AUGMENTATION VARIETIES AND DISK POTENTIALS I

KENNETH BLAKEY, SOHAM CHANDA, YUHAN SUN, AND CHRIS WOODWARD

Abstract. This is the first in a sequence of papers where we show that Lagrangian fillings such as the Harvey-Lawson filling in any dimension define augmentations of Chekanov-Eliashberg differential graded algebras by counting configurations of holomorphic disks connected by gradient trajectories, as in Aganagic-Ekholm-Ng-Vafa [2]; we also prove that for Legendrian lifts of monotone tori, the augmentation variety is the zero level set of the Landau-Ginzburg potential of the Lagrangian projection, as suggested by Dimitroglou-Rizell-Golovko [19]. In this part, we set up the analytical foundation of moduli spaces of pseudoholomorphic buildings.

Contents

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1. Introduction

Contact homology as introduced by Eliashberg, Givental, and Hofer [27] is a theory whose generators come from closed Reeb orbits and whose differential counts holomorphic curves in the symplectization of a contact manifold. Its relative version, Legendrian contact homology, has been developed by Ekholm-Etnyre-Ng-Sullivan [25] and related works for the case that the contact manifold fibers over an exact symplectic manifold. This is the first of a sequence of papers in which we develop a version of Legendrian contact homology in case that the contact manifold fibers over a monotone symplectic manifold, extending work of Sabloff [43], Asplund [34], and Dimitroglou-Rizell-Golovko [19]. In particular, we give a definition of augmentation variety of Legendrians such as the lift of a Clifford torus in any dimension.

Our definition of the Legendrian contact homology is related to work of Aganagic-Ekholm-Ng-Vafa [2]. They suggested that in order for Lagrangian fillings such as the Harvey-Lawson filling to define augmentations of Chekanov-Eliashberg differential graded algebra, one should count configurations of holomorphic disks connected by gradient trajectories. Such configurations are otherwise known as treed disks or clusters. By counting such configurations, we give a construction of the Chekanov-Eliashberg dga in which the zeroes of the corresponding vector fields as generators, similar to the definition of immersed Lagrangian Floer homology developed by Akaho-Joyce [3]. The resulting version of the Chekanov-Eliashberg algebra gives rich Legendrian isotopy invariants, which distinguish many Legendrian lifts of Lagrangians in monotone symplectic manifolds such as Vianna's exotic tori; the non-isotopy of these tori was conjectured by Dimitroglou-Rizell-Golovko [19].

In this first part, we construct the moduli spaces of treed disks with punctures, and prove results about their compactness, regularization and orientations. In sequels [6, 7], these moduli spaces will be used to:

(a) Define the Chekanov-Eliashberg algebra, and show the resulting Legendrian contact homology is independent of various choices made;
(b) For a tame Lagrangian cobordism between Legendrians, define a chain map between the Chekanov-Eliashberg algebras. Particularly, define an augmentation variety for a Legendrian and show it is a Legendrian isotopy invariant.
(c) Establish a concrete relation between the disk potential functions of certain monotone Lagrangians such as Vianna's tori and the augmentation varieties of their Legendrian lifts, hence distinguish infinitely many Legendrians that are mutually not Legendrian isotopic.
(d) Compute the augmentation varieties for the Clifford-type and Hopf-type Legendrians, and the augmentation variety in particular for disconnected Legendrians, in relation to the conjectures of [2] on the parametrization of components by partitions.

Now we describe our geometric settings and moduli spaces in a more detailed way. Consider a Legendrian Λ in a compact contact manifold Z that is a negatively-curved circle-fibration over a symplectic base Y; we have in mind especially the example that Z is some unit complex line bundle over a Fano projective toric variety Y, and Λ is the horizontal lift of a Lagrangian in the base. One fundamental example is the horizontal lift of the Clifford torus in complex projective space $Y = \mathbb{C}P^{n-1}$ to the unit tautological line bundle. This lift is a Legendrian torus in the standard contact sphere $Z = S^{2n-1}$:

$$\Lambda_{\mathrm{Cliff}} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{l} |z_1|^2 = \ldots = |z_n|^2 = \frac{1}{n} \\ z_1 \ldots z_n \in (0, \infty) \end{array} \right\} \cong (S^1)^{n-1} \subset S^{2n-1}. \tag{1}$$

As in Dimitroglou-Rizell-Golovko in [19], this Legendrian is the horizontal lift of the Clifford torus

$$\Pi_{\mathrm{Cliff}} = \left\{ [z_1, \ldots, z_n] \;\middle|\; \sum_{i=1}^{n} |z_i|^2 = 1 \right\} \cong (S^1)^{n-1} \subset \mathbb{C}P^{n-1}.$$

The map

$$\mathbb{C}^n - \{0\} \to \mathbb{C}P^{n-1}, \quad z \mapsto \mathrm{span}(z)$$

restricts an n-fold cover of Π_{Cliff}. Similar examples are given by taking Π to be a Lagrangian in a toric variety Y, $Z \to Y$ a circle bundle with connection with rational holonomies and Λ the horizontal lift of Π.

An example of a Lagrangian filling is given by the Harvey-Lawson filling of the Legendrian in the previous paragraph. Let

$$a_1, \ldots, a_n \in \mathbb{R}$$

be real numbers with exactly two zero. Define as in Joyce [35, (37)]

$$L_{(1)} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{l} |z_1|^2 - a_1^2 = |z_2|^2 - a_2^2 = \ldots = |z_n|^2 - a_n^2 \\ z_1 \ldots z_n \in [0, \infty) \end{array} \right\}. \tag{2}$$

To understand this *Harvey-Lawson filling* better, consider the map

$$\pi_m : \mathbb{C}^n \to \mathbb{C}, \;\; \pi_m(z_1, \ldots, z_n) = z_1 z_2 \ldots z_n$$

which is a symplectic fibration away from the critical locus. The symplectic horizontal bundle of the fibration π_m is the symplectic complement of the vertical bundle. A direct computation shows that the symplectic horizontal bundle H_z at a point $z \in \mathbb{C}^n$ is given as

$$H_z = \mathrm{Span}_{\mathbb{C}} \left\{ \left(\frac{z_1}{|z_1|^2}, \ldots, \frac{z_n}{|z_n|^2} \right) \right\}.$$

Consider the torus

$$T_1 = \{ z \in \pi_m^{-1}(1) \mid |z_1|^2 - a_1^2 = |z_2|^2 - a_2^2 = \ldots = |z_n|^2 - a_n^2 \}.$$

Since the derivative of $|z_i|^2 - |z_j|^2$ along any vector in H_z vanishes, we conclude that the parallel transport of T_1 along $[0, \infty)$ lies in $L_{(1)}$. Thus $L_{(1)}$ can be viewed as the union of parallel transports of T_1 along $[0, \infty)$.

