohomology
\[
H(\mu^1_b) = \frac{\ker(\mu^1_b)}{\text{im}(\mu^1_b)}.
\]
An \( A_\infty \) algebra is weakly unobstructed if there exists a solution to the weak Maurer-Cartan equation.
We now introduce a notion of gauge equivalence for solutions to the weak Maurer-Cartan equation. For $b_1, \ldots, b_n \in A^+$ define

$$
(34) \quad \mu_{b_0, b_1, \ldots, b_n}^n(a_1, \ldots, a_n) =
\sum_{i_1, \ldots, i_{n+1}} \mu^{n+i_1+\ldots+i_{n+1}}_{b_0, \ldots, b_0, a_1, b_1, \ldots, b_1, a_2, b_2, \ldots, b_2, \ldots, a_n, b_n, \ldots, b_n, i_{n+1}).
$$

Two cochains $b_0, b_1$ are gauge equivalent iff

$$
\exists h \in A^{++}, \ b_1 - b_0 = \mu_{b_0, b_1}^1(h),
$$

see (30). We then write $b_0 \sim_h b_1$.

Gauge equivalence is an equivalence relation, by a discussion parallel to that in Seidel [30, Section 1h]. To show transitivity, if $b_0 \sim_{h_{01}} b_1$ and $b_1 \sim_{h_{12}} b_2$ then $b_0 \sim_{h_{02}} b_2$ where

$$
h_{02} = h_{01} + h_{12} + \mu_{b_0, b_1, b_2}^2(h_{01}, h_{12}).
$$

To prove symmetry define suppose that $b_0 \sim_{h_{01}} b_1$. Define

$$
\phi(h_{10}) = h_{10} - (-1)^{|h_{10}|} \mu_{b_1, b_0, b_1}^2(h_{10}, h_{01})
$$

$$
\psi(h_{11}) = h_{11} + (-1)^{|h_{11}|} \mu_{b_2, b_1, b_0}^2(h_{01}, h_{11}).
$$

From the identities

$$
\mu_{b_1, b_0}^1 = \mu_{b_1, b_1}^1 - \mu_{b_1, b_0, b_1}^2(\cdot, b_1 - b_0), \quad \mu_{b_1, b_1}^1 = \mu_{b_0, b_1}^1 + \mu_{b_0, b_1, b_1}^2(b_1 - b_0, \cdot)
$$

one sees that $\phi, \psi$ are chain maps:

$$
\phi \mu_{b_1, b_0}^1 = \mu_{b_1, b_1}^1 \phi, \quad \psi \mu_{b_1, b_1}^1 = \mu_{b_0, b_1}^1 \psi.
$$

Indeed,

$$
\phi \mu_{b_1, b_0}^1(h_{10}) = \mu_{b_1, b_0}^1(h_{10}) - (-1)^{|h_{10}|} \mu_{b_1, b_0, b_1}^2(\mu_{b_1, b_0}^1(h_{10}), h_{01})
$$

$$
= \mu_{b_1, b_1}^1(h_{10}) + \mu_{b_1, b_0, b_1}^2(h_{10}, \mu_{b_0, b_1}^1(h_{01})) - (-1)^{|h_{10}|} \mu_{b_1, b_0, b_1}^2(\mu_{b_1, b_0}^1(h_{10}), h_{01})
$$

$$
= \mu_{b_1, b_1}^1(h_{10}) - (-1)^{|h_{10}|} \mu_{b_1, b_1}^1(\mu_{b_0, b_0, b_1}^1(h_{10}, h_{01}))
$$

$$
= \mu_{b_1, b_1}^1(\phi(h_{10}))
$$

and similarly for $\psi$. Since the $g = 0$ part of $\mu_{b_1, b_0, b_1}^2(\cdot, h_{01})$ resp. $\mu_{b_1, b_1, b_0}^2(h_{01}, \cdot)$ has negative degree, the maps $\phi, \psi$ are invertible. Furthermore,

$$
(35) \quad \phi(b_0 - b_1) = \mu_{b_1, b_1}^1(h_{01})
$$

$$
(36) \quad \psi(\mu_{b_1, b_1}^1(h_{01})) = (-1)^{|h_{01}|} \mu_{b_0, b_1}^1 \mu_{b_0, b_1, b_1}^2(h_{01}, h_{01}) + b_0 - b_1.
$$

Hence if we define

$$
h_{10} := (\psi \circ \phi)^{-1}(-h_{01} - (-1)^{|h_{01}|} \mu_{b_0, b_1, b_1}^2(h_{01}, h_{01}))
$$
then \( b_1 \sim_{h_{0,0}} b_0 \):

\[
\mu_{b_1,b_0}^1(h_{10}) = \mu_{b_1,b_0}^1 \phi^{-1} \psi^{-1}(-h_{01} - (-1)^{|h_{01}|} \mu_{b_0,b_1,b_1}^2(h_{01},h_{01})) = \phi^{-1} \psi^{-1} \mu_{b_0,b_1}^1(-h_{01} - (-1)^{|h_{01}|} \mu_{b_0,b_1,b_1}^2(h_{01},h_{01})) = \phi^{-1} \psi^{-1}((-1)^{|h_{01}|} \mu_{b_1,b_1}^1 \mu_{b_0,b_1,b_1}^2(h_{01},h_{01}) + b_0 - b_1) = b_0 - b_1.
\]

Also \( b \sim b \) for any \( b \), hence \( \sim \) is reflexive.

We now define the potential of the algebra as a function on the moduli space of solutions to the weak Maurer-Cartan equation. Denote by \( MC(A) \) the quotient of \( \widetilde{MC}(A) \) by the gauge equivalence relation,

\[ MC(A) = \widetilde{MC}(A)/\sim \]

which we call the moduli space of solutions to the weak Maurer-Cartan equation. Define a potential

\[ \widetilde{W} : \widetilde{MC}(A) \to \Lambda \]

on the space of solutions to the weak Maurer-Cartan equation by

\[ \widetilde{W}(b)_{1_A} = \mu_{b}^0(1). \]

**Lemma 2.30.** The potential \( \widetilde{W} \) is gauge-invariant and so descends to a potential \( W : MC(A) \to \Lambda. \)

**Proof.** We have

\[
\mu_{b_1}^1(1) - \mu_{b_0}^1(1) = \sum_{i,j} \mu_{i,j}^{i+j+1}(b_0, \ldots, b_0, b_1 - b_0, b_1, \ldots, b_1) = \sum_{i,j} \mu_{i,j}^{i+j+1}(b_0, \ldots, b_0, \mu_{b_1-b_0}(h), b_1, \ldots, b_1) = \sum_{i,j,k} \mu_{i,j,k}^{i+j+k+2}(b_0, \ldots, b_0, \mu_{b_0}^1(1), b_0, \ldots, b_0, h, b_1, \ldots, b_1) + (-1)^{|h|+|b_1|+j+1} \sum_{i,j,k} \mu_{i,j,k}^{i+j+k+1}(b_0, \ldots, b_0, h, b_1, \ldots, b_1, \mu_{b_1}^1(1), b_1, \ldots, b_1) = \mu^2(\widetilde{W}(b_0)e_A, h) - (-1)^{|h|} \mu^2(h, \widetilde{W}(b_1)e_A) = (\widetilde{W}(b_0) - \widetilde{W}(b_1))h.
\]

It follows that

\[ (e_A + h)\widetilde{W}(b_1) = (e_A + h)\widetilde{W}(b_0). \]

Since \( h \in A^{++} = A_{<0} + \Lambda_{>0}A_{>0} \) (that is, the components of non-negative degree have positive \( q \)-degree) the sum \( e_A + h \) is non-zero. Hence \( \widetilde{W}(b_1) = \widetilde{W}(b_0) \) as claimed. \( \square \)

**Corollary 2.31.** If \( b_0 \sim_h b_1 \), then \( \mu_{b_0,b_1}^1 \) is a differential.
Proof. Using the \( A_\infty \) relations and strict unitality we have
\[
(m_{b_0, b_1}^1)^2(a) = m^2(m_{b_0}^0(1), a) - (-1)^{|a|} m^2(a, m_{b_1}^0(1)) = (\tilde{W}(b_1) - \tilde{W}(b_0))a = 0
\]
as claimed. \( \square \)

The cohomology of an \( A_\infty \) algebra is a complex of bundles over the space of solutions to the weak Maurer-Cartan equation. For any \( b \in \tilde{MC}(A) \) define
\[
H(b) := H(m_{b}^1) = \frac{\ker(m_{b}^1)}{\text{im}(m_{b}^1)}.
\]
The cohomology complex is the resulting complex of sheaves over \( \tilde{MC}(A) \):
\[
A \times \tilde{MC}(A) \to A \times \tilde{MC}(A), \quad (a, b) \mapsto (m_{b}^1(a), b)
\]
The complex (37) may be viewed as an object in the derived category of bounded complexes of coherent sheaves of \( \mathcal{O}_{\tilde{MC}(A)} \)-modules. Since we are working over Novikov fields the algebraic geometry here is non-standard and we do not attempt to discuss it here. The “stalks” of the cohomology complex fit together to the “cohomology sheaf”
\[
H(A) := \bigcup_{b \in \tilde{MC}(A)} H(b).
\]

Lemma 2.32. The cohomology sheaf \( H(A) \) is gauge-equivariant in the sense that if \( b_0 \sim_{h_{10}} b_1 \), then \( H(b_0) \simeq H(b_1) \).

Proof. One can verify that
\[
\mu^2_{b_1, b_0, b_0}(h_{10}, a) = \sum_{n_1, n_2, n_3} \mu^{2+n_1+n_2+n_3}(b_{1, \ldots, b_{1}}, h_{10}, b_{0, \ldots, b_{0}}, a, b_{0, \ldots, b_{0}})
\]
satisfies
\[
\mu_{b_1, b_0, b_0}^2(h_{10}, \mu_{b_0}^1(a)) - \mu_{b_1, b_0}^1(\mu_{b_1, b_0, b_0}^2(h_{10}, a)) = \mu_{b_0}^1(a) - \mu_{b_1, b_0}^1(a).
\]
Hence the operator
\[
\mu^2_{b_1, b_0, b_0}(h_{10}, \_)(A, \mu_{b_0}^1) : (A, \mu_{b_0}^1) \to (A, \mu_{b_1, b_0}^1),
\]
is a chain morphism, see (35). For the same reasons,
\[
\mu^2_{b_0, b_1, b_0}(h_{10}, \_)(A, \mu_{b_1}^1) : (A, \mu_{b_1}^1) \to (A, \mu_{b_0}^1),
\]
is a chain morphism. Consider the map
\[
A \to A, \quad a \mapsto H^1_{b_0, b_1, b_0}(h_{01}, h_{10}, a)
\]
where
\[
H_{b_0,b_1,b_0}(h_{01},h_{10},a) := \sum_{n_1,n_2,n_3,n_4} \mu_{n_1+n_2+n_3+n_4}^{b_0,b_1,b_0}(b_0,\ldots,b_0,h_{01},
\underbrace{b_1,\ldots,b_1}_n,h_{10},\underbrace{b_0,\ldots,b_0}_m,a,\underbrace{b_0,\ldots,b_0}_n).
\]

This map is a chain homotopy between
\[
(\mu_{b_0,b_1,b_0}^2(h_{10},\_)-\text{Id}(\_)) \circ (\mu_{b_0,b_1,b_0}^2(h_{10},\_)-\text{Id}(\_))
\]
and the chain map
\[
\Phi(a) = a + \mu_{b_0}^2(h_{01} + h_{10} + \mu_{b_0,b_1,b_0}^2(h_{01},h_{10}),a).
\]
The latter can be seen to induce an isomorphism of $H(b_0)$. In the same way, their reverse composition is homotopic to a map inducing an isomorphism of
\[
H(b_1,b_0) := H(A,\mu_{b_1,b_0}^1).
\]
Similarly, $\mu_{b_1,b_0}^2(h_{10},\_)-\text{Id}(\_)$ and $\mu_{b_1,b_0}^2(h_{01},\_)-\text{Id}(\_)$ define a chain morphism from $(A,\mu_{b_1}^1)$ to $(A,\mu_{b_1,b_0}^1)$ and $(A,\mu_{b_1,b_0}^1)$ to $(A,\mu_{b_1}^1)$, respectively. This shows that $H(b_1,b_0) \cong H(b_1)$. Hence $H(b_0) \cong H(b_1,b_0) \cong H(b_0)$ as claimed.

Morally speaking, the proposition implies that the cohomology bundle descends to the Maurer-Cartan moduli space but we are unsure about whether this point of view is really helpful. An $A_\infty$ algebra $A$ is said to have non-zero cohomology if the cohomology complex $H(A)$ has a non-zero stalk at some point.

We show that the perturbation system may be chosen so that the Fukaya algebra associated to a Lagrangian is convergent:

**Definition 2.33.** A perturbation system $P = (P_\Gamma)$ is **convergent** if for each energy bound $E$, there exists a constant $C(E)$ such that for any $\Gamma$ and any treed $J_\Gamma$-holomorphic disk $u : C \to X$ of type $\Gamma$, the total Maslov index $I(u) := \sum I(u_i)$ satisfies
\[
(E(u) < E) \implies (I(u) < c(E)).
\]

**Lemma 2.34.** Any convergent, coherent, regular, stabilizing perturbation system $P = (P_\Gamma)$ defines a convergent Fukaya algebra $\widehat{CF}(L,P)$.