We briefly describe the moduli spaces of pseudoholomorphic curves needed for the definition of the differential associated to the Legendrian and the chain maps associated to Lagrangian cobordisms. The construction is a special case of symplectic field theory with Lagrangian boundary conditions, where the contact manifold is circle-fibered. The simplifications in this case are comparable to that of genus-zero relative Gromov-Witten theory compared to full symplectic field theory with contact boundary. In particular, we use Cieliebak-Mohnke [14] perturbations to regularize the moduli spaces. By a *punctured disk* we mean a surface obtained by removing a finite collection of points on the boundary. More precisely our configurations are *treed punctured disks* obtained by adding a finite collection of trajectories of some vector fields, on both the space of Reeb chords and the Legendrian itself, as in Figure 1, and to be explained below. In Figure 1, the dotted lines represent Reeb chords, and the trajectories can either connect the Reeb chords or connect points on the Lagrangian boundary condition of the disks. The disk shown would contribute to the coefficient of a word of length eight in the differential applied to a Reeb chord. The natural chain complex $CE(\Lambda)$ is generated by words in chains on the

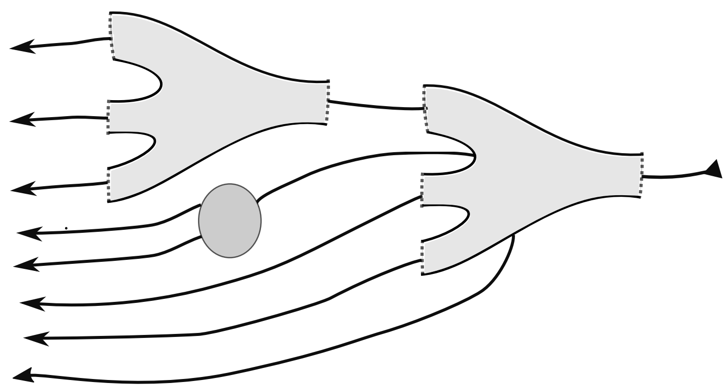

FIGURE 1. A treed punctured disk

Reeb chords and chains on the Legendrian; the addition of these *classical generators* does not occur in the version of Legendrian contact homology over exact symplectic manifolds in Ekholm-Etnyre-Sullivan [24]. The generators arising from chains on the Legendrian arise because of nodes developing along points in the interior of the cobordism, which then lead to trajectories as in Ekholm-Ng [26]; in particular [26] assigns augmentations to the Harvey-Lawson filling in the three-dimensional setting of knot contact homology while the construction here is intended to work in all dimensions, under some restrictions.

The current paper first (Section 2) introduces geometric constructions of many circle-fibered contact manifolds and their Legendrians. In particular, we construct Lagrangian fillings of certain Legendrian lifts of Vianna's tori, which may be of independent interest. Then we set up moduli spaces of treed punctured disks solving the Cauchy-Riemann equation (Section 3), and prove their compactness, regularity and orientability (Section 4).

2. Fibered contact manifolds and Lagrangian cobordisms

In this section, we describe various constructions of Legendrians in circle-fibered contact manifolds and Lagrangian cobordisms. The Harvey-Lawson filling is described both as a (non-exact) asymptotic filling and as a compact filling of a Legendrian with respect to a perturbation of the standard contact structure on the sphere.

2.1. **Fibered contact manifolds and Legendrians.** The contact manifolds we consider are circle fibrations over a compact symplectic manifold. Let n be a positive integer and Z a smooth manifold of dimension $2n-1$. Let $\Omega(Z) = \bigoplus_j \Omega^j(Z)$ be the algebra of smooth differential forms. Recall, as in for example Geiges [30], that a *contact form* on Z is a one-form

$$\alpha \in \Omega^1(Z), \quad (\alpha \wedge (\mathrm{d}\alpha)^{n-1})(z) \neq 0 \quad \forall z \in Z.$$

The kernel of α is the *contact distribution*

$$\xi := \ker(\alpha) \subset TZ$$

associated to α. The form α defining ξ is unique up to multiplication by a non-zero function.

A *Legendrian submanifold* of Z is a submanifold $\iota : \Lambda \subset Z$ of dimension $n-1$ on which α vanishes:

$$\iota^*\alpha = 0 \in \Omega^1(Z).$$

We will assume our submanifolds are embedded, rather than immersed, unless otherwise stated.

More generally, as in Cieliebak-Volkov [15], a *stable Hamiltonian structure* on Z is a pair of a two-form and a one-form

$$\omega \in \Omega^2(Z), \quad \alpha \in \Omega^1(Z)$$

such that

$$\alpha \wedge \omega^{n-1} \neq 0, \quad \ker(\omega) \subset \ker(\mathrm{d}\alpha) \subset TZ.$$

A manifold with a stable Hamiltonian structure will be called a *stable Hamiltonian manifold.*

A *Legendrian submanifold* of a stable Hamiltonian manifold is a maximal isotropic submanifold for ω on which α vanishes. If $\omega = -\mathrm{d}\alpha$ then the pair (Z, α) is a contact manifold.

A *fibered contact manifold* is a manifold Z with a contact form $\alpha \in \Omega^1(Z)$ so that Z is the total space of a principal S^1-fiber bundle

$$p : Z \to Y, \quad p^{-1}(y) \cong S^1, \quad \forall y \in Y$$

so that α is a connection one-form; recall that this means that α is invariant under the S^1-action on Z and the contraction $\alpha(\partial_\theta) \in \Omega^0(Z)$ of the generating vector field $\partial_\theta \in \mathrm{Vect}(Z)$ is equal to 1. A fibered stable Hamiltonian manifold is defined similarly.

Lemma 2.1. *Any Legendrian submanifold Λ in a fibered contact manifold Z projects to a (possibly immersed) Lagrangian submanifold*

$$\Pi := p(\Lambda) \subset Y$$

with respect to the symplectic form $\omega_Y := -\operatorname{curv}(\alpha)$ on Y.

Proof. By the constant rank theorem, $\Pi \subset Y$ is an immersed submanifold of dimension $\dim(\Pi) = n = \dim(Y)/2$. Since α vanishes on Λ, so does $\mathrm{d}\alpha$. Thus the symplectic form $\omega_Y = -\operatorname{curv}(\alpha)$ vanishes on Π as desired. □

Example 2.2. The following example arises from truncating the Harvey-Lawson Lagrangian by cutting off the complement of the unit ball. For

$$\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n$$

let

$$\Lambda_\epsilon := \left\{ \ | z_1|^2 + \epsilon_1 = \dots = |z_n|^2 + \epsilon_n \mid z_1 \dots z_n \in (0, \infty) \right\} \cap S^{2n-1}.$$

The subset Λ_ϵ is a Legendrian submanifold for the stable Hamiltonian structure defined by a connection one-form α_ϵ on $S^{2n-1} \to \mathbb{C}P^{n-1}$ that is trivial on Λ_ϵ. For example, one could take

$$\alpha_\epsilon := \frac{1}{1 + \sum_{i=1}^n \epsilon_i} \sum_{j=1}^n (r_j^2 + \epsilon_j) \mathrm{d}\theta_j \in \Omega^1(S^{2n-1})$$

in an open neighborhood of Λ_ϵ. Indeed, the contraction of the vector fields spanning the tangent space of $T\Lambda_\epsilon$

$$\frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_j} \in \operatorname{Vect}(S^{2n-1}), \quad i, j \in \{1, \dots, n\}$$

with the one-form α_ϵ is

$$\iota \left(\frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_j} \right) \alpha_\epsilon = \frac{(r_i^2 + \epsilon_i) - (r_j^2 + \epsilon_j)}{1 + \sum_{i=1}^n \epsilon_i} = 0$$

for all i, j. If α_ϵ is sufficiently close to α in C^1 then $\mathrm{d}\alpha_\epsilon$ is non-degenerate. So α_ϵ defines a contact structure.