**Proof.** We claim that the sequence $E_m := \inf \{E(u) | I(u) \geq m\}$ converges to infinity. To prove the claim, note that otherwise there would exist a sequence $u_m$ with $I(u_m) \geq m$ and $E(u_m)$ bounded, which would violate (40). The convergence condition (2.33) follows.

**Proposition 2.35.** There exist convergent, coherent, regular, stabilizing perturbations $P = (P_\Gamma)$. 
Proof. First, note that for a fixed almost complex structure $J_D$ and energy bound $E$, the perturbation system given by taking $J_\Gamma$ constant equal to $J_D$ is convergent. Indeed, the number of disk and sphere components is bounded by $E/\hbar$, while the number of homotopy classes of disks and spheres with energy at most $E$ is finite by Gromov compactness.

Next, we may suppose that the surface part $S_\Gamma$ of the universal bundle is equipped with a metric so that $C^2$ norms are well-defined. Recall that any fiber $S_\Gamma$ of $S_\Gamma$ breaks up into disjoint components $S_\Gamma_i$ with number of interior markings $n(\Gamma_i, k)$. There exists a function $d(n)$ such that if $\|J_\Gamma - J_D\|_{C^2(S_\Gamma_i \times X, \text{End}(TX))} \leq d(n(\Gamma_i, k))$ then the homotopy classes of spheres and disks appearing in $P_\Gamma = (J_\Gamma, F_\Gamma)$-holomorphic treed disks of type $\Gamma$ are contained in those for $P = (J_D, F)$. Indeed, otherwise there exists a sequence of types $\Gamma_\nu$, a sequence $J_\Gamma, \nu, F_\Gamma, \nu$ of perturbation data, an integer $i$, and a sequence of components $S_{\Gamma_i, \nu} \subset S_{\Gamma_i}$ with $n(\Gamma_i, k)$ markings such that $J_{\Gamma_i, \nu} \to J_D$ in $C^2$ on $S_{\Gamma_i, \nu}$ such that the homotopy type of spheres and disks in $u_\nu|S_{\Gamma_i, \nu}$ does not occur for $J_D$-holomorphic spheres and disks with $n(\Gamma_i, k)$ markings. By Gromov compactness, $S_{\Gamma_i, \nu}$ converges to a stable disk or sphere and $u_\nu$ converges to a limit $u : S \to X$ with the same homotopy type as $u_\nu$ for $\nu$ sufficiently large, a contradiction. Hence the inequality (40) holds for $(J_\Gamma, F_\Gamma)$-holomorphic treed disks as well, as long as the perturbations are chosen $C^2$-small in the above sense. Since the $C^2$ bound depends only on the number of markings $n(\Gamma_i, k)$ on the connected components of the fibers $S \subset S_\Gamma$, this restriction is compatible with the coherence and regularity conditions. Furthermore, since the condition is open then the restriction does not affect the construction of regular almost complex structures either. The statement of the proposition follows. \qed

Consider the space of solutions to the weak Maurer-Cartan equation:

$$\tilde{MC}(L) := \mu^{-1}(\Lambda e_L) \subset CF(L), \quad e_L = \langle x_M^* \rangle$$

The following Lemma will be used elsewhere to show that Lagrangians are weakly unobstructed.

**Lemma 2.36.** Suppose that $\mu^0(1) \in \Lambda x_M^*$ and there are no non-constant disks of non-positive Maslov index. Then $\tilde{MC}(L)$ is non-empty.

*Proof.* Suppose $\mu^0(1) = W x_M^*$ and the condition in the Lemma holds. Equation (19) becomes $\mu^1(x_M^*) = x_M^* - x_M^*$. Hence

$$\mu(W x_M^*) = \mu^0(1) + W \mu^1(x_M^*) = W x_M^* + W(x_M^* - x_M^*) = W x_M^* \in \Lambda x_M^*$$

which implies $W x_M^* \in \tilde{MC}(L)$. \qed
3. Homotopy invariance

We show that the Fukaya algebra constructed above is independent, up to $A_\infty$ homotopy invariance, of the choice of perturbation system. The argument uses moduli spaces of quilted treed disks, introduced without trees in [26], which are a particular realization of Stasheff’s multiplihedron [33].

3.1. $A_\infty$ morphisms. We begin by recalling the definitions of $A_\infty$ morphisms and related notions.

**Definition 3.1.** (a) ($A_\infty$ morphisms) Let $A_0, A_1$ be $A_\infty$ algebras. An $A_\infty$ morphism $F$ from $A_0$ to $A_1$ consists of a sequence of maps

$$F^d : A_0^k \to A_1^{1-d}, \quad d \geq 0$$

such that the following holds:

$$\sum_{i+j \leq d} (-1)^{i+j} a_d \mu_{A_0}^{i+j} (F_{a_1}, \ldots, a_i, a_{i+1}, \ldots, a_{i+j}, a_{i+j+1}, \ldots, a_d) = \sum_{i_1+\ldots+i_m=d} \mu_{A_1}^m (F_{a_1}, \ldots, a_{i_1}, \ldots, F_{a_d}, \ldots, a_{d-i_m+1}, \ldots, a_d)$$

where the first sum is over integers $i, j$ with $i + j \leq d$, the second is over partitions $d = i_1 + \ldots + i_m$.

We say that an $A_\infty$ morphism $F$ is unital iff

$$F^1(e_{A_0}) = e_{A_1}, \quad F^k(a_1, \ldots, a_i, e_{A_0}, a_{i+2}, \ldots, a_k) = 0$$

for every $k \geq 2$ and every $0 \leq i \leq k - 1$.

(b) (Composition of $A_\infty$ morphisms) The composition of $A_\infty$ morphisms $F_0, F_1$ is defined by

$$F_0 \circ F_1^d(a_1, \ldots, a_d) = \sum_{i_1+\ldots+i_m=d} F_0^m (F_{a_1}^{i_1}(a_1, \ldots, a_{i_1}), \ldots, F_{a_d}^{i_m}(a_{d-i_m+1}, \ldots, a_d)).$$

(c) (Homology morphism) Any $A_\infty$ morphism $F : A_0 \to A_1$ defines an ordinary morphism

$$H(F) : H(A_0) \to H(A_1), \quad [a] \mapsto (-1)^{|a|} [F(a)].$$

(d) ($A_\infty$ natural transformations) Let $F_0, F_1 : A_0 \to A_1$ be $A_\infty$ morphisms. A pre-natural transformation $T$ from $F_0$ to $F_1$ consists of for each $d \geq 0$ a multilinear map

$$T^d : A_0^d \to A_1.$$
Let $\text{Hom}(\mathcal{F}_0, \mathcal{F}_1)$ denote the space of pre-natural transformations from $\mathcal{F}_0$ to $\mathcal{F}_1$. Define a differential on $\text{Hom}(\mathcal{F}_0, \mathcal{F}_1)$ by

\begin{equation}
(\mu^1_{\text{Hom}(\mathcal{F}_0, \mathcal{F}_1)})^d(a_1, \ldots, a_d) = \sum_{k,m} \sum_{i_1, \ldots, i_m} (-1)^i \mu^{m^1}_{A^1} (\mathcal{F}_0^1(a_1, \ldots, a_{i_1}), \mathcal{F}_0^{i_2}(a_{i_1+1}, \ldots, a_{i_1+i_2}), \ldots, T^{i_k}(a_{i_1+\ldots+i_{k-1}+1}, \ldots, a_{i_1+\ldots+i_k}), T^{i_{k+1}}(a_{i_1+\ldots+i_k+1}, \ldots, \mathcal{F}_1^{i_{k+1}}(\ldots)), \ldots, \mathcal{F}_1^{i_{i-1}}(\ldots)), \ldots, \mathcal{F}_1^m(a_{d-i_m}, \ldots, a_d))
\end{equation}

where

\[ \| = (|T| - 1)(|a_1| + \ldots + |a_{i_1+\ldots+i_{k-1}}| - i_1 - \ldots - i_{k-1}) \]

A natural transformation is a closed pre-natural transformation. We say that an $A^\infty$ (pre-)natural transformation $\mathcal{T}$ from a unital morphism $\mathcal{F}_0$ to a unital morphism $\mathcal{F}_1$ is unital if $\mathcal{T}^k(a_1, \ldots, a_i, e_{A_0}, a_{i+2}, \ldots, a_k) = 0$ for every $k \geq 1$ and every $0 \leq i \leq k - 1$.

Let $\text{Hom}(A_0, A_1)$ denote the space of $A^\infty$ morphisms from $A_0$ to $A_1$, with morphisms given by pre-natural transformations. Higher compositions give $\text{Hom}(A_0, A_1)$ the structure of an $A^\infty$ category [15, 10.17], [24, 8.1], [30, Section 1d].

(f) (Homology natural transformations) Any $A^\infty$ natural transformation $\mathcal{T} : \mathcal{F}_0 \to \mathcal{F}_1$ induces a natural transformation of the corresponding homological morphisms $H(\mathcal{F}_0) \to H(\mathcal{F}_1)$.

(g) ($A^\infty$ homotopies) Suppose that $\mathcal{F}_1, \mathcal{F}_2 : A_0 \to A_1$ are morphisms. A homotopy from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a pre-natural transformation $\mathcal{T} \in \text{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ such that

\begin{equation}
\mathcal{F}_1 - \mathcal{F}_2 = \mu^1_{\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)}(\mathcal{T}).
\end{equation}
where \( \mu^1_{\text{Hom}(\mathcal{F}_i, \mathcal{F}_j)}(T) \) is defined in (43). If a homotopy exists we say that \( \mathcal{F}_1 \) is homotopic to \( \mathcal{F}_2 \) and write \( \mathcal{F}_1 \equiv \mathcal{F}_2 \). As shown in [30, Section 1h], homotopy of \( A_\infty \) morphisms is an equivalence relation.

(h) (Composition of homotopies) Given homotopies \( T_1 \) from \( \mathcal{F}_0 \) to \( \mathcal{F}_1 \), and \( T_2 \) from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \), the sum

\[
T_2 \circ T_1 := T_1 + T_2 + \mu^2(T_1, T_2) \in \text{Hom}(\mathcal{F}_0, \mathcal{F}_2)
\]

is a homotopy from \( \mathcal{F}_0 \) to \( \mathcal{F}_2 \).

(i) (Composition of morphisms) Let \( A_0, A_1, A_2 \) be \( A_\infty \) algebras. Given a morphism \( \mathcal{F}_{12} : A_1 \to A_2 \) resp. \( \mathcal{F}_{01} : A_0 \to A_1 \), right composition with \( \mathcal{F}_{12} \) resp. left composition with \( \mathcal{F}_{01} \) define \( A_\infty \) morphisms

\[
\mathcal{R}_{\mathcal{F}_{12}} : \text{Hom}(A_0, A_1) \to \text{Hom}(A_0, A_2) \\
\mathcal{L}_{\mathcal{F}_{01}} : \text{Hom}(A_0, A_2) \to \text{Hom}(A_1, A_2).
\]

The action on pre-natural transformations is given as follows [30, Section 1e]: Let \( \mathcal{F}_{01}, \mathcal{F}_{01}'' : A_0 \to A_1 \) be \( A_\infty \) morphisms and \( T_{01} \) a pre-natural transformation from \( \mathcal{F}_{01} \) to \( \mathcal{F}_{01}'' \). Define

\[
(\mathcal{R}_{\mathcal{F}_{12}}(T_{01}))_d(a_1, \ldots, a_d) = \sum_{r,j} \sum_{i_1 + \ldots + i_r = d} (-1)^i \mathcal{F}_{12}(\mathcal{F}_{01}'(a_1, \ldots, a_{i_1}), \ldots, \mathcal{F}_{01}'(\ldots), T_{01}(a_{i_1+\ldots+ij-1+1}, \ldots, a_{i_1+\ldots+ij}), \mathcal{F}_{01}''(\ldots), \ldots, \mathcal{F}_{01}''(a_{i_1+\ldots+ir-1+1}, \ldots, a_d)).
\]

(j) (Homotopy equivalence of \( A_\infty \) algebras) We say that \( A_\infty \) algebras \( A_0, A_1 \) are homotopy equivalent if there exist morphisms \( \mathcal{F}_{01} : A_0 \to A_1 \) and \( \mathcal{F}_{10} : A_1 \to A_0 \) such that \( \mathcal{F}_{01} \circ \mathcal{F}_{10} \) and \( \mathcal{F}_{10} \circ \mathcal{F}_{01} \) are homotopic to the respective identities. Homotopy equivalence of \( A_\infty \) algebras is an equivalence relation: Symmetry and reflexivity are immediate. For transitivity note that by the previous item, if \( \mathcal{F}_{01}' \) and \( \mathcal{F}_{01}'' \) are homotopy equivalent then so are \( \mathcal{F}_{12} \circ \mathcal{F}_{01}' \) and \( \mathcal{F}_{12} \circ \mathcal{F}_{01}'' \); repeating the argument for left composition, if \( \mathcal{F}_{01}' \) and \( \mathcal{F}_{01}'' \) are homotopic and also \( \mathcal{F}_{12}' \) and \( \mathcal{F}_{12}'' \) then \( \mathcal{F}_{12}' \circ \mathcal{F}_{01}' = \mathcal{F}_{12}'' \circ \mathcal{F}_{01}''. \)

Hence if \( \mathcal{F}_{01} : A_0 \to A_1, \mathcal{F}_{10} : A_1 \to A_0 \) and \( \mathcal{F}_{12} : A_1 \to A_2, \mathcal{F}_{21} : A_2 \to A_1 \) are homotopy equivalences then

\[
(F_{10} \circ F_{21}) \circ (F_{12} \circ F_{01}) = (F_{10} \circ (F_{21} \circ F_{12}) \circ F_{01} \\
\cong F_{10} \circ F_{01} \cong \text{Id}
\]

and similarly for \( (F_{12} \circ F_{01}) \circ (F_{10} \circ F_{21}) \).