Remark 2.3. We also wish to consider immersed as well as embedded Lagrangians in the base. In the case that Π is not embedded, a standard argument using the Sard-Smale theorem implies the existence of a Legendrian isotopy of Λ for which Π is transversally self-intersecting. However, if Π is cleanly self-intersecting, then we may prefer not to perturb.

More generally, we will consider Legendrians obtained as horizontal lifts of Lagrangians. Denote by

$$\mathcal{A}(\Pi) = \langle \{ A(u), u : S \to Y, \ u(\partial S) \subset \Pi \} \rangle \subset \mathbb{R}$$

the subgroup generated by the areas

$$A(u) = \int_S u^* \omega$$

of surfaces $u : S \to Y$ bounding Π.

Proposition 2.4. *Suppose that $p : Z \to Y$ is a fibered contact manifold, Y is simply connected and and $\Pi \subset Y$ is a connected Lagrangian submanifold. There exists an embedded Legendrian submanifold Λ projecting to Π if and only if the subgroup $\mathcal{A}(\Pi)$ is discrete. In this case, suppose that the areas in $\mathcal{A}(\Pi)$ are integer multiples of $1/k$ for some non-zero integer $k \in \mathbb{Z}$. Then the order of any fiber divides k.*

Proof. The proof is an application of the usual relationship between integral of the curvature and holonomy. Let $y \in \Pi$ be a base point, $Z_y \subset Z$ the fiber over y and $z \in Z_y$. The restriction of Z to Π is flat and so

$$\ker(\alpha) \subset TZ$$

is an integrable distribution on $TZ|_\Pi$. Let

$$\Lambda \subset Z, \quad T\Lambda = \ker(\alpha)|_\Lambda$$

be the leaf of the associated foliation through z. The holonomy of Λ around any loop γ bounding a disk $u : D \to Y$ may be computed as follows. Let τ denote a trivialization of the pull-back bundle.

$$\tau : D \times S^1 \to u^* Z.$$

The trivializing section is section $\tau(\cdot, 1) : D \to u^* Z$. Then

$$\begin{aligned} \mathrm{Hol}_\Lambda(\gamma) &= \exp\left(2\pi i \int_{\partial D} \tau(\cdot,1)^* \tilde{u}^* \alpha\right) \\ &= \exp\left(2\pi i \int_D \mathrm{d}\tau(\cdot,1)^* \tilde{u}^* \alpha\right) \\ &= \exp\left(2\pi i \int_D \tau(\cdot,1)^* \tilde{u}^* \mathrm{d}\alpha\right) \\ &= \exp\left(2\pi i \int_D u^* \omega\right) \end{aligned}$$

where $\tilde{u} : u^* Z \to Z$ is the natural lift. Suppose the area of any disk $u : D \to Y$ is divisible by k. Let $\gamma : S^1 \to Y$ be a loop. Since Y is simply connected, γ bounds a disk smooth disk $u : D \to Y$, and the holonomy of ∂u is the exponential of $2\pi i$ times the area of u. Since the area is divisible by k, the holonomy is an k-th root of unity by assumption. It follows that the fiber Λ_y contains $\mathbb{Z}_k z$, so k divides $|\Lambda_y|$. □

Example 2.5. The Clifford Legendrian of (1) is a Legendrian in the standard contact sphere S^{2n-1} that is an n-to-1 cover of the Clifford Lagrangian in projective space $\mathbb{C}P^{n-1}$. Consider the fibration $S^{2n-1} \to \mathbb{C}P^{n-1}$. The symplectization of S^{2n-1} may be identified with punctured affine space

$$\mathbb{R}^n \times S^{2n-1} \cong \mathbb{C}^n - \{0\}.$$

The Maslov index two disks bounding the Clifford torus Π_{Cliff} are those induced by the maps to $\mathbb{C}^n - \{0\}$

$$u(z) = \frac{1}{\sqrt{n}}(1, \ldots, 1, z, 1, \ldots, 1)$$

and have areas $1/n$, as in Cho-Oh [13]. The Clifford Legendrian Λ_{Cliff} is an n-to-1 cover of Π_{Cliff}. For example, in the case $n = 2$ the disks correspond to the two hemispheres of $\mathbb{C}P^1 = S^2$ and the Legendrian lift is

$$\Lambda_{\mathrm{Cliff}} = \left\{ |z_1| = |z_2| = \frac{1}{\sqrt{2}}, z_1 z_2 \in (0, \infty) \right\}$$

is a double cover of the equator Π_{Cliff} with $\mathbb{Z}_2$ action on the fiber given by scalar multiplication by -1.

The following are standard constructions on fibered contact/stable Hamiltonian manifolds, that is, circle bundles with connection.

Lemma 2.6. *(a)* (Unions) *If $Z_1 \to Y_1$ and $Z_2 \to Y_2$ are circle-fibered contact/stable Hamiltonian manifolds then so is the disjoint union, written $Z_1 \sqcup Z_2 \to Y_1 \sqcup Y_2$. If $\Lambda_1 \subset Z_1$ and $\Lambda_2 \subset Z_2$ then are Legendrians then the disjoint union $\Lambda_1 \sqcup \Lambda_2$ is a Legendrian in $Z_1 \times Z_2$.*

(b) (Exterior tensor products) *If $Z_1 \to Y_1$ and $Z_2 \to Y_2$ are circle-fibered contact/stable Hamiltonian manifolds then so is the exterior tensor product*

$$(Z_1 \boxtimes Z_2) = (Z_1 \times Z_2)/S^1 \to Y_1 \times Y_2$$

where the S^1-action is the anti-diagonal action. If $\Lambda_1 \subset Z_1$ and $\Lambda_2 \subset Z_2$ are Legendrians then the image

$$\Lambda_1 \boxtimes \Lambda_2 = (\Lambda_1 \times \Lambda_2)/S^1 \subset Z_1 \boxtimes Z_2$$

of $\Lambda_1 \times \Lambda_2$ in $Z_1 \boxtimes Z_2$ is Legendrian.

(c) (Finite covers) *Let $\mathbb{Z}_m \subset S^1$ denote the finite subgroup of order m. If $Z \to Y$ is a circle-fibered contact/stable Hamiltonian manifold then so is the quotient $Z/\mathbb{Z}_m \to Y$. Note that the transition maps of $Z/\mathbb{Z}_m$ are the m-th powers of those of Z and so the Chern classes are related by*

$$c_1(Z/\mathbb{Z}_m) = mc_1(Z).$$

If $\Lambda \subset Z$ is Legendrian then the image $\underline{\Lambda}$ of Λ in $Z/\mathbb{Z}_m$ is a Legendrian.