The following is a version of the material in [17, Chapter 4], which we will use in order to obtain homotopy invariance of the Maurer-Cartan moduli space. Let \( A_0, A_1 \) be convergent \( A_\infty \) algebras. Let \( \mathcal{F} : A_0 \to A_1 \) be an \( A_\infty \) morphism or pre-natural transformation. We write

\[
\mathcal{F}^n = \sum_{m} \mathcal{F}^{n,m}, \quad \mathcal{F}^{n,m} = \pi_{1-n+m} \circ \mathcal{F}^n.
\]
We say that $\mathcal{F}$ is convergent iff $\mathcal{F}^0(1) \in A_{>0}A$ and there exists a sequence $E_m \to \infty$ such that

$$\mathcal{F}^{n,m}(A_0) \subset q^{E_m}A_1, \quad \forall n \geq 0.$$ 

**Lemma 3.2.** (Map between Maurer-Cartan moduli spaces) Suppose that $A_0, A_1$ are convergent strictly-unital $A_\infty$ algebras and $\mathcal{F} : A_0 \to A_1$ is a convergent unital $A_\infty$ morphism. Then

$$\phi_{\mathcal{F}}(b_0) = \mathcal{F}^0(1) + \mathcal{F}^1(b_0) + \mathcal{F}^2(b_0, b_0) + \mathcal{F}^3(b_0, b_0, b_0) + \ldots$$

defines a map from $A_0^+$ to $A_1^+$ and restricts to a map $\widetilde{MC}(A_0) \to \widetilde{MC}(A_1)$ of the moduli spaces of solutions to the weak Maurer-Cartan equation and descends to a map $MC(A_0)$ to $MC(A_1)$. That is: For every $b_0 \in MC(A_0)$, $\phi_{\mathcal{F}}(b_0) \in MC(A_1)$; and

$$(b_0 \sim b_0') \implies (\phi_{\mathcal{F}}(b_0) \sim \phi_{\mathcal{F}}(b_0')),$$ \quad $\forall b_0, b_0' \in MC(A_0).$

Moreover, if $\mathcal{F}_0, \mathcal{F}_1 : A_0 \to A_1$ are unital $A_\infty$ morphisms that are homotopic by a convergent $A_\infty$ homotopy $T : A_0 \to A_1$ then $\mathcal{F}_0, \mathcal{F}_1$ induce the same map on Maurer-Cartan moduli spaces, that is, $\phi_{\mathcal{F}_0} = \phi_{\mathcal{F}_1}$. In particular, if $\mathcal{F} : A \to A$ is convergent and convergent-homotopic to the identity then $\mathcal{F}$ induces the identity on $MC(A)$. Hence the set $MC(A)$ is an invariant of the homotopy type of $A$.

**Proof.** The proof that the sum

$$\mathcal{F}^0(1) + \mathcal{F}^1(b_0) + \mathcal{F}^2(b_0, b_0) + \mathcal{F}^3(b_0, b_0, b_0) + \ldots$$

converges is essentially the same as that for Lemma 2.29 and left to the reader. Regarding gauge invariance, we first notice that for every $b \sim b' \in \widetilde{MC}(A_0)$ with $h \in A_0$ such that $b - b' = \mu^1_{b,b'}(h)$,

$$\phi_{\mathcal{F}}(b) - \phi_{\mathcal{F}}(b') = \sum_{n_0, n_1} \mathcal{F}^{n_0+n_1+1}(b, \ldots, b, b - b', b', \ldots, b')$$

$$= \sum_{n_0, n_1, n_2, n_3} (-1)^{n_0+n_0|b|} \mathcal{F}^{n_0+n_1+n_1+1}(b, \ldots, b, \mu^{n_2+n_3+1}_{b, b, b, b} h, b', \ldots, b', b')$$

$$= \mu^1_{\phi_{\mathcal{F}}(b), \phi_{\mathcal{F}}(b')} \left( \sum_{n_4, n_5} \mathcal{F}^{n_4+n_5+1}(b, \ldots, b, h, b', \ldots, b') \right)$$

so that $\phi_{\mathcal{F}}(b) \sim \phi_{\mathcal{F}}(b')$. Note that the third equality above uses the fact that $b, b' \in MC(A_0)$ and the unitality of $\mathcal{F}$.

The second part of the claim follows from the transitivity of $\sim$, the above calculation, plus the fact that if $\mathcal{F}_0 - \mathcal{F}_1 = \mu^1_{\text{Hom}(\mathcal{F}_0, \mathcal{F}_1)}(T)$ for some unital pre-natural
transformation $\mathcal{T}$, then
\[
\phi_{\mathcal{F}_0}(b) - \phi_{\mathcal{F}_1}(b) = \sum_{k \geq 0} (\mu_{\text{Hom}(\mathcal{F}_0, \mathcal{F}_1)}^1)^k(b, \ldots, b)
\]
\[
= \sum_{k \geq 0} \sum_{m} \sum_{i_1 + \ldots + i_m = k} (-1)^{i_1} \mu_{\mathcal{A}^2}(\mathcal{F}_0^{i_1}(b, \ldots, b), \mathcal{F}_0^{i_2}(b, \ldots, b), \ldots),
\]
\[
\mathcal{T}^k(b, \ldots, b, \mathcal{F}_1^{i_1+k+1}(b, \ldots, b), \ldots, \mathcal{F}_1^{i_m}(b, \ldots, b))
\]
\[
- \sum_{i, e} (-1)^{i+\sum_{j=1}^{i} |b|+|\mathcal{T}|-1} T_{k+e+1}^{i, e}(b, \ldots, b, \mu_{\mathcal{A}^1}(b, \ldots, b), b, \ldots, b, \lambda e_{\mathcal{A}^1}, b, \ldots, b)
\]
\[
= \sum_{k \geq 0} \sum_{m} \sum_{i_1 + \ldots + i_m = k} (-1)^{i_1} \mu_{\mathcal{A}^2}(\mathcal{F}_0^{i_1}(b, \ldots, b), \mathcal{F}_0^{i_2}(b, \ldots, b), \ldots),
\]
\[
\mathcal{T}^k(b, \ldots, b, \mathcal{F}_1^{i_1+k+1}(b, \ldots, b), \ldots, \mathcal{F}_1^{i_m}(b, \ldots, b))
\]
\[
= \mu_{\phi_{\mathcal{F}_0}(b), \phi_{\mathcal{F}_1}(b)}^1(\sum_k \mathcal{T}^k(b, \ldots, b)).
\]

Since $\mathcal{T}$ is convergent and $b \in A^+ = A_{\leq 0} + \Lambda_{>0} A_{\geq 0}$ the sum
\[
h_{01} := \sum_k \mathcal{T}^k(b, \ldots, b)
\]
exists in $A$. Furthermore since $\mathcal{T}^k$ has degree $-k$ and $\mathcal{T}^0(1) \in \Lambda_{>0} A$ we have
\[
h_{01} \in A^{++} = A_{<0} + \Lambda_{>0} A_{\geq 0}.
\]
Hence $\phi_{\mathcal{F}_0}(b) \sim_{h_{01}} \phi_{\mathcal{F}_1}(b)$ as claimed. \hfill \Box

Similarly one has a homotopy invariance property of the cohomology vector bundle introduced in (38):

**Lemma 3.3.** (Maps of cohomology bundles) Any convergent $A_{\infty}$ morphism $\mathcal{F} : A_0 \rightarrow A_1$ induces a morphism of stalks $H(\mathcal{F}) : H(A_0) \rightarrow H(A_1)$. If $\mathcal{F}_0, \mathcal{F}_1 : A_0 \rightarrow A_1$ are convergent morphisms related by a convergent homotopy then $H(\mathcal{F}_0)$ is equal to $H(\mathcal{F}_1)$ up to gauge transformation. In particular, if there exist convergent $A_{\infty}$ maps $\mathcal{F}_{01} : A_0 \rightarrow A_1$ and $\mathcal{F}_{10} : A_1 \rightarrow A_0$ such that $\mathcal{F}_{01} \circ \mathcal{F}_{10}$ and $\mathcal{F}_{10} \circ \mathcal{F}_{01}$ are homotopic to the identities via convergent homotopies then $H(A_0)$ is isomorphic to $H(A_1)$ up to gauge equivalence in the sense that
\[
H(b_0) \cong H(\mathcal{F}_{01}(b_0)), \quad H(b_1) \cong H(\mathcal{F}_{10}(b_1))
\]
for any $b_0 \in \hat{MC}(A_0), b_1 \in \hat{MC}(A_1)$. 


The proof is similar to that of Lemma 2.32 and omitted. Thus having a non-trivial cohomology is an invariant of the homotopy type of a convergent, strictly unital $A_\infty$ algebra.

3.2. Multiplihedra. The terms in the $A_\infty$ morphism axiom correspond to codimension one cells in a cell complex called the multiplihedron introduced by Stasheff [33]. Stasheff’s definition identifies the $n$-multiplihedron as the cell complex whose vertices correspond to total bracketings of $x_1, \ldots, x_n$, together with the insertion of expressions $f(\cdot)$ so that every $x_j$ is contained in an argument of some $f$. For example, the second multiplihedron is an interval with vertices $f(x_1)$, $f(x_2)$, $f(x_1x_2)$.

A geometric realization of this polytope was given by Boardman-Vogt [5] in terms of what we will call quilted metric trees. A quilted metric planar tree is a planar metric tree $T = (T, e_0 \in \text{Edge}_i(T), \ell : \text{Edge}_f(T) \to [0, \infty])$ together with a subset $\text{Vert}^{\text{col}}(T) \subset \text{Vert}(T)$ of colored vertices such that every simple path from the root to a leaf meets precisely one colored vertex, satisfying the (Balanced lengths condition) For any two colored vertices $v_1, v_2 \in \text{Vert}^{\text{col}}(T)$,

$$
\sum_{e \in P_+(v_1, v_2)} \ell(e) = \sum_{e \in P_-(v_1, v_2)} \ell(e)
$$

where $P(v_1, v_2)$ is the (finite length) oriented non-self-crossing path from $v_1$ to $v_2$ and $P_+(v_1, v_2)$ resp. $P_-(v_1, v_2)$ is the part of the path towards resp. away from the root edge, see Ma’u-Woodward [26].

The set of combinatorially finite resp. seminfinite edges is denoted $\text{Edge}_f(T)$ resp. $\text{Edge}_i(T)$; the latter are equipped with a labelling by integers $0, \ldots, n$. A quilted tree is stable if each colored vertex has valence at least two and any non-colored vertex has valence at least three. One can consider broken quilted trees as in the non-quilted case, but this time making sure that any simple path from the root of the broken tree to a leaf still meets only one colored vertex. There is a natural notion of convergence of quilted trees, in which edges whose length approaches zero are contracted and edges whose lengths go to infinity are replaced by broken edges.

A different realization of the multiplihedron is the moduli space of stable quilted disks in Ma’u-Woodward [26]. In this realization, one obtains Stasheff’s cell structure on the multiplihedron naturally. Namely in [26] a quilted disk was defined as a datum $(S, Q, x_0, \ldots, x_n \in \partial S)$ consisting of a marked complex disk $(S, x_0, \ldots, x_n \in \partial S)$ (the points are required to be in counterclockwise cyclic order) together with a circle $Q \subset S$ (here we take $S$ to be a ball in the complex plane, so the notion of circle makes sense) tangent to the 0-th marking $x_0$. An isomorphism of quilted disks from $(S, Q, x_0, \ldots, x_n)$ to $(S', Q', x'_0, \ldots, x'_n)$ is an isomorphism of holomorphic disks $S \to S'$ mapping $Q$ to $Q'$ and $x_0, \ldots, x_n$ to $x'_0, \ldots, x'_n$. The moduli space of stable quilted disks with $n + 1$ boundary markings admits a compactification, isomorphic to the multiplihedron as a cell complex, that allows the interior circle $Q$ to “bubble
“out” into the extra disk bubbles, or disk bubbles without interior circles to form when the points come together. (The homotopy invariance of the $A_\infty$ algebra of a Lagrangian is proved in Fukaya et al [17] using a moduli space of weighted stable disks; we find the use of quilted stable disks more natural because it reproduces exactly Stasheff’s cell structure, although the quilting has no geometric meaning in the current paper.) The open stratum may be identified with the set of sequences $0 = x_1 < \ldots < x_n$; the bubbles form either when the points come together, in which case a disk bubble forms, or when the markings go to infinity, in which case one rescales to keep the maximum distance between the markings constant and a collection of quilted disk bubbles form. Moreover, one can consider the addition of $m$ interior markings as in the unquilted case.

There is a combined moduli space of quilted marked treed disks given by (i) a quilted rooted metric planar tree (ii) for each non-colored vertex of the tree a marked disk with boundary markings whose number is the valence of the given vertex (iii) for each colored vertex, a marked quilted disk with number of boundary markings equal to the valence that of the colored vertex. From this datum one defines a space obtained by attaching the endpoints of the segments of the quilted metric tree to the marked points on the disks corresponding to the vertices. A marked quilted treed disk is stable if its underlying tree is stable and each of its quilted and unquilted disks is stable. Let $\mathcal{M}_{n,m,1}$ denote the moduli space of stable marked quilted treed disks with $n$ leaves and $m$ interior markings. See Figure 13 for a picture of $\mathcal{M}_{2,0,1}$. The quilted disks are those with two shadings; while the ordinary disks have either light or dark shading depending on whether they can be connected to the zero-th edge without passing a colored vertex. The hashes on the line segments indicate nodes connecting segments of infinite length, that is, broken segments.

![Figure 13. Moduli space of stable quilted treed disks](image)

The moduli space $\mathcal{M}_{n,m,1}$ will now admit two types of forgetful maps given by forgetting some leaf edges and forgetting the quilting disks. Each main stratum of $\mathcal{M}_{n,m,1}$ can be oriented by noticing that the stratum made of quilted treed disks having a single disk is naturally isomorphic to $\mathbb{R}$ times that of $\mathcal{M}_{n,m}$, the extra factor corresponding to the quilting parameter. The boundary of the moduli space is naturally isomorphic to a union of moduli spaces:

\[
\partial \mathcal{M}_{n,m,1} \cong \bigcup_{i,j} (\mathcal{M}_{n-i+1,m_1,1} \times \mathcal{M}_i,m_2) \cup \bigcup_{i_1,\ldots,i_r} (\mathcal{M}_{r,m_0} \times \prod_{j=1}^r \mathcal{M}_{i_j,m_j,1}) .
\]
By construction, for the facet of the first type, the sign of the inclusions of boundary strata are the same as that for the corresponding inclusion of boundary facets of \( M_{n,m,1} \), that is, \((-1)^{(n-i)-j} \). For facets of the second type, the gluing map is
\[
(0, \infty) \times M_{r,m_0} \times \bigoplus_{j=1}^{r} M_{|I_j|,m_j,1} \to M_{n,m,1}
\]
given for boundary markings by
\[
(50) \quad \left( \delta, x_1, \ldots, x_r, (x_{1,j} = 0, x_{2,j}, \ldots, x_{|I_j|,j})_{j=1}^{r} \right) \mapsto (x_1, x_1 + \delta^{-1} x_{2,1}, \ldots, x_1 + \delta^{-1} x_{|I_1|,1}, \ldots, x_r, x_r + \delta^{-1} x_{2,r}, \ldots, x_r + \delta^{-1} x_{|I_r|,r}).
\]
This map changes orientations by \( \sum_{j=1}^{r} (r-j)(|I_j|-1) \); in case of non-trivial weightings, \(|I_j|\) should be replaced by the number of incoming markings or non-trivial weightings on the \( j \)-th component.

The combinatorial type of a quilted disk is the graph obtained as in the unquilted case by replacing each quilted disk component with its combinatorial tree (now having colored vertices), each unquilted disk or sphere component with its combinatorial tree, and each edge being identified as infinite, semi-infinite, finite non-zero or zero. We also wish to allow disconnected types. In this case, labelling of the unquilted components by \( \{0, 1\} \). Morphisms of graphs (Cutting infinite length edges, collapsing edges, making lengths/weights finite/non-zero, and forgetting tails) induce morphisms of moduli spaces of stable quilted treed disks as in the unquilted case. In the space case of cutting an edge of infinite length, one of the pieces will be quilted and the other unquilted. The \( \{0,1\}\)-labelling of the unquilted components takes value 0 resp. 1 if component is further away from the root resp. closer to the root than the quilted components with respect to any non-self-crossing path of components. For any combinatorial type \( \Gamma \) of quilted disk there is a universal quilted treed disk \( \mathcal{U}_\Gamma \to \mathcal{M}_\Gamma \) which is a cell complex whose fiber over \( C \) is isomorphic to \( C \), and splits into surface and tree parts \( \mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma \).

One can furthermore consider labels and weights on the semi-infinite ends of quilted tree disks as in the case of treed disks. We suppose there is a partition of the semi-infinite edges
\[
\text{Edge}^*(T) \sqcup \text{Edge}^0(T) \sqcup \text{Edge}^*(T) = \text{Edge}_{i}(T)
\]
into weighted resp. forgettable resp. unforgettable edges as in the unquilted case, except that now the root of the quilted tree with one weighted leaf and no marking is weighted with the same weight as the leaf, see Figure 14. The moduli space with a single weighted leaf and no markings is then a point.

The morphisms of moduli spaces of different type for quilted treed disks are defined as in the unquilted case, except that the (Cutting edges) resp. (Collapsing edges) resp. (Making an edge length or weight finite/non-zero) axiom now allows one to cut resp. collapse resp. change the length of a collection of edges of infinite resp. zero resp. infinite or zero length above a quilted treed disk component. In
particular, in the case of making a collection of such edges finite the corresponding morphism is the inclusion of the second union $\overline{\mathcal{M}}_{r,m_0} \times \prod_{j=1}^{r} \overline{\mathcal{M}}_{i_j,m_j,1}$ in (49) as a component of $\partial \overline{\mathcal{M}}_{n,m,1}$.

3.3. **Quilted holomorphic disks.** To prove that the Fukaya algebra is independent of the perturbation system, one considers two systems of perturbations and extends them to a set of perturbations for the moduli space of quilted disks with interior markings.

In the next few sections we deal only with divisors of the same degree, built from homotopic sections of the same line bundle; the general case is treated in Section 3.6 below. Suppose that $D^0, D^1$ are stabilizing divisors for $L$ with respect to compatible almost complex structure $J^0, J^1$, of the same degree and built from homotopic unitary sections over $L$. Choose a path $J^t$ from $J^0$ to $J^1$. By Lemma 2.21 above, there exists a path of $J^t$-stabilizing divisors $D^t, t \in [0, 1]$ connecting $D^0, D^1$, and a path $J^t_D$ of compatible almost complex structures such that $D^t$ contains no $J^t_D$-holomorphic spheres.

In order to specify which divisor of the above family we use at a given point of a quilted domain, we choose a smooth function

\[(51) \delta : [-\infty, \infty] \to [0, 1]\]

such that $\delta$ is equal to 0 resp. 1 on a neighborhood of $-\infty$ resp. $\infty$. For every point $z \in C$ of a quilted treed disk $C$, let

\[d(z) : \pm \sum_{e \text{ edge to seams}} \ell(e) \in [-\infty, \infty]\]

be the distance of $z$ to the quilted components of $C$ (with respect to the lengths of the edges) times 1 resp. $-1$ if $z$ is above resp. below the quilted components (that is, further from resp. closer to the root than the quilted components).

Given perturbation data $P^0$ and $P^1$ with respect to metrics $G^0, G^1 \in \mathcal{G}(L)$ over unquilted treed disks for $D^0$ resp. $D^1$, a **perturbation morphism** $P^{01}$ from $P^0$ to $P^1$ for the quilted combinatorial type $\Gamma$ consists of

(a) a function $\delta^{01} : [-\infty, \infty] \to [0, 1]$ as in (51),
(b) a smooth domain-dependent choice of metric
\[ G^{01}_\Gamma : \mathcal{T}_\Gamma \to \mathbb{R} \]
constant to \( G^0 \) resp. \( G^1 \) on the neighborhood of the points at infinity of the semi-infinite edges for which \( d = -\infty \) resp. \( d = \infty \),

(c) a domain-dependent Morse function
\[ F^{01}_\Gamma : \mathcal{T}_\Gamma \to \mathbb{R} \]
constant to \( F^0 \) resp. \( F^1 \) on the neighborhood of the endpoints for which \( d = -\infty \) resp. \( d = \infty \) and equal to \( F^0_{\Gamma_0} \) resp. \( F^1_{\Gamma_1} \) on the (unquilted) treed disks components of type \( \Gamma_0, \Gamma_1 \) for which \( d = -\infty \) resp. \( d = \infty \), and

(d) a domain-dependent almost complex structure
\[ J^{01}_\Gamma : \mathcal{S}_\Gamma \to J_\tau(X) \]
with the property that for any surface component \( C_i \) of \( C \), \( J^{01}_\Gamma \) is equal to the given \( J \) away from the compact part:
\[ J_{\Gamma}|_{S_\Gamma - S_\Gamma_{cp}} = \pi_2^* J \]
where \( \pi_2 \) is the projection on the second factor in (5), and equal to the complex structures \( J^{01}_{\Gamma_0} \) resp. \( J^{01}_{\Gamma_1} \) on the (unquilted) treed disks components of type \( \Gamma_0, \Gamma_1 \) for which \( d = -\infty \) resp. \( d = \infty \): Let \( \iota_k : \mathcal{S}_{\Gamma_k} \to \mathcal{S}_\Gamma \) denote the inclusion of the unquilted components. Then we require
\[ J_{\Gamma}|_{\iota_k} : \mathcal{S}_{\Gamma_k} = J^{01}_{\Gamma_k}, \quad k \in \{0, 1\}. \]

(e) One can also require the following invariance property: A perturbation system is \textit{quilt-independent} if \( G^{01}_\Gamma, F^{01}_\Gamma \), and \( J^{01}_\Gamma \) are pull-backs under the forgetful morphism forgetting the quilting on the quilted disk components.

To obtain a well-behaved moduli space of quilted holomorphic treed disks we impose a stability condition and quotient by an equivalence relation. Given a quilted treed disk \( C \), we obtain a stable quilted treed disk by collapsing unstable surface and tree components. The result may be identified with a fiber of a universal curve of some type \( s(\Gamma) \). By pullback we obtain a triple on \( C \), still denoted \((\delta^{01}_\Gamma, J^{01}_\Gamma, F^{01}_\Gamma)\).

A \textit{holomorphic quilted treed disk} \( u : C \to X \) of combinatorial type \( \Gamma \) is a continuous map from a quilted treed disk \( C \) that is smooth on each component, \( J^{01}_\Gamma \)-holomorphic on the surface components and a \( F^{01}_\Gamma \)-Morse trajectory with respect to the metric \( G^{01}_\Gamma \) on each tree segment. A quilted disk \( u : C \to X \) is \textit{stable} if

(a) each unquilted disk component on which \( u \) is constant has at least three special boundary points or one special boundary point and one interior special point
\[ du(C_i) = 0, \quad C_i \text{ unquilted disk} \quad \implies \quad 2\#\{z_k, w_k \in \text{int}(C_i)\} + \#\{w_k \in \partial C_i\} \geq 3; \]

(b) each quilted disk component on which \( u \) is constant has at least two special points on the boundary or one interior special point:
\[ du(C_i) = 0, \quad C_i \text{ quilted disk} \quad \implies \quad \#\{z_i, w_j \in \text{int}(C_i)\} + \#\{w_k \in \partial C_i\} \geq 2 \]
(c) each sphere $C_i \subset C$ component on which $u$ is constant has at least three special points:

$$du(C_i) = 0, \quad C_i \text{ sphere} \implies \# \{z_i, w_j \in C_i\} \geq 3.$$ 

As in the unquilted case, the stability condition is not quite the same as requiring no automorphisms, because of the exceptional case in the last item. A stable holomorphic quilted tree disk is adapted iff

(a) each sphere component $C_i$ of $C$ that maps to $D^{\delta_{01};od(C_i)}$ is constant,
(b) each the interior marking $z_i$ maps to $D^{\delta_{01};od(z_i)}$ and
(c) for each $t \in [0, 1]$, each component of $u^{-1}(D^t) \cap (\delta_{01}^{-1}(t))$ contains a marking.

We remark that the condition that $\delta_{01}$ is constant on each disk or sphere implies that the union

$$D^{\delta_{01}} = \bigcup_{z \in S_\Gamma} \left( \{z\} \times D^{\delta_{01};od(z)} \right)$$

is an almost complex submanifold of $S_\Gamma \times X$. In particular, the intersection multiplicity of $u : C \to X$ with $D^{\delta_{01}}$ at $z_i$ is positive.

In order to obtain a moduli space of holomorphic treed quilted disks we quotient by an equivalence relation. Two stable weighted disks $u_0 : C_0 \to X, u_1 : C_1 \to X$ are isomorphic if there exists an isomorphism $\phi : C_0 \to C_1$ intertwining $u_0$ and $u_1$.