(d) (Interior tensor products) *If $Z_1 \to Y$ and $Z_2 \to Y$ are circle-fibered contact/stable Hamiltonian manifolds*

$$Z_1 \otimes Z_2 := \Delta^*(Z_1 \boxtimes Z_2 \to Y \times Y)$$

where $\Delta : Y \to Y \times Y$ is the diagonal, is a circle-fibered contact manifold over Y. If $\Lambda_1 \subset Z_1$ and $\Lambda_2 \subset Z_2$ are Legendrians then the image of $\Lambda_1 \times_Y \Lambda_2$ in $Z_1 \otimes Z_2$ is Legendrian.

(e) (Tensor powers) *As a special case of the previous item, if $Z \to Y$ is a circle-fibered contact/stable Hamiltonian manifold then so is the k-fold tensor product*

$$Z^{\otimes k} := (Z \times_Y \ldots \times_Y Z)/(S^1)^{k-1}.$$

(f) (Symplectic quotients) *Suppose $Z \to Y$ is a circle-fibered contact/stable Hamiltonian manifold with an action of a Lie group H preserving the contact*

form α and commuting with the action of S^1. Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{h}^\vee$ its dual. The moment map *for the action of H is the map*

$$\Phi : Z \to \mathfrak{h}^\vee, \quad z \mapsto (\xi \mapsto \alpha(\xi_Z(z))).$$

The map Φ is the pull-back of a moment map $\overline{\Phi} : Y \to \mathfrak{h}^\vee$ for the action of H on Y. The symplectic quotient of Z is

$$Z /\!\!/ H = \Phi^{-1}(0)/H.$$

If H acts freely on $\overline{\Phi}^{-1}(0)$ then $Z /\!\!/ H$ is a circle-fibered contact manifold over

$$Y /\!\!/ H := \overline{\Phi}^{-1}(0)/H.$$

Let $\Lambda \subset Z$ be a H-invariant Legendrian. Then Λ is contained in $\Phi^{-1}(0)$ and Λ/H is a Legendrian in $Z /\!\!/ H$.

Proof. Most of these claims are immediate and left to the reader. In the case of products, suppose that the connection forms are $\alpha_1 \in \Omega^1(Z_1)$ and $\alpha_2 \in \Omega^1(Z_2)$. The one-form

$$\pi_1 \alpha_1^* + \pi_2 \alpha_2^* \in \Omega^1(Z_1 \times Z_2)$$

is basic (that is, invariant and vanishes on the vertical subspace) for the anti-diagonal S^1 action. Thus it descends to a one-form on $Z_1 \boxtimes Z_2$. An easy check shows that this one-form is contact. Since $\pi_1 \alpha_1^* + \pi_2 \alpha_2^*$ vanishes on $\Lambda_1 \times \Lambda_2$, its image $\Lambda_1 \boxtimes \Lambda_2$ is Legendrian. $\square$

Example 2.7. We discuss the order of the covering in our running example: Let Π_{Cliff} be the Clifford Lagrangian in $\mathbb{C}P^{n-1}$.

(a) Let $Z = S^{2n-1} \to \mathbb{C}P^{n-1}$ be the unit tautological bundle, that is, the space of unit vectors in the total space of the line bundle

$$\mathcal{O}(-1) := \{(z, \ell) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} | z \in \ell\}$$

with respect to the standard metric. The horizontal lift of Π_{Cliff} is an n-fold cover $\Lambda_{\mathrm{Cliff}} \to \Pi_{\mathrm{Cliff}}$, described in (1).

(b) Let $Z \to \mathbb{C}P^{n-1}$ be the unit canonical bundle, that is, the set of unit vectors in the total space of the line bundle

$$\mathcal{O}(-n) = \Lambda_{\mathbb{C}}^{n-1} T\mathbb{C}P^{n-1}$$

with respect to the metric induced by the Fubini-Study metric on $\mathbb{C}P^{n-1}$. The horizontal lift Λ is diffeomorphic to Π under the projection; the horizontal lift $\Lambda \to \Pi$ is defined by setting

$$s(y) \in \Lambda_{\mathbb{R}}^{n-1} T_y \Pi \cong \Lambda_{\mathbb{C}}^{n-1} T_y Y$$

the unique unit volume form defining the orientation on Π.

For a given contact/stable Hamiltonian form α on Z, the *Reeb vector field R_α* is the unique vector field on Z defined by

$$\omega(R_\alpha) = 0, \quad \alpha(R_\alpha) = 1.$$

A *Reeb orbit* is a closed orbit of the Reeb vector field

$$\gamma : [0,T] \to Z, \quad \mathrm{d}\alpha\left(\frac{d}{dt}\gamma\right) = 0, \quad \alpha\left(\frac{d}{dt}\gamma\right) = 1, \quad \gamma(0) = \gamma(T).$$

A *Reeb chord* with boundary on a Legendrian Λ is a path starting and ending at the Legendrian:

$$\gamma : [0,T] \to Z, \quad \mathrm{d}\alpha\left(\frac{d}{dt}\gamma\right) = 0, \quad \alpha\left(\frac{d}{dt}\gamma\right) = 1, \quad \gamma(\{0,T\}) \subset \Lambda.$$

The number T is also called the *angle* of the Reeb orbit/chord.

In our circle-fibered case, all Reeb orbits are multiple covers of circle fibers. Denote by $\mathcal{R}(\Lambda)$ the set of Reeb chords:

$$\mathcal{R}(\Lambda) = \{\, \gamma : [0,1] \to Z \mid \gamma(\{0,1\}) \subset \Lambda \,\}\,.$$

Lemma 2.8. *There is a diffeomorphism* $\mathcal{R}(\Lambda) \to (\Lambda \times_\Pi \Lambda) \times \mathbb{Z}_{\geq 0}$.

Proof. Given a Reeb chord γ, the starting and ending point $\gamma(0), \gamma(1)$ of the chords together with the number of times $\gamma(t)$ crosses the endpoint $\gamma(0)$ as γ winds around the fiber $Z_{p(\gamma)}$ uniquely specifies the chord. □

Corollary 2.9. *If* Π *is embedded then each component of* $\mathcal{R}(\Lambda)$ *is diffeomorphic to* Λ*. If* Π *is immersed with transverse self-intersection then the components are either points or diffeomorphic to* Λ*. In either case, the fibers of the starting points resp. ending point map* $\mathcal{R}(\Lambda) \to \Lambda$ *are identified with discrete subsets of* $\mathbb{R}_{>0}$ *via the length map.*

2.2. **Lagrangian cobordisms.** In this section, we define symplectic and Lagrangian cobordisms and introduce assumptions that guarantee that the counts of pseudo-holomorphic disks give rise to chain maps between Legendrian dga's.

Definition 2.10. Fix two circle-fibered stable Hamiltonian manifolds $(Z_\pm, Y_\pm, \alpha_\pm, \omega_\pm)$. A *symplectic cobordism* from $(Z_+, Y_+, \alpha_+, \omega_+)$ to $(Z_-, Y_-, \alpha_-, \omega_-)$ is a symplectic manifold with boundary $(\tilde{X}, \partial\tilde{X}, \omega)$ whose boundary is equipped with a partition

$$\partial\tilde{X} = \partial_+\tilde{X} \cup \partial_-\tilde{X}$$

and diffeomorphisms

$$\iota_\pm : Z_\pm \to \partial_\pm\tilde{X}$$

so that

(a) the two forms satisfy the proportionality relation

$$\iota_\pm^*\omega \mid_{\partial_\pm\tilde{X}} = \lambda_\pm\omega_\pm$$

for some positive constants $\lambda_\pm \in \mathbb{R}$, and

(b) the natural isomorphism between the normal bundle of $Z_\pm$ in $\tilde{X}$ and the kernel of $Dp : TZ_\pm \to TY_\pm$ is orientation preserving resp. reversing for the positive resp. negative boundary.

Denote by X the interior of $\widetilde{X}$, viewed as a manifold with cylindrical ends, and, abusing terminology, call X the cobordism.

Remark 2.11. In the definition of cobordism the one-forms $\alpha_\pm \in \Omega^1(Z_\pm)$ are not required to extend over X, and will not, in our basic example of the Harvey-Lawson filling. For the most part, the reader could restrict to the case that $Z_\pm$ are contact manifolds, keeping in mind that the truncation of asymptotically-cylindrical manifolds such as the Harvey-Lawson filling are only Legendrian with respect to stable Hamiltonian structures on the boundary of the truncation.

Remark 2.12. Because of the assumption on the orientations, the notion of symplectic cobordism is not symmetric. That is, a cobordism from Z_- to Z_+ is not also a cobordism from Z_+ to Z_-.

Example 2.13. Truncations of symplectizations are particular examples of symplectic cobordisms. The symplectization of a contact manifold with one-form (Z, α) is the symplectic manifold

$$\mathbb{R} \times Z, \quad \omega := \mathrm{d}(e^s \alpha)$$

where s is the coordinate on $\mathbb{R}$. For real numbers $\sigma_- < \sigma_+$ The truncation

$$X = (\sigma_-, \sigma_+) \times Z$$

is a symplectic cobordism from (Z, α) to itself, with proportionality constants for the two-forms given by $e^{\sigma_\pm}$.

Definition 2.14. The *compactification* of a symplectic cobordism X is the symplectic manifold $\overline{X}$ obtained by collapsing the null foliation on the boundary:

$$\overline{X} = \widetilde{X}/\sim \quad = (\widetilde{X} - \partial\widetilde{X}) \sqcup (\partial\widetilde{X}/S^1) \tag{3}$$

where $\sim$ is the trivial equivalence relation on $\widetilde{X} - \partial\widetilde{X}$ and the equivalence relation

$$z \sim e^{i\theta} z, \quad z \in Z_\pm, \theta \in \mathbb{R}$$

generated by the circle action on the boundary.

Lemma 2.15. *The compactification $\overline{X}$ is a smooth symplectic manifold.*

Proof. The statement of the Lemma is a special case of the symplectic cut construction in Lerman [36]: Let $U_\pm \subset X$ be an open neighborhood of $Z_\pm$ in X, which may be identified with a subset of $\mathbb{R} \times Z_\pm$ by the coisotropic embedding theorem. Consider the product $\mathbb{R} \times Z_\pm \times \mathbb{C}$ with the diagonal S^1-action with moment map

$$\Phi : \mathbb{R} \times Z_\pm \times \mathbb{C} \to \mathbb{R}, \quad (s, z, w) \mapsto s \pm |w|^2/2.$$

The quotient of the zero level set of the moment map is naturally the quotient described in (3):

$$(U_\pm \times \mathbb{C}) /\!\!/ S^1 \cong (Z_\pm \times \{0\})/S^1 \cup (U_\pm - Z_\pm) \cong \overline{U_\pm}.$$

The compactification $\overline{X}$ is obtained from $\mathbb{R} \times Z_\pm$ near the boundary by taking the symplectic quotient by a free action. The quotient is smooth by the Meyer-Marsden-Weinstein theorem. □

Remark 2.16. The unit normal bundle $N_{\pm,1}$ of the codimension-two submanifold $Y_\pm = Z_\pm/S^1$ in $\overline{X}$ is naturally identified with $Z_\pm$ up to a reversal of the action for the incoming boundary. It follows that $N_{\pm,1}$ as a circle bundle with connection is positively resp. negatively curved for the incoming resp. outgoing boundary.

Conversely, compact symplectic manifolds equipped with disjoint symplectic hypersurfaces induce cobordisms, under the assumption of suitable positivity/negativity of the normal bundles:

Lemma 2.17. *Let* $(\overline{X}, \overline{\omega})$ *be a closed symplectic manifold with two disjoint symplectic hypersurfaces* $Y_\pm$*. Suppose that the unit normal bundles* $Z_\pm \to Y_\pm$ *are equipped with connection one-forms* $\alpha_\pm$ *so that*

$$\mathrm{d}\alpha_\pm = \pm \lambda_\pm p_\pm^* \omega_{Y,\pm} \in \Omega^2(Z_\pm)$$

for some positive constants $\lambda_\pm > 0$*. Then the real blow up of* $\overline{X}$ *at* $Y_\pm$ *has the structure of a symplectic cobordism from* $(Z_-, \alpha_-, p_-^* \omega_{Y,-})$ *to* $(Z_+, \alpha_+, p_+^* \omega_{Y_+})$*.*

Proof. The real blow-up $\tilde{X}$ is obtained by replacing the $Y_\pm$ by their unit normal bundles and admits a blow-down map to $\overline{X}$. The two-form $\tilde{\omega}$ on $\widetilde{X}$ is obtained by pull-back of the form $\overline{\omega}$ on $\overline{X}$. □

We view the interiors of the cobordisms as manifolds with cylindrical ends. Denote the interior of the cobordism $X = \widetilde{X} - \partial \widetilde{X}$. Tubular neighborhoods give rise to proper embeddings

$$\kappa_\pm : (\sigma_{\pm,0}, \sigma_{\pm,1}) \times Z_\pm \to X.$$

Equip $(\sigma_{\pm,0}, \sigma_{\pm,1}) \times Z_\pm$ with the two form

$$\omega_\pm := \pi_{Y_\pm}^* \omega_{Y_\pm} + \mathrm{d}(e^s \alpha_\pm).$$

Lemma 2.18. *For intervals* $(\sigma_{\pm,0}, \sigma_{\pm,1})$ *that are sufficiently small, the forms* $\omega_\pm$ *are symplectic and the maps* $\kappa_\pm$ *may be taken to be symplectomorphisms.*

Proof. The local model for constant rank embeddings in Marle [38] implies the existence of the claimed symplectomorphism, since the symplectic forms by definition agree at $Z_\pm$. □

Definition 2.19. For a symplectic cobordism X from Z_+ to Z_- and two Legendrian submanifolds $\Lambda_\pm \subset Z_\pm$ so that X is equipped with cylindrical ends, a *Lagrangian cobordism cylindrical near infinity* from Λ_+ to Λ_- is a Lagrangian submanifold $L \subset X$ such that

$$L \cap ((\sigma_{\pm,0}, \sigma_{\pm,1}) \times Z_\pm) = (\sigma_{\pm,0}, \sigma_{\pm,1}) \times \Lambda_\pm$$

for some intervals $(\sigma_{\pm,0}, \sigma_{\pm,1}) \subset \mathbb{R}$.