Note that if $C_0, C_1$ have a single unmarked quilted disk component and a single leaf, the $u_0, u_1$ equivalence does not involve weightings on the leaf. On the set of isomorphism classes we quotient by a further equivalence relation as follows: Given a non-constant holomorphic quilted treed disk $u : C \to X$ with leaf $e_i \in \text{Edge}^*(\Gamma)$ on which there is a weighting $\rho(e_i) = 0$ resp. $\infty$, we declare $u$ to be equivalent to the holomorphic treed disk $u' : C \to X$ obtained by replacing the asymptotic critical point at $e_i$ by $x^*_M$ resp. $x^\circ_M$ and adding a constant segment from $x^*_M$ to $x^\circ_M$ above that leaf as in the unquilted case in Figure 8. In other words, forgetting constant infinite segments and replacing them by the appropriate weights gives equivalent holomorphic treed disks. For any combinatorial type $\Gamma$ of quilted disks we denote by $\overline{M}_\Gamma(L, D)$ the compactified moduli space of equivalence classes of adapted quilted holomorphic treed disks.

The moduli space of quilted disks breaks into components depending on the limits along the root and leaf edges. Denote by $\mathcal{M}_\Gamma(L, D, \underline{x}) \subset \overline{M}_\Gamma(L, D)$ the moduli space of isomorphism classes of stable adapted holomorphic quilted treed disks with boundary in $L$ and limits $\underline{x}$ along the root and leaf edges, where $\underline{x} = (x_0, \ldots, x_n) \in \hat{I}(L)$ satisfies the requirement:

(a) (Label axiom)

(i) If $x_0 = x^*_M$ resp. $x_0 = x^\circ_M$, then there is a single leaf reaching $x^*_M$ resp. $x^\circ_M$ and no interior marking (in which case the moduli space will be a point).
(ii) If \( x_i = x_M^* \) resp. \( x_M^\circ \) resp. \( x_M^\bullet \), for some \( i \geq 1 \) then the \( i \)-th leaf is required to be weighted resp. forgettable resp. unforgettable and the limit along this leaf is required to be \( x_M \).

(iii) If \( x_i \notin \{ x_M^*, x_M^\circ, x_M^\bullet \} \), then the \( i \)-th leaf is required to be unforgettable.

(b) (Outgoing edge axiom) The outgoing edge \( e_0 \) is weighted (resp. forgettable) only if there is a single incoming leaf, which is weighted (resp. forgettable) with the same weight and the configuration has no interior markings (so there is a single quilted disk with no markings.)

In order to obtain moduli spaces with the expected boundary, we introduce a coherence condition. We say that a collection \( P^{01} = (P^{01}_\Gamma) \) of perturbation morphisms is \textit{coherent} if \( P^{01}_\Gamma \) is compatible with the morphisms of moduli spaces as before:

(a) (Cutting edges) If \( \Gamma' \) is obtained from \( \Gamma \) by cutting an edge or a collection of edges of infinite length, then \( P^{01}_\Gamma \) is the pushforward of \( P^{01}_{\Gamma'} \).

(b) (Collapsing edges/making an edge/weight finite/non-zero) If \( \Gamma' \) is obtained from \( \Gamma \) by collapsing an edge or edges, then \( P^{01}_\Gamma \) is the pullback of \( P^{01}_{\Gamma'} \).

(c) (Products) If \( \Gamma \) is the union of a quilted type \( \Gamma_1 \) and a non-quilted type \( \Gamma_0 \), then \( P^{01}_\Gamma \) is obtained from \( P^{01}_{\Gamma_1} \) and \( P^{01}_{\Gamma_0} \) as follows: Let \( \pi_k : \mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_0} \to \mathcal{M}_{\Gamma_0} \) denote the projection on the \( k \)-factor, so that \( \mathcal{U}_\Gamma \) is the union of \( \pi_1^* \mathcal{U}_{\Gamma_1} \) and \( \pi_0^* \mathcal{U}_{\Gamma_0} \). Then we require that \( P^{01}_\Gamma \) is equal to the pullback of \( P^{01}_{\Gamma_1} \) on \( \pi_1^* \mathcal{U}_{\Gamma_1} \) and to the pullback of \( P^{01}_{\Gamma_0} \) on \( \pi_0^* \mathcal{U}_{\Gamma_0} \).

Similarly, if \( \Gamma \) is the union of a non-quilted type \( \Gamma_1 \) and quilted types \( \cup_i \Gamma_{0,i} \) and quilted types with interior markings, then \( P^{01}_\Gamma \) is equal to the pullback of \( P^{01}_{\Gamma_1} \) on \( \pi_1^* \mathcal{U}_{\Gamma_1} \) and to the pullback of \( P^{01}_{\Gamma_{0,i}} \) on \( \pi_{0,i}^* \mathcal{U}_{\Gamma_{0,i}} \).

The case of constant quilted types requires special treatment. If any of the types \( \Gamma_{0,i} \) have no interior markings and a single weighted incoming leaf then we label the corresponding incoming leaf of \( \Gamma_1 \) with the same weight. This guarantees that the moduli spaces are of expected dimension.

(d) (Infinite weights) Whenever a weight parameter \( \rho(e_i) \) is equal to infinity, then the \( P^{01}_\Gamma \) is pulled back under the forgetful map forgetting the \( e_i \) semi-infinite edge and stabilizing from the perturbation morphism \( P^{01}_\Gamma \) given by (Forgetting tails).

For generic perturbation morphisms close to the given base structure the moduli space of holomorphic adapted quilted treed disks has compactness and transversality properties similar to those for unquilted disks. A perturbation morphism is \textit{stabilized} if it satisfies a condition analogous to that in Definition 2.23. A perturbation morphism is \textit{convergent} if it satisfies a condition analogous to Definition 2.33. For a comeager subset of perturbation morphisms extending those chosen for unquilted disks, the uncrowded moduli spaces of expected dimension at most one are smooth and of expected dimension. For sequential compactness, it suffices to consider a sequence \( u_\nu : C_\nu \to X \) of quilted treed disks of fixed combinatorial type \( \Gamma_\nu \) constant in \( \nu \). Coherence of the perturbation morphism implies the existence of a stable limit \( u : C \to X \) which we claim is adapted. In particular, the (Marking Property) which
is justified as follows. For each component \( C_i \subset C \), the almost structure \( J_\Gamma|C_i \) is constant near the almost complex submanifold \( D_{\delta_{\Gamma}^i}^{01}\od(C_i) \). We suppose that \( C_i \) is a component of the limit of some sequence of components \( C_{i,\nu} \) of \( C_\nu \). Coherence for the parameter \( \delta_{\Gamma}^i \) implies that \( D_{\delta_{\Gamma}^i}^{01}\od(C_i) \) is the limit of the divisors \( D_{\delta_{\Gamma,\nu}^i}^{01}\od(C_{i,\nu}) \). Then local conservation of intersection degree implies that any component of \( u^{-1}(D_{\delta_{\Gamma}^i}^{01}\od(C_i)) \) contains a limit point of some markings \( z_{i,\nu} \in u^{-1}(D_{\delta_{\Gamma,\nu}^i}^{01}\od(C_{i,\nu})) \). For types of index at most one, each component of \( u^{-1}(D_{\delta_{\Gamma}^i}^{01}\od(C_i)) \) is a limit of a unique component of \( u_{\nu}^{-1}(D_{\delta_{\Gamma,\nu}^i}^{01}\od(C_{i,\nu})) \), otherwise the intersection degree would be more than one which is a codimension two condition. Since non-trivial sphere bubbling is a codimension two condition and ghost bubbling is impossible unless two markings come together, this implies that \( u^{-1}(D_{\delta_{\Gamma}^i}^{01}\od(C_i)) = \{z_i\} \) is also a marking.

Using the additional condition (e) of definition 3.3 can help to understand the above morphism moduli spaces. By the argument in the proof of Proposition 3.7, in their 0-dimensional strata, all of the quilted disks are mapped to points (and thus they are unmarked in the domains). The 1-dimensional strata will be of two types. First, they can again be 1-dimensional families of quilted trajectories where every quilted disk is constant. However, they can also be a 1-dimensional family for which \( \phi \) is unmarked in the domains). The 1-dimensional strata will be of two types. First, they can again be 1-dimensional families of quilted trajectories where every quilted disk is constant. However, they can also be a 1-dimensional family for which only the quilting parameter on a single non-constant quilted disk varies from \(-\infty\) to \(\infty\). We also note that under (e) of definition 3.3, the above morphism setting could be seen as a generalization of that of [4].

3.4. Morphisms of Fukaya algebras. Given a regular, stabilized, convergent, coherent perturbation morphism \( \hat{P}^0 \) from \( P^0 \) to \( P^1 \), define

\[
\phi^n : \widehat{CF}(L;P^0)^{\otimes n} \rightarrow \widehat{CF}(L;P^1)
\]

\[
(x_1, \ldots, x_n) \mapsto \sum_{x_0, [u] \in \mathcal{M}_L(E,L,D,x_0,\ldots,x_n)} (-1)^q \epsilon([u]) (\sigma([u])1)^{-1} q^{|E([u])|} \text{Hol}_L(u)x_0
\]

where the sum is over combinatorial types \( \Gamma \) of quilted disks.

Remark 3.4. (Lowest energy terms) Note that for \( x \in \hat{\mathcal{F}}(L) \), the element \( \phi^1(x^*_M) \) resp. \( \phi^1(x_M) \) resp. \( \phi^1(x_M^\ast) \) has a \( x^*_M \) resp. \( x_M^* \) resp. \( x_M^\ast \) term coming from the count of a quilted treed disk with no interior marking, that is, a treed disk with only one disk that is quilted and mapped to a point. In the latter case, it will be the only term with \( x_M^\ast \) output, by the (Label axiom).

Remark 3.5. The codimension one strata are of several possible types: either there is one (or a collection of) edge of length infinity, there is one (or a collection of) edge of length zero, or equivalently, boundary nodes, or there is an edge with zero or infinite weight. The case of an edge of zero or infinite weighting is equivalent to breaking off a constant trajectory, and so may be ignored. In the case of edges of infinite length(s), then either \( \Gamma \) is

(a) (Breaking off an uncolored tree) a pair \( \Gamma_1 \uplus \Gamma_2 \) consisting of a colored tree \( \Gamma_1 \) and an uncolored tree \( \Gamma_2 \); necessarily the breaking must be a leaf of \( \Gamma_1 \); or
(b) (Breaking off colored trees) a collection consisting of an uncolored tree $\Gamma_0$ containing the root and a collection $\Gamma_1, \ldots, \Gamma_r$ of colored trees attached to each of its $r$ leaves. Such a stratum $\mathcal{M}_\Gamma$ is codimension one because of the (Balanced Condition) which implies that if the length of any edge between $e_0$ to $e_i$ is infinite for some $i$ then the path from $e_0$ to $e_i$ for any $i$ has the same property.

**Figure 15.** Breaking off an unquilted treed disk

**Figure 16.** Breaking off a collection of quilted disks

In the case of a zero length(s), one obtains a fake boundary component with normal bundle $\mathbb{R}$, corresponding to either deforming the edge(s) to have non-zero length or deforming the node(s).

**Theorem 3.6.** ($A_\infty$ morphisms via quilted disks) For any coherent, stabilizing, regular convergent collection $P_0 \Rightarrow P_1$ of perturbations morphisms from $P^0$ to $P^1$, the collection of maps $\phi = (\phi^n)_{n \geq 0}$ constructed above by counting quilted adapted treed holomorphic disks is a convergent unital $A_\infty$ morphism from $\widehat{CF}(L, P^0)$ to $\widehat{CF}(L, P^1)$.

**Proof.** By counting the ends of the one-dimensional moduli spaces we obtain the relation (20) but with $T_{n,m}$ replaced by the set of types $T_{n,m,1}$ of quilted treed disks with $n$ leaves and $m$ interior markings. The true boundary strata are those described in Remark 3.5 and correspond to the terms in the axiom for $A_\infty$ morphisms (41). The signs for the terms of the type $\phi^{n-d+1}(\ldots, \mu^d(\ldots), \ldots)$ are similar to those
for the $A_\infty$ axiom, see (28), and will be omitted. For terms of the second type $\mu^r(\phi^i(\ldots), \ldots, \phi^r(\ldots))$ we verify the sign of the gluing map

$$R \otimes \mathcal{M}_{r,m_0}(y_0, \ldots, y_r) \times \bigoplus_{j=1}^r \mathcal{M}|_{I_j},m_j,1(y_i,x_{I_j}) \rightarrow \mathcal{M}_{n,m_1}(y_0, x_1, \ldots, x_n).$$

An orientation of the former is determined by an orientation of

$$R \oplus T\mathcal{M}_{r,m_0} \oplus TL \oplus T^+_y \oplus T^+_y \cdots \oplus T^+_y \oplus \bigoplus_{j=1}^r \left( T\mathcal{M}|_{I_j},m_j,1 \oplus TL \oplus T^+_y \oplus \bigoplus_{k \in I_j} T^-_{y_k} \right)$$