Given a Lagrangian cobordism that is cylindrical near infinity, we form a Lagrangian in the compactified symplectic cobordism by closure. Let $L \subset X$ be a Lagrangian cobordism.

Lemma 2.20. *The closure* $\overline{L} \subset \overline{X}$ *is contained in a cleanly self-intersecting Lagrangian. The intersection*

$$\Pi_\pm := \overline{L} \cap Y_\pm \subset \overline{X}$$

is equal to the projection of $\Lambda_\pm$ *and is a smooth immersed Lagrangian in* $Y_\pm$*.*

Proof. The cleanly-self-intersecting manifold is given as follows. A tubular neighborhood furnishes a local diffeomorphism from an open neighborhood of the zero section of the normal bundle $Z_\pm \times_{S^1} \mathbb{C}$ to $\overline{X}$. In the local model, the Lagrangian is diffeomorphic to the quotient $Z_\pm \times_{S^1} \mathbb{R}_{\geq 0}\Lambda_\pm$. The closure $\overline{L}$ is therefore a submanifold of the subset $Z_\pm \times_{S^1} \mathbb{R}\Lambda_\pm$, which is a cleanly-self-intersecting but non-compact Lagrangian submanifold in the local model. □

Lemma 2.21. *If $\Lambda_\pm$ are isotopic Legendrians in (Z, α) then there exists a Lagrangian cobordism from Λ_+ to Λ_-.*

Proof. (c.f. Chantraine [10] and Golovko [31, Example 4.1].) Consider a Legendrian isotopy from Λ_- to Λ_+. By definition, such an isotopy consists of a family of Legendrians

$$\Lambda_\rho \subset Z, \rho \in \mathbb{R}, \quad \Lambda_\rho = \Lambda_-, \rho \ll 0, \quad \Lambda_\rho = \Lambda_+, \rho \gg 0.$$

Choose a family of diffeomorphisms

$$\psi_s : Z \to Z, \quad s \in \mathbb{R}$$

so that

$$\psi_s(\Lambda_-) = \Lambda_s, \forall s, \quad \partial_s \psi_s = 0, |s| \gg 0.$$

Consider the family of contact forms

$$\alpha_\rho = \psi_\rho^* \alpha \in \Omega^1(Z).$$

Define a one-form $\tilde{\alpha} = (\alpha_s)_{s \in \mathbb{R}} \in \Omega^1(\mathbb{R} \times Z)$ and a two-form

$$\omega| = \mathrm{d}_{\mathbb{R}\times Z}(e^s \tilde{\alpha}) = (\mathrm{d}s) \wedge (e^s \alpha_s + e^s \partial_s \alpha_s) + e^s \mathrm{d}_Z \alpha_s \in \Omega^2(\mathbb{R} \times Z).$$

The top exterior power of the two-form is

$$\omega^n = n(\mathrm{d}s) \wedge (e^s \alpha_s + e^s \partial_s \alpha_s) \wedge (\mathrm{d}_Z \alpha_s)^{n-1}. \tag{4}$$

By taking α_s to be slowly varying, we may assume that the following inequality holds point-wise with respect to some Riemannian metric on the bundle of forms:

$$\|\mathrm{d}s \wedge \alpha_s \wedge \mathrm{d}_Z(\alpha_s)^{n-1}\| > \|ds \wedge \partial_s \alpha_s \wedge \mathrm{d}_Z(\alpha_s)^{n-1}\|. \tag{5}$$

Since $ds \wedge \alpha_s \wedge \mathrm{d}_Z(\alpha_s)^{n-1}$ is non-vanishing, equations (4) and (5) imply that ω^n is non-vanishing. Thus, ω is a symplectic form on $\mathbb{R} \times Z$. Denote the inclusion

$$\iota : \mathbb{R} \times \Lambda \to \mathbb{R} \times Z.$$

The pull-back two-form is

$$\iota^* \omega = \mathrm{d}(e^s \iota^* \tilde{\alpha}) = 0.$$

Thus $(\mathbb{R} \times \mathbb{Z}, \omega)$ is a symplectic manifold containing $\mathbb{R} \times \Lambda_-$ as a Lagrangian submanifold. The diffeomorphisms ψ_ρ identify the ends with the symplectizations $\pm(|\rho|, \infty) \times Z$ for $|\rho|$ sufficiently large. By symplectic cutting on $\mathbb{R} \times Z$, we obtain a compact symplectic manifold $\overline{X}$ containing $Y_\pm$ as symplectic submanifolds, up to re-scaling of the symplectic forms $\omega_\pm$. □

Our motivating examples of Lagrangian cobordisms are asymptotically-cylindrical rather than cylindrical near infinity. To explain the asymptotically-cylindrical definition, recall that the space of submanifolds has a natural C^0-topology. In this topology, open neighborhoods of a given submanifold are those submanifolds corresponding to sections of the normal bundle contained in some tubular neighborhood. Choose tubular neighborhoods of $\Lambda_\pm$ in Z given by diffeomorphisms of the normal bundle $N\Lambda_\pm$ onto an open neighborhood of $\Lambda_\pm$. The tubular neighborhoods of $\Lambda_\pm$ induce tubular neighborhoods of $\mathbb{R} \times \Lambda_\pm$ given by maps $\mathbb{R} \times N\Lambda_\pm \to \mathbb{R} \times Z_\pm$. Let

$$\tau_\sigma : \mathbb{R} \times \mathbb{Z} \to \mathbb{R} \times Z, \quad (s, z) \mapsto (s + \sigma, z)$$

be the translation map.

Definition 2.22. Let $L_\nu \subset \mathbb{R} \times Z_\pm, \nu \in \mathbb{N}$ be a sequence of submanifolds given as the graphs of sections

$$s_{L_\nu} : \mathbb{R} \times \Lambda_\pm \to \mathbb{R} \times N\Lambda_\pm.$$

The sequence L_ν C^ℓ-*converges* to $(\sigma_-, \sigma_+) \times \Lambda_\pm$ if the sequence of sections s_{L_ν} converges to 0 in the C^ℓ-norm uniformly on $(\sigma_-, \sigma_+) \times \Lambda_\pm$.

A non-compact Lagrangian $L \subset X$ is *asymptotically cylindrical* if the submanifolds

$$\tau_{-\sigma}(L \cap ((\sigma, \sigma + 1) \times Z)) \subset (0, 1) \times Z$$

converge in C^1 on $(0, 1) \times Z$ in the sense above.

Example 2.23. For the Clifford Legendrian Λ_{Cliff} in (1), a natural asymptotic filling was introduced by Harvey-Lawson [33], see also Joyce [35]. Let

$$a_1, \ldots, a_n > 0$$

be positive constants exactly two of which are zero (without loss of generality, the last two $a_{n-1} = a_n = 0$.) The *Harvey-Lawson asymptotic filling* of the Clifford Legendrian is given by (2). The filling $L_{(1)}$ is asymptotic to $\mathbb{R}_{>0} \times \Lambda_{\text{Cliff}}$ at infinity in $\mathbb{C}^n$. Each asymptotic filling defines a filling of the Legendrian

$$\Lambda_\epsilon = \left\{ \begin{array}{l} |z_1|^2 - a_1^2 = |z_2|^2 - a_2^2 = \ldots = |z_n|^2 - a_n^2 \\ z_1 z_2 \ldots z_n \in (0, \infty) \end{array} \right\} \cap e^\sigma S^{2n-1}.$$

The projection is the Lagrangian torus orbit torus

$$\{|z_1|^2 - a_1^2 = |z_2|^2 - a_2^2 = \ldots = |z_n|^2 - a_n^2\} \cap e^\sigma S^{2n-1}/S^1. \tag{6}$$

This Lagrangian is a toric moment fiber over a point close, but not equal to, the barycenter $e^\sigma(1, 1, \ldots, 1)/n$ of the moment polytope

$$\Delta_n = \{\lambda_1 + \ldots + \lambda_n = 1\} \cap \mathbb{R}^n_{\geq 0}.$$

In particular, if L is an asymptotically cylindrical Lagrangian with limits $\Lambda_\pm$ that have monotone projections to $Y_\pm$, the truncations Λ_{σ_+} may not have monotone projections.

For technical reasons, in the construction of moduli spaces of pseudoholomorphic curves it is convenient to have cylindrical-near-infinity Lagrangians rather than asymptotically cylindrical Lagrangians. Presumably, the gluing and compactness results in symplectic field theory hold equally well for asymptotically cylindrical Lagrangians, but these results have yet to appear in the literature. The following lemma allows us to replace asymptotically cylindrical Lagrangians with those cylindrical near infinity:

Lemma 2.24. (Straightening Lemma) *Suppose that X is a smooth symplectic manifold with ends modelled on $\mathbb{R} \times Z_\pm$ with symplectic form $\mathrm{d}(e^s\alpha)$. Let $L \subset X$ be an asymptotically cylindrical Lagrangian with limits Legendrians $\Lambda_\pm \subset Z_\pm$. Suppose that $\Lambda_\pm$ fiber over Lagrangians $\Pi_\pm \subset Y$ with finite covering group*

$$\Upsilon \cong \mathbb{Z}_k \subset S^1.$$

For $\sigma > 0$ let

$$X_{[\sigma_-,\sigma_+]} := X - \kappa_+^{-1}((\sigma_+,\infty) \times Z_+) - \kappa_-^{-1}(-(\infty,\sigma_-) \times Z_-)$$

denote the result of truncating the cylindrical ends at $s = \sigma_\pm$. For $\sigma \in \mathbb{R}$ let

$$\Lambda_\sigma := (\{\sigma\} \times Z_\pm) \cap L$$

be the intersection with the σ-level set. For

$$\sigma_\pm \in \mathrm{im}(0,\infty)$$

there exist connection one-forms $\alpha_\pm$ on $Z_\pm$ for which $\Lambda_\pm$ are Legendrian and so that

$$L_{[\sigma_-,\sigma_+]} := X_{[\sigma_-,\sigma_+]} \cap L$$

can be isotoped to a cylindrical-near-infinity Lagrangian cobordism from (Z_-,Λ_{σ_-}) to (Z_+,Λ_{σ_+}) for some stable Hamiltonian manifolds $(Z_\pm,\alpha_\pm,\omega_\pm)$.

Proof. Moser isotopy can be used to create a cylindrical end for which the Lagrangian is cylindrical near infinity. It suffices to consider the case that X has only a positive end Z_+, as the case of a negative end is similar. By assumption, the submanifold L is C^1 close to $\mathbb{R} \times \Lambda_+$ on the end. It follows that Λ_{σ_+} is smooth and is a codimension one submanifold of L. The projection

$$\Pi^{\sigma_+} = p(\Lambda_{\sigma_+}) \subset Z_{\sigma_+}$$

is an (in general immersed) isotropic submanifold, since $\omega = \mathrm{d}(e^s\alpha_\pm)$ vanishes on Λ_{σ_+}. Because L is Υ-invariant, $\Pi^{\sigma_+} \subset Z$ is embedded and the map $\Lambda^{\sigma_+} \to \Pi^{\sigma_+}$ is an Υ-cover.

The first step is to choose a connection one-form for which the slice is horizontal. At any point $z \in \Lambda_{\sigma_+}$ there is a unique one-form

$$\alpha_{\sigma_+} \in \mathrm{Hom}(T_zZ_+,\mathbb{R})$$

for which $T_z\Lambda_{\sigma_+}$ is horizontal and for which $\alpha_{\sigma_+}(\frac{\partial}{\partial\theta}) = 1$. By S^1-invariance, α_{σ_+} extends to a form on the orbit of z. Since Λ_{σ_+} is by assumption Υ-invariant, $T_{hz}\Lambda_{\sigma_+}$ is horizontal for any $h \in \Upsilon$. This construction defines a one-form

$$\alpha_{\sigma_+} \in \Omega^1(Z|_{\Pi_{\sigma_+}})$$

of Z to Π_{σ_+}. Using a S^1-equivariant tubular neighborhood one may extend α_{σ_+} first to an open neighborhood of $Z|_{\Pi_{\sigma_+}}$, then to all of Z using a patching argument and convexity of the space of connection one-forms.

We remark that the curvature of the connection constructed in the previous paragraph may be taken to be positive. Indeed, for σ_+ is sufficiently large, α_{σ_+} is C^1-close to α, and so the form $\mathrm{d}\alpha_{\sigma_+}$ is C^0 close to $\mathrm{d}\alpha = \pi^*\omega$. It follows that $\mathrm{d}\alpha_{\sigma_+}$ is positive on the horizontal subspace in TZ_+. Since $\mathrm{curv}(\alpha_{\sigma_+}) \in \Omega^2(Y_+)$ pulls back to $\mathrm{d}\alpha_{\sigma_+} \in \Omega^2(Z_+)$, the curvature $\mathrm{curv}(\alpha_{\sigma_+})$ is positive. The submanifold Λ_{σ_+} is then Legendrian for the contact structure defined by $\alpha_{\sigma_+} \in \Omega^1(Z_+)$.