(where $T^+_y$ denotes $T_y W^+_y$ etc.). Transposition to

$$R \oplus T\mathcal{M}_{r,m_0} \oplus TL \oplus T^+_y \oplus \bigoplus_{j=1}^r \left( T\mathcal{M}|_{I_j},m_j,1 \oplus T^-_y \oplus TL \oplus T^+_y \oplus \bigoplus_{k \in I_j} T^-_{y_k} \right)$$

changes sign by $(-1)$ to the power $\sum_{j=1}^r (|I_j| - 1)|y_j|$. Using $T^-_y \oplus T^+_y \cong TL$ these three factors disappear. Transposing the factors $T\mathcal{M}|_{I_j},m_j,1$ and $T^-_{y_k}$ for $k < \min I_j$ contributes a number of signs $\sum_{j=1}^r (|I_j| - 1)(\sum_{k < \min I_j} |x_k|)$. Transposing $TL \oplus T^+_y$ past $\bigoplus_{j=1}^r T\mathcal{M}|_{I_j},m_j,1$ gives $|y_0| \sum_{j=1}^r (|I_j| - 1)$ additional signs. Finally we have a contribution from the signs in the definition of $\phi|_{I_j}$ and the sign from the definition of $\mu^r$

$$\sum_{k=1}^n k|x_k| + \sum_{j=1}^r \sum_{i=1}^{|I_j|} i|x_i| + \sum_{j=1}^r j|y_j|.$$
Using these the total sign is \((-1)^{\Theta}\) where

\[
\Theta = \sum_{j=1}^{r}(|I_j| - 1)|y_j| + \sum_{j=1}^{r}(|I_j| - 1) \sum_{k<\min I_j} |x_k| + |y_0| \sum_{j=1}^{r}(|I_j| - 1)
\]

\[
+ \sum_{k=1}^{n} k|x_k| + \sum_{j=1}^{r} \sum_{i=1}^{I_j} i|x_i| + \sum_{j=1}^{r} j|y_j| + \sum_{j=1}^{r} (r - j)(|I_j| - 1)
\]

\[
= \sum_{j=1}^{r}(|I_j| - 1)|y_j| + \sum_{j=1}^{r}(|I_j| - 1) \left( \sum_{k<\min I_j} |x_k| \right) + |y_0| \sum_{j=1}^{r}(|I_j| - 1)
\]

\[
+ \sum_{j=1}^{r} (|I_j| \left( \sum_{k>\max I_j} |x_k| \right)) + \sum_{j=1}^{r} j|y_j| + \sum_{j=1}^{r} (r - j)(|I_j| - 1)
\]

\[
= - \left( \sum_{j=1}^{r} \sum_{k<\min I_j} |x_k| \right) + \sum_{j=1}^{r}(|I_j| - 1)|y_j| + |y_0| \sum_{j=1}^{r}(|I_j| - 1) + \sum_{j=1}^{r} j|y_j| + n(|y_0| + n - 2)
\]

\[
- \sum_{j=1}^{r} (|I_j||y_j| + |I_j| - 1)) + \sum_{j=1}^{r} (r - j)(|I_j| - 1)
\]

\[
= - \sum_{j=1}^{r}(|y_j|r - |y_j|j + (|I_j| - 1)(r - j)) + \sum_{j=1}^{r}(|I_j| - 1)|y_j| + |y_0| \sum_{j=1}^{r}(|I_j| - 1) + \sum_{j=1}^{r} j|y_j|
\]

\[
+ n(|y_0| + n - 2) - \sum_{j=1}^{r} (|I_j|(|y_j| + |I_j| - 1)) + \sum_{j=1}^{r} (r - j)(|I_j| - 1)
\]

\[
= (r + 1) \left( \sum_{j=1}^{r} |y_j| - |y_0| \right) + y_0 + n(n - 2) + 2 \sum_{j=1}^{r} j|y_j|
\]

\[
= (r + 1)(r + 2) + (n - 2) + \sum_{k=1}^{n} |x_k| + n(n - 2) + 2 \sum_{j=1}^{r} j|y_j| = 2 \sum_{k=1}^{n} |x_k|.
\]

It follows that the sign induced by gluing the broken configuration of the second type is the same as that induced by the first type in (28). The case of breaking off a trivial trajectory is similar to that in the proof the \(A_\infty\) axiom Theorem 2.28 and left to the reader.

Finally we prove the assertion on the strict units. This is a consequence of the assumption of the existence of forgetful maps for infinite values of the weights. By assumption the \(\phi^n\) products involving \(x_M^2\) as inputs involve counts of quilted treed disks using perturbation that are independent of the disk boundary incidence points of the lines marked \(x_M^2\) asymptotic to \(x_M \in X\). Since forgetting that semi-infinite edge gives a configuration of negative expected dimension, if non-constant, the only
configurations contributing to these terms must be the constant maps. Hence
\[ \phi^1(x_M^2) = x_M^2, \quad \phi^n(\ldots, x_M^2, \ldots) = 0, \ n \geq 2. \]
In other words, the only regular quilted trajectories from the maximum, considered as \( x_M^2 \), being regular are the ones reaching the other maximum that do not have interior markings (i.e. nonconstant disks). The proof of convergence is similar to that of Proposition 2.35 and left to the reader. □

**Proposition 3.7.** Suppose that \( P^0 = P^1 \) is a regular, coherent perturbation datum for unquilted treed disks. For each type \( \Gamma \) of quilted treed disk, let \( \Gamma' \) denote the corresponding type of unquilted treed disk obtained by forgetting the quilting and collapsing unstable components. Pulling back \( P_{\Gamma',0} = P_{\Gamma',1} \) to a perturbation morphism \( P_{01}^{\Gamma'} \) for quilted treed disks gives a regular, coherent perturbation morphism for quilted disks such that the corresponding \( A_\infty \) morphism is the identity.

**Proof.** Any regular perturbation system for unquilted disks induces a regular perturbation system for quilted disks by pullback under the forgetful map forgetting the quilting and collapsing unstable components. Given a non-trivial configuration contributing to \( \phi^n \) in the moduli space of expected dimension 0, one obtains a configuration contributing to \( \mu^n \) in the moduli space of expected dimension \(-1\) via the forgetful map. Therefore, the only configurations contribution to \( \phi^n \) are the constant configurations. Hence \( \phi^1 \) is the identity and all other maps \( \phi^n, n > 0 \) vanish. □

### 3.5. Homotopies.

The morphism of homotopy-associative algebras constructed above is a homotopy equivalence of \( A_\infty \) algebras by an argument using twice-quilted disks. A twice-quilted disk is defined in the same way as once-quilted disks, but with two interior circles that are either equal or with the second contained inside the first, say with radii \( \rho_1 < \rho_2 \). The moduli space of twice-quilted treed disks is a cell complex constructed in a similar way to the space of once-quilted treed disks. We denote the moduli space with \( n \) leaves and two quiltings by \( \mathcal{M}_{n,2} \). We show in Figure 17 the moduli space of twice-quilted stable disks \( \mathcal{M}_{n,2} \) without trees in the case \( n = 2 \). The moduli space \( \mathcal{M}_{2,2} \) is a pentagon whose vertices correspond to the expressions
\[
\begin{align*}
&f(g(x_1 x_2)), f(g(x_1)g(x_2)), f(g(x_1))f(g(x_2)), (f(g)(x_1))(f(g)(x_2)), (fg)(x_1 x_2).
\end{align*}
\]

The effect of adding trees is to add more cells corresponding to the boundary strata, with non-zero lengths attached to the nodes. The combinatorial type of a twice-quilted disk is a tree \( \Gamma = (\text{Vert}(\Gamma), \text{Edge}(\Gamma)) \) equipped with subsets \( \text{Vert}^1(\Gamma), \text{Vert}^2(\Gamma) \subset \text{Vert}(\Gamma) \) corresponding to the quilted components; the set
\[ \text{Vert}^{12}(\Gamma) := \text{Vert}^1(\Gamma) \cap \text{Vert}^2(\Gamma) \]
corresponds to the twice-quilted components. The ratios
\[ \lambda_S(v) = \rho_2(v)/\rho_1(v), \quad v \in \text{Vert}^{12}(\Gamma) \]
of the radii of the interior circles with radii \( \rho_2(v), \rho_1(v), v \in \text{Vert}^{12}(\Gamma) \) are required to be equal for each twice-quilted disk in the configuration, if the configuration has
twice-quilted components:

\[
\lambda_S(v_1) = \lambda_S(v_2), \quad \forall v_1, v_2 \in \text{Vert}^{12}(\Gamma).
\]

Figure 17. Twice-quilted disks

There is a similar marked weighted treed version $\overline{\mathcal{M}}_{n,m,2}$, where the lengths to each colored vertex satisfy the balanced condition (48) for each color:

(a) For any two vertices of the same color $v_1, v_2 \in \text{Vert}^k(\Gamma)$,

\[
\sum_{e \in P_+(v_1,v_2)} \ell(e) = \sum_{e \in P_-(v_1,v_2)} \ell(e)
\]

where $P(v_1,v_2)$ is the (finite length) oriented non-self-crossing path from $v_1$ to $v_2$ and $P_+(v_1,v_2)$ resp. $P_-(v_1,v_2)$ is the part of the path pointing towards resp. away from the root edge, and

(b) for two vertices of different colors $v_1 \in \text{Vert}^1(\Gamma)$ and $v_2 \in \text{Vert}^2(\Gamma)$ for which there is a (finite length) oriented non-self-crossing path $P(v_1,v_2)$ from $v_1$ to $v_2$, let

\[
\lambda_T(v_1,v_2) = \sum_{e \in P(v_1,v_2)} \ell(e).
\]

Then $\lambda_T(v_1,v_2)$ is independent of the choice $v_1 \in \text{Vert}^1(\Gamma)$ and $v_2 \in \text{Vert}^2(\Gamma)$.

We suppose that divisors and perturbations for unquilted and once-quilted disks have already been chosen. That is, there are given compatible almost complex structures $J_0, J_1, J_2$, metrics $G_0, G_1, G_2$, divisors $D_0, D_1, D_2$ and perturbation systems $P^0, P^1, P^2$ for unquilted disks. Furthermore, there are given paths $J^t_{01}, J^t_{12}, J^t_{02}$
of compatible almost complex structure from $J_0$ to $J_1$, $J_1$ to $J_2$ and $J_0$ to $J_2$. In our application we are particularly interested in the case $D_0 = D_2$ and the constant path $D_{02} = D_0 = D_2$.) We suppose there are given perturbation data $P^{ij}$ for once-quilted disks giving rise to morphisms

$$
\phi_{ij} : \widehat{CF}(L, P^i) \to \widehat{CF}(L, P^j), \quad 0 \leq i < j \leq 2.
$$

We have in mind especially the case that $D_0 = D_2$, $D_{01} = D_{12}^{-1}t$, and $D_{02}$ is the constant path. In this case one may take $P_{12}^{12} = P_{01,1-t}^{01}$ and $P_{02}$ the perturbation system pulled back by the forgetful map forgetting the quilting as in Proposition 3.7.

We begin by extending the families of stabilizing divisors over the universal twice-quilted disks. A \textit{domain-dependent parameter} for twice quilted disks is a smooth map

$$
\delta^{012} : \Delta \equiv \{(t_1, t_2) \in [-\infty, \infty]^2 \mid t_2 \leq t_1\} \to [0, 2].
$$

For every point $z \in C$ of a twice quilted treed disk $C$, let $d(z) = (t_1, t_2) \in \Delta$ with $t_1$ being the signed distance of $z$ to the lowest quilted components of $C$ and $t_2$ being the signed distance of $z$ to the highest quilted components of $C$.

**Definition 3.8.** A perturbation $P_t^{012}$ for twice-quilted treed disks from quilted perturbation systems $P^{01} \times P^{12}$ to $P^{02}$ consists of

(a) a domain-dependent parameter $\delta_t^{012}$ that agrees

- with $\delta^{01}_{t_{01}}$ on $[-\infty, \infty] \times \{-\infty\}$,
- with $\delta^{12}_{t_{12}}$ on $(-\infty) \times [-\infty, \infty]$ and
- with $\delta^{02}_{t_{02}}$ on $\{(t_1, t_2) \in [-\infty, \infty]^2 \mid t_1 = t_2\}$,

where $\Gamma_{ij}, 0 \leq i \leq j \leq 2$ is the corresponding type of once-quilted disk;

(b) a smooth family of metrics $G_t^{012}$ constant equal to $G^0$ resp. $G^1$ resp. $G^2$ on a neighborhood of the endpoints for which $d = (-\infty, -\infty)$ resp. $d = (\infty, -\infty)$ resp. $d = (\infty, \infty)$ and that agrees

- with $G_{t_{01}}^{01}$ on $[-\infty, \infty] \times \{-\infty\}$, with $G_{t_{12}}^{12}$ on $\{+\infty\} \times [-\infty, \infty]$ and
- with $G_{t_{02}}^{02}$ on $\{(t_1, t_2) \in [-\infty, \infty]^2 \mid t_1 = t_2\}$,

where $\Gamma_{ij}, 0 \leq i \leq j \leq 2$ is the corresponding type of once-quilted disk;

(c) a domain-dependent Morse function $F_t^{012}$ equal to $F^0$ resp. $F^1$ resp. $F^2$ on in a neighborhood of the endpoints at $d = (-\infty, -\infty)$ resp. $d = (\infty, -\infty)$ resp. $d = (\infty, \infty)$ and that agrees with $F_{t_{01}}^{01}$ resp. $F_{t_{12}}^{12}$ resp. $F_{t_{02}}^{02}$ on the once quilted treed disk components of type $\Gamma_{01}$ resp. $\Gamma_{12}$ resp. $\Gamma_{02}$ containing the root resp. the leaves resp. where the quilting radii coincide,

(d) a domain-dependent almost-complex structure $\bar{J}_t^{012}$ such that for every surface component $C_i$, it is equal to $J^{012}_{\odot d(z)}$ on $D^{012}_{\odot d(z)}$, in a neighborhood of the spherical nodes, the interior markings and on the boundary of $C_i$, that is
equal to \( J^0_{\Gamma_0} \), resp. \( J^1_{\Gamma_1} \) resp. \( J^2_{\Gamma_2} \) on the unquilted components of type \( \Gamma_0 \) resp. \( \Gamma_1 \) resp. \( \Gamma_2 \) at \( d = (-\infty, -\infty) \) resp. \( d = (\infty, -\infty) \) resp. \( d = (\infty, \infty) \) and

goes with \( J^{01}_{\Gamma_{01}} \) resp. \( J^{12}_{\Gamma_{12}} \) resp. \( J^{02}_{\Gamma_{02}} \) on the once quilted treed disk components of type \( \Gamma_{01} \) resp. \( \Gamma_{12} \) resp. \( \Gamma_{02} \) containing the root resp. the leaves resp. where the quilting radii coincide.

(e) One can also require the following invariance property: A perturbation datum is quilting-independent if \( G^{012}_{\Gamma}, F^{012}_{\Gamma}, J^{012}_{\Gamma} \) are pull-backs under the forgetful morphism forgetting the quivilings on each once-quilted or twice-quilted disk.

Given a treed twice-quilted treed disk \( C \) of type \( \Gamma \), one obtains perturbation data by pull-back from the stabilization of \( C \), which may be identified with a fiber of the universal twice-quilted disk. A holomorphic twice-quilted treed disk is a twice-quilted disk \( C \) a map \( u : C \to X \) that is \( J^{012}_{\Gamma} \)-holomorphic on surface components and a \( F^{012}_{\Gamma} \)-Morse trajectory on tree segments with respect to the metrics \( G^{01}_{\Gamma}, G^{12}_{\Gamma}, G^{02}_{\Gamma} \). Stable and adapted twice-quilted treed disks are defined as in the once-quilted case. In particular, each interior marking \( z_i \) maps to the divisor \( D^{012}_{\delta}(\varphi) \). Assuming the perturbations satisfy coherence and stabilized conditions similar to those for quilted disks, the moduli spaces of adapted stable twice-quilted disks are compact for each uncrowded combinatorial type of expected dimension at most one. The property (e) of Definition 3.8 ensures that the latter zero dimensional spaces will not contain non-constant (either once or twice) quilted disks and that the one dimensional strata may involve at most one once quilted disk and if it does, it is a constant family over which the latter quilting radius varies freely.

In order to obtain transversality the fiber products involved in the definition of the universal twice-quilted disks in (53) must be perturbed, using delay functions, as in Seidel [30] and Ma’u-Wehrheim-Woodward [26] which we follow closely. Again, the problem is a variation on the multiple cover problem: if an isolated twice-quilted component bubbles off with ratio given by some \( \lambda_i \), then configurations with the same bubble repeated also appear, again with the same ratio. But transversality with the diagonal in (53) implies that at most one such twice-quilted component can be isolated in its moduli space.

We define a map incorporating both the condition on ratios and distances between quilted components as follows. We identify \( \mathcal{M}_{1,0,2} \) with \([0, \infty]\) as in Figure 18. For \( n \geq 1, m \geq 0 \) denote by

\[
\lambda : \mathcal{M}_{n,m,2} \to \mathcal{M}_{1,0,2} \cong [0, \infty]
\]

the forgetful morphism forgetting all but the first marking; note that on the interval consisting of only once-quilted disks, \( \lambda \) is essentially equivalent to the map \( \lambda_T \) of (55) while on the interval with twice-quilted disks \( \lambda \) is given by the map \( \lambda_S \) of (53). We note that \( \lambda \) is also defined in the case \( n = 0 \), by combining the maps (55) and (53). This is also true if furthermore \( m = 0 \) although the latter twice-quilted treed disks are unstable and will not appear in the moduli spaces considered later.
We combine the maps from the previous paragraph as follows. Let $\Gamma$ be a combinatorial type of twice-quilted disks. Define $\mathcal{M}_{\Gamma}^{\text{pre}}$ as the product of moduli spaces for the vertices,

$$\mathcal{M}_{\Gamma}^{\text{pre}} = \prod_{v \in \text{Vert}(\Gamma)} \mathcal{M}_v.$$ 

Let $k$ denote the number of twice-quilted vertices and

$$\lambda_{\Gamma} : \mathcal{M}_{\Gamma}^{\text{pre}} \to \mathbb{R}^k, \quad (r_v) \mapsto \prod_{v \in \text{Vert}(\Gamma)} \lambda(r_v)$$

the map combining the forgetful maps for the twice-quilted components.

$$\mathcal{M}_{\Gamma} = \lambda_{\Gamma}^{-1}(\Delta)$$

where $\Delta \subset \mathbb{R}^k$ is the diagonal. A delay function for $\Gamma$ is a collection of smooth functions depending on $r \in \mathcal{M}_{\Gamma}^{\text{pre}}$

$$\tau_{\Gamma} = (\tau_e \in C^\infty(\mathcal{M}_{\Gamma}^{\text{pre}}))_{e \in \text{Edge}(\Gamma_0)}.$$ 

Letting $\lambda_i := \lambda(r_{v_i})$ where $\lambda(r_{v_i})$ is the ratio of the radii circles for $r_{v_i}$, the delayed evaluation map is

$$\lambda_{\tau_{\Gamma}} : \prod_{v \in \text{Vert}(\Gamma)} \mathcal{M}_v \to \mathbb{R}^k$$

$$(r_v, u_v)_{v \in \text{Vert}(\Gamma)} \mapsto \left( \lambda_i \exp \left( \sum_{e \in p_i} \tau_e(r) \right) \right)_{i=1, \ldots, k}$$

which is the sum of delays along each path $p_i$ to a twice-quilted disk component. We call $\tau_{\Gamma}$ regular if the delayed evaluation map $\lambda_{\tau_{\Gamma}}$ is transverse to the diagonal $\Delta \subset \mathbb{R}^k$. Given a regular delay function $\tau_{\Gamma}$, we define

$$\mathcal{M}_{\Gamma} := \lambda_{\tau_{\Gamma}}^{-1}(\Delta).$$

For a regular delay function $\tau_{\Gamma}$, the delayed fiber product has the structure of a smooth manifold, of local dimension

$$\dim \mathcal{M}_{\Gamma} = 1 - k + \sum_{v \in \text{Vert}(\Gamma)} \dim \mathcal{M}_v.$$
where \( k \) is the number of twice-quilted disk components. A collection \( \{ \tau^d \}_{d \geq 1} \) of delay functions is compatible if the following properties hold. Let \( \Gamma \) be a combinatorial type of twice-quilted disk and \( v_0, \ldots, v_k \).

(a) \( \text{(Subtree property)} \) Suppose that the root component \( v_0 \) is not a twice-quilted disk. Let \( \Gamma_1, \ldots, \Gamma_{|v_0|-1} \) denote the subtrees of \( \Gamma \) attached to \( v_0 \) at its leaves; then \( \Gamma_1, \ldots, \Gamma_{|v_0|-1} \) are combinatorial types for nodal twice-quilted disks. Let \( r_i \) be the component of \( r \in \mathcal{M}_{\Gamma}^{\text{pre}} \) corresponding to \( \Gamma_i \). We require that 
\[
\tau_{\Gamma}(r)_{|\Gamma_i} = \tau_{\Gamma_i}(r_i), \]
i.e., for each edge \( e \) of \( \Gamma_i \), the delay function \( \tau_{\Gamma,e}(r) \) is equal to \( \tau_{\Gamma_i,e}(r_i) \).

(b) \( \text{(Refinement property)} \) Suppose that the combinatorial type \( \Gamma' \) is a refinement of \( \Gamma \), in other words there is a surjective morphism \( f : \Gamma' \to \Gamma \) of trees; let \( r \) be the image of \( r' \) under gluing. We require that \( \tau_{\Gamma}(r)_{|U} = \tau_{\Gamma'}(r') \) as follows: for each \( e \in \text{Edge}(\Gamma) \), and \( r \in U \), the delay function is given by the formula
\[
\tau_{\Gamma,e}(r) = \tau_{\Gamma',e} + \sum_{e'} \tau_{\Gamma',e'}(r')
\]
where the sum is over edges \( e' \) in \( \Gamma' \) that are collapsed under gluing and the \( e \) is the next-furthest-away edge from the root vertex.
In the case that the collapsed edges connect twice quilted components with unquilted components, this means that the delay functions are equal for both types, as in the Figure 21.

(c) (Core property) If two combinatorial types say \( \Gamma \) and \( \Gamma' \), have the same core \( \Gamma_0 \), let \( r, r' \) be disks of type \( \Gamma \) resp. \( \Gamma' \). Then \( \tau_{\Gamma,e}(r) = \tau_{\Gamma',e}(r') \). (That is, the delay functions depend only on the region between the root vertex and the bicolored vertices.)

A collection of compatible delay functions is positive if, for every vertex \( v \in \Gamma_0 \) with \( k \) leaves labeled in counterclockwise order by \( e_1, \ldots, e_k \), their associated delay functions satisfy \( \tau_{e_1} < \tau_{e_2} < \ldots < \tau_{e_k} \).

![Figure 21. The (Refinement property), second case](image)

For each combinatorial type \( \Gamma \) we may find regular, positive delay functions. The (Subtree property) implies that all the delay functions in \( \tau_{\Gamma} := \tau_{\Gamma}(L) \) except those for the finite edges adjacent to \( v_0 \), the root component, are already fixed. It remains to find regular delay functions for the finite edges adjacent to the root component of each combinatorial type, in a way that is also compatible with conditions (Refinement property). We ensure compatibility by proceeding inductively on the number of leaves of the root component, as follows. We suppose that we have constructed inductively regular delay functions for types \( \Gamma \) corresponding to strata of \( \mathcal{M}_{e,m,2} \) for \( e < d \), as well as for types \( \Gamma' \) appearing in the (Refinement property) for \( \Gamma \), and now construct inductively on \( n \geq 2 \) a collection of regular delay functions \( \tau_{\Gamma} := \tau_{\Gamma}(L) \) for all combinatorial types \( \Gamma \) whose root component has \( 2 \leq k \leq n \) leaves. We may assume that \( \Gamma \) has no components “beyond the twice-quilted components” since by the (Core property) the delay functions are independent of the additional components. Consider a combinatorial type whose root component has \( n \) leaves. There is an open neighborhood \( U \) of \( \partial \mathcal{M}_{\Gamma}^{\text{pre}} \) in \( \overline{\mathcal{M}_{\Gamma}^{\text{pre}}} \) in which the delay functions \( \tau_{\Gamma} \) for the leaves adjacent to the root vertex are already determined by the compatibility condition (Refinement property) and the inductive hypothesis. We need to show that we may extend the \( \tau_{i} \) over the interior of \( \overline{\mathcal{M}_{\Gamma}^{\text{pre}}} \). To set up the relevant function spaces let \( l \geq 0 \) be an integer and let \( f \) be a given \( C^l \) function on \( \overline{U} \). Let \( C^l(\mathcal{M}_{\Gamma}^{\text{pre}}) \) denote the Banach manifold of functions with \( l \) bounded derivatives on \( \mathcal{M}_{\Gamma}^{\text{pre}} \), equal to \( f \) on \( \overline{U} \). Let \( \Gamma_i, i = 1, \ldots, n \) be the trees attached to the root vertex \( v_0 \). Consider
the evaluation map

\[ ev : M_{\Gamma_1} \times \ldots \times M_{\Gamma_n} \times M_{\tau_0} \times \prod_{i=1}^{n} C_{\tau_i}^d (M_{\Gamma}^{\text{pre}}) \rightarrow \mathbb{R}^{n-1} \]

\[ \left( (r_1, u_1), \ldots, (r_n, u_n), (r_0, u_0), \tau_1, \ldots, \tau_n \right) \mapsto \left( \lambda_{\Gamma_j}(r_j) \exp(\tau_j(r)) - \lambda_{\Gamma_{j+1}}(r_{j+1}) \exp(\tau_{j+1}(r)) \right)_{j=1}^{n-1} \]

where \( r = (r_0, \ldots, r_n) \). Note that 0 is a regular value. The Sard-Smale theorem implies that for \( l \) sufficiently large the regular values of the projection

\[ \Pi : ev^{-1}(0) \rightarrow \prod_{i=1}^{n} C_{\tau_i}^d (M_{\Gamma}^{\text{pre}}) \]

form an open dense set. Taking the intersection over \( l \) sufficiently large gives that the set of smooth regular delay functions is comeager. Both the positivity condition and the regularity condition (b) are open conditions given an energy bound. It follows from the monotonicity condition that an energy bound for quilts of dimension zero or one exists, and therefore the set of smooth, positive, compatible, delay functions that are regular for a given energy bound is non-empty and open. Taking the intersection of these sets over all possible energy bounds we obtain a comeager set of delay functions for which all moduli spaces are regular. By induction, there exists a smooth, positive, compatible, regular delay function \( \tau_{\Gamma} \) for each combinatorial type of twice-quilted \( d+1 \)-marked disk, and hence a regular compatible collection \( \tau^d \).

Given a collection of regular compatible delay functions and a collection of perturbation data \( P^0, P^1, P^2, P^{01}, P^{12} \) and \( P^{02} \) for unquilted and once-quilted disks, perturbations \( P^{012} = (P_{\Gamma}^{012}) \) for twice-quilted disks are constructed inductively using the gluing construction and the Sard-Smale theorem. For each stratum, we first use the construction of the previous paragraph to find regular delay functions for the boundary strata, then (after replacing the fiber products in the boundary strata by the delayed fiber products of boundary strata using the delay functions chosen) extend the perturbations by the gluing construction. A Sard-Smale argument shows that for perturbations in a comeager subset, the uncrowded moduli spaces of index at most one are regular of expected dimension. We say that a system \( P^{012} \) is convergent if it satisfies (40).

**Theorem 3.9.** (\( A_\infty \) homotopies via twice-quilted disks) Given convergent, regular, coherent, stabilizing perturbation systems \( P^{01}, P^{12}, P^{02} \) and morphisms \( \phi_{ij} : CF(L, P^i) \rightarrow CF(L, P^j) \), \( 0 \leq i < j \leq 2 \) and \( P^{012} \) as above, counting treed holomorphic twice-quilted disks defines a convergent \( A_\infty \) homotopy between \( \phi_{02} \) and \( \phi_{01} \circ \phi_{12} \).

**Proof.** Consider the map \( \lambda : M_{n,2} \rightarrow [1, \infty] \) giving the ratio of radii of the inner circles of the twice-quilted disks, after subtracting the delay functions. For generic values of \( \lambda \) the moduli space \( \overline{M}_{n,2}^\lambda(L, D) = \psi^{-1}(\lambda) \) is smooth of expected dimension. We let \( \phi_{02}^\lambda \) denote the \( A_\infty \) morphism obtained by counting twice-quilted disks with...
ratio of radii \( \lambda \in [1, \infty) \). After fixing an energy bound \( E \) and a number of leaves \( n \) we may divide the interval \([0, 1]\) into subintervals \([\lambda_i, \lambda_{i+1}]\), \(i = 0, \ldots, k - 1\) so that each is sufficiently small so that there is a single singular value \( \lambda \in [\lambda_i, \lambda_{i+1}] \), contained in the interior of the interval, for which there exist twice-quilted disks of expected dimension zero of energy at most \( E \) and \( n \) leaves and no such disks with fewer number of leaves. Define \( T_{02}^{\lambda_i, \lambda_{i+1}, \leq E} \) by counting such twice-quilted disks,

\[
(59) \quad (T_{02}^{\lambda_i, \leq E})^n : C\hat{F}(L; P^0)^\otimes n \to C\hat{F}(L; P^1)
\]

\[
(x_1, \ldots, x_n) \mapsto \sum_{x_0, [u] \in \mathcal{M}_{\lambda_i}^\circ (L, D; x_0, \ldots, x_n)} (-1)^{\gamma} \epsilon([u])[\sigma([u])!]^{-1} q^{E([u])}\text{Hol}_L(u)x_0
\]

where the sum is over combinatorial types \( \Gamma \) of twice-quilted disks. The difference

\[
(\phi_{02}^{\lambda_i} - \phi_{02}^{\lambda_{i+1}})^n(x_1, \ldots, x_n)
\]

is a count of configurations either involving an unquilted disk breaking off, or a collection of twice-quilted treed disks with \( i_1, \ldots, i_r \) leaves breaking off from an unquilted treed disk, see [25, Section 7]. For degree reasons, because of the fiber product with the diagonal exactly one of these twice-quilted disks lies in the moduli space of expected dimension zero, while the rest have index one. We suppose that the twice-quilted configuration in the expected-dimension-zero is the \( i + 1 \)-st twice-quilted treed disk attached to the unquilted treed disk. Using positivity of the delay functions one obtains that the moduli space of twice quilled disks \( \lambda_i \) and \( \lambda_{i+1} \) and expected dimension zero are cobordant:

\[
\mathcal{M}_{ij,2}^{\lambda_j \tau_j}(L, D) \sim \mathcal{M}_{ij,2}^{\lambda_{i+1} \tau_j}(L, D), j > i + 2 \quad \mathcal{M}_{ij,2}^{\lambda_j \tau_j}(L, D) \sim \mathcal{M}_{ij,2}^{\lambda_{i+1} \tau_j}(L, D), j \leq i.
\]

It follows that

\[
\phi_{02}^{\lambda_i} - \phi_{02}^{\lambda_{i+1}} = \mu_2^{1 \phi_{02}^{[\lambda_i, \lambda_{i+1}]} (T_{02}^{\lambda_i, \lambda_{i+1}, \leq E})}.
\]

The facets of \( \mathcal{M}_{ij,2} \) with ratio \( \lambda = 1 \) or \( \lambda = \infty \) correspond to either to terms in the definition of composition of \( A_\infty \) maps \( \phi_{12} \circ \phi_{01} : C\hat{F}(L, P^0) \to C\hat{F}(L, P^2) \), to the components contributing to \( \phi_{02} : C\hat{F}(L, P^0) \to C\hat{F}(L, P^2) \) or to terms corresponding to the bubbling off of some markings on the boundary which define a homotopy operator for the difference \( \phi_{12} \circ \phi_{01} - \phi_{02} \). In case \( P^0 = P^2 \) composition using (46) produces a homotopy

\[
T_{02}^{\leq E} := \mu_2 (T_{02}^{\lambda_0, \lambda_1 \leq E} \hat{\otimes} \mu_2 (T_{02}^{\lambda_1, \lambda_2} \hat{\otimes} \mu_2 (T_{02}^{\lambda_2, \lambda_3} \hat{\otimes} \mu_2 (T_{02}^{\lambda_3, \lambda_4} \hat{\otimes} \mu_2 (T_{02}^{\lambda_4, \lambda_5} \cdots \cdots \cdots )))
\]

between \( \phi_{12} \circ \phi_{01} \) and the identity modulo terms involving powers \( q^E \). Taking the limit \( E \to \infty \) defines a homotopy

\[
T_{02} := \lim_{E \to \infty} T_{02}^{\leq E}
\]

between \( \phi_{02} \) and \( \phi_{12} \circ \phi_{01} \). Convergence is similar to Proposition 2.35. In particular \( T_{02}^{0}(1) \in \Lambda_\infty \) \( C\hat{F}(L, P^0) \) since any contributing configuration must contain a non-trivial disk. \( \square \)
Corollary 3.10. For any two regular, stabilized, coherent, convergent collections of perturbation data $P^0, P^1$ the Fukaya algebras $\widehat{CF}(L, P^0)$ and $\widehat{CF}(L, P^1)$ are homotopy equivalent via convergent maps and homotopies in the sense of Lemma 3.3.

Proof. Taking $D_{02}$ to be constant and $P^0_{02}$ to be pulled back by the map forgetting the quilting, one obtains a homotopy between the composition of morphisms

$$\phi_{01} : \widehat{CF}(L, P^0) \to \widehat{CF}(L, P^1), \quad \phi_{10} : \widehat{CF}(L, P^1) \to \widehat{CF}(L, P^0)$$

and the identity morphism as in Proposition 3.7. □

Define the Floer cohomology as the stalks of the cohomology complex of (37):

$$HF(L) := \bigcup_{b \in MC} HF(L)_b, \quad HF(L)_b := H(\hat{CF}(L), \partial_b).$$

We say that a Lagrangian brane $L$ is Floer non-trivial if some fiber $HF(L)_b$ is non-zero; that is, the space of solutions to the weak Maurer-Cartan equation is non-empty and for at least one solution the cohomology is non-zero. By the Corollary 3.10 and Lemma 3.3, Floer non-triviality of $L$ is independent of all choices.

3.6. Stabilization. In this section we complete the proof of homotopy invariance of the Fukaya algebras constructed above in the case that the algebras are defined using divisors are not of the same degree or built from homotopic sections.

For this we need to recall some results about existence of a Donaldson hypersurface transverse to a given one. Recall from [9, Lemma 8.3] that for a constant $\epsilon > 0$, two divisors $D, D'$ intersect $\epsilon$-transversally if at each intersection point $x \in D \cap D'$ their tangent spaces $T_xD, T_xD'$ intersect with angle at least $\epsilon$. A result of Cieliebak-Mohnke [9, Theorem 8.1] states that there exists an $\epsilon > 0$ such that given a divisor $D$, there exists a divisor $D'$ of sufficiently high degree $\epsilon$-transverse to $D$. Moreover, for any $\theta > 0$, $\omega$-tamed almost complex structures $\theta$-close to $J$ making $D, D'$ almost complex exist (provided that the degrees are sufficiently large).

We apply the result of the previous paragraph as follows. Suppose that $D^0, D^1$ are stabilizing divisors for $L$, possibly of different degrees. By the previous paragraph, there exists a pair $D^{0'}_1, D^{1'}_0$ of higher degree stabilizing divisors built from homotopic unitary sections over $L$ that are $\epsilon$-transverse to $D^0$ and $D^1$, respectively. Let $P^0_k, P^1_k$ be perturbation systems for $D^{0'}, D^{1'}$. We have already shown that

$$\widehat{CF}(L, P^0_0) \cong \widehat{CF}(L, P^1_1)$$

are homotopy equivalent. It remains to show:

Theorem 3.11. For any convergent, coherent, regular, stabilized perturbation systems $P_k, P'_k, k = 0, 1$ as above, the Fukaya algebras $\widehat{CF}(L, P_k)$ and $\widehat{CF}(L, P'_k)$ are homotopy equivalent.

Sketch of proof. Denote by $J^+(X, D^k \cup D^{k'}; J_{P^0}, \theta, E)$ the subset of $J^+(X, D^k, J, \theta, E)$ of almost complex structures close to $J$ preserving $TD^{k'}$ as well. By [9, Corollary
there exists a path-connected, open, dense set in $\mathcal{J}^*(X, D^k \cup D^{k'}; J_{D^k}, \theta, E)$ with the property that for any $J \in \mathcal{J}^*(X, D^k \cup D^{k'}; J_{D^k}, \theta, E)$, neither $D^k$ nor $D^{k'}$ contain any non-constant holomorphic spheres of energy at most $E$, and each holomorphic sphere meets both $D^k$ and $D^{k'}$ in at least three points. Fix such an almost complex structure $J_{D^k, D^{k'}}$, and an associated perturbation system $P''_k$ using the divisor $D^k$ such that the almost complex structures $J_{k, \Gamma}$ are equal to $J_{D^k, D^{k'}}$ on $D^k \cup D^{k'}$. The argument in the previous section (keeping the divisor constant but changing the almost complex structures) shows that the associated Fukaya algebras

$$\hat{CF}(L, P_k) \cong \hat{CF}(L, P''_k)$$

are homotopy equivalent. Similarly, choose a perturbation system $P'_k$ using the divisor $D^{k'}$. We claim that the Fukaya algebras

$$\hat{CF}(L, P'_k) \cong \hat{CF}(L, P''_k)$$

are homotopy equivalent. To see this, we define adapted stable maps adapted to the pair $(D^k, D^{k'})$: a map is adapted if each interior marking maps to either $D^k$ or $D^{k'}$, and the first $n_k$ markings map to $D^k$ and the last $n'_k$ markings map to $D^{k'}$. A perturbation datum morphism $P = (P_T)$ is coherent if it is compatible if with the morphisms of moduli spaces as before: (Cutting edges) axiom, (Collapsing edges or Making an edge/weight finite/non-zero) axiom, and satisfies the (Infinite weights) axiom and (Product) axiom, where now on the unquilted components above resp. below the quilted components the perturbation system is required to depend only on the first $n_k$ resp. last $n'_k$ points mapping to $D^k$ resp. $D^{k'}$ (that is, pulled back under the forgetful map forgetting the first $n_k$ resp. last $n'_k$ markings). Then the same arguments as before produce the required homotopy equivalence; taking the perturbations to be $C^2$-small implies that the analog of (40) holds. Putting everything together we have homotopy equivalences

$$\hat{CF}(L, P_k) \cong \hat{CF}(L, P'_k) \cong \hat{CF}(L, P''_k).$$

Applying (60) completes the proof. \hfill $\square$

**Corollary 3.12.** For any stabilizing divisors $D^0, D^1$ and any convergent, coherent, regular, stabilized perturbation systems $P_0, P_1$, the Fukaya algebras $\hat{CF}(L, P_0)$ and $\hat{CF}(L, P_1)$ are convergent-homotopy-equivalent.

**Proof.** Since homotopy equivalence of $A_\infty$ algebras is an equivalence relation by Definition 3.1 (i), combining Theorems 3.6 and 3.11 gives the result. \hfill $\square$

**References**


DEPARTMENT OF MATHEMATICS, BARNARD COLLEGE - COLUMBIA UNIVERSITY, MC 4433, 2990 BROADWAY, NEW YORK, NY 10027

E-mail address: charlest@math.columbia.edu

MATHEMATICS-HILL CENTER, RUTGERS UNIVERSITY, 110 FRELINGHUYSEN ROAD, PICATAWAY, NJ 08854-8019, U.S.A.

E-mail address: ctw@math.rutgers.edu