We construct a Lagrangian cobordism that is cylindrical-near-infinity by a patching argument. The submanifold $L_{\sigma_+} = \mathbb{R} \times \Lambda_{\sigma_+}$ is isotropic with respect to the closed two-form

$$\omega_{\sigma_+} := \pi_{\sigma_+}^*\omega + \mathrm{d}((s-\sigma_+)\alpha_{\sigma_+}) \in \Omega^2(\mathbb{R} \times Z_+)$$

where $\pi_\sigma : \mathbb{R} \times Z_\pm \to X$ is the map obtained by composing projection onto $Z_\pm$ with inclusion of $Z_\pm$ into X. Furthermore, ω_{σ_+} is symplectic in a tubular neighborhood of $\{\sigma_+\} \times Z_+$. Moser isotopy implies that the forms ω and ω_{σ_+} are symplectomorphic on the truncation

$$X_{(\sigma_0,\sigma_1)} := (\sigma_0, \sigma_1) \times Z \subset (0, \infty) \times Z$$

for $\sigma_0 < \sigma_+ < \sigma_1$ where σ_i is close enough to σ_+, by a symplectomorphism

$$\psi : (X_{(\sigma_0,\sigma_1)}, \omega) \to (X_{(\sigma_0,\sigma_1)}, \omega_\sigma)$$

equal to the identity on Λ_{σ_+}. The image $\psi(L)$ of L under ψ is a Lagrangian with respect to ω_{σ_+} and agrees with L_{σ_+} at Λ_{σ_+}. It follows that $\psi(L)$ is an exact deformation of L_{σ_+} in an open neighborhood of Λ_{σ_+}, given by the Hamiltonian flow of L_{σ_+} of some function

$$H : X_{(\sigma_0,\sigma_1)} \to \mathbb{R}.$$

A cylindrical-near-infinity Lagrangian

$$L' := \psi'(L) \subset X$$

is given as Hamiltonian flow ψ' of L for ρH where ρ is a function whose derivative is supported near $\{\sigma_+\} \times \Lambda_{\sigma_+}$, equal to 0 in an open neighborhood of $\{\sigma_1\} \times \Lambda_{\sigma_+}$ and 1 in an open neighborhood of $\{\sigma_0\} \times \Lambda_{\sigma_-}$. Gluing together L' with L (the map ψ is the identity outside of a small open neighborhood of $\{\sigma_+\} \times Z_+$) gives the required Lagrangian. □

Remark 2.25. We remark on the possibility of obtaining a cobordism between contact manifolds, rather than stable Hamiltonian manifolds. If $\sigma_+, -\sigma_-$ are sufficiently large then the forms $\alpha_\pm$ are contact forms. However, we may not be able to choose a cobordism from (Z_-, α_-) to (Z_+, α_+) in the sense that the forms $\mathrm{d}\alpha_\pm$ extend as symplectic forms over the cobordism and L is Lagrangian with respect to the extension. Indeed, the action of loops in the Legendrian Λ as $\mathrm{d}\alpha_\pm$ may not equal those of $\omega_{\sigma_\pm}$.

Example 2.26. We specialize to the case of the Harvey-Lawson filling. The straightening Lemma 2.24 implies that after a Hamiltonian isotopy we may assume that $L_{(1)} \cong S^1 \times \mathbb{R}^{n-1} \subset \mathbb{C}^n$ is invariant under dilation in a neighborhood of some sphere $e^\sigma S^{2n-1} \cap L_{(1)}$.

2.3. **Reductions of symplectic fillings.** We introduce various constructions that will be used to produce fillings later. In particular, the following Lemma produces symplectic fillings by symplectic reduction:

Lemma 2.27. *Let X be a symplectic cobordism between Y_- and Y_+ equipped with a Hamiltonian action of a Lie group H with moment map Φ such that H acts freely on $\Phi^{-1}(0)$. Then $X /\!\!/ H := \Phi^{-1}(0)/H$ is a symplectic cobordism from $Y_- /\!\!/ H$ to $Y_+ /\!\!/ H$. If $L \subset X$ is an invariant Lagrangian cobordism from Λ_- to Λ_+ then $L /\!\!/ H$ is a Lagrangian cobordism from $\Lambda_- /\!\!/ H$ to $\Lambda_+ /\!\!/ H$.*

The proof is an immediate consequence of the Meyer-Marsden-Weinstein theorem, applied to Hamiltonian actions on manifolds with boundary for which the proof is the same. We obtain fillings of toric moment fibers in toric varieties by symplectic reduction as follows. Let Y be a a Fano projective toric variety Y equipped with a symplectic form corresponding to the anticanonical class. Let $H = (S^1)^n$ denote the n-torus acting in Hamiltonian fashion with moment map

$$\Psi : Y \to \mathbb{R}^n.$$

The moment polytope $P = \Psi(Y)$ may be written as the set of points in $\mathbb{R}^n$ satisfying a finite list of linear inequalities

$$P = \{\lambda \in \mathfrak{h}^\vee | \langle \lambda, \nu_i \rangle \geq -1, \ i = 1, \dots, k\} \subset \mathbb{R}^n.$$

We may assume by translation that P contains the 0 vector.

A singular symplectic filling is given by the cone construction. By the cone $\mathrm{Cone}(Y)$ on Y we mean the (usually singular) affine toric variety with a completely integrable action

$$\hat{H} \times \mathrm{Cone}(Y) \to \mathrm{Cone}(Y)$$

of the torus $\hat{H} = H \times S^1$ of one dimension higher. The moment map for this action

$$\hat{\Psi} : \mathrm{Cone}(Y)^* \to \hat{\mathfrak{h}}^\vee$$

has moment polytope given by the cone on the polytope P above:

$$\mathrm{Cone}(P) \subset \mathfrak{h}^\vee \times \mathbb{R}.$$

For example, $\mathrm{Cone}(\mathbb{P}^{n-1}) = \mathbb{C}^n$. In particular, the complement

$$\mathrm{Cone}^*(Y) \subset \mathrm{Cone}(Y)$$

of the vertex $0 \in \mathrm{Cone}(Y)$ is a smooth symplectic cobordism from Y to Y. We produce fillings corresponding to the choice of codimension two faces; this generalizes the Harvey-Lawson construction which depends on a choice of two coordinates. Let $Q \subset P$ be a codimension two face, given as the intersection of facets with normal vectors $\nu_1, \nu_2 \in \mathfrak{h}$, and $\mathrm{hull}(Q, 0) \subset P$ the convex hull of Q with the zero vector.

Lemma 2.28. *For any vector λ in the interior of* $\mathrm{hull}(Q, 0)$ *there exists a Lagrangian L invariant under a codimension one sub-group of H whose image $\Psi(L)$ under Ψ is the intersection of P with the line $\{\lambda\} \times \mathbb{R}$, and whose intersection $L \cap \hat{\Psi}^{-1}(\lambda)$ projects to the Lagrangian torus orbit $\Psi^{-1}(\lambda)$ in Y.*

In the case $Y = \mathbb{C}P^{n-1}$, the image $\Psi(L)$ is shown in Figure 2.

Proof of Lemma 2.28. We construct the Lagrangian by symplectic reduction. Choose a subspace $\mathfrak{h}_1$ of codimension one in $\mathfrak{h}$ intersecting the span of ν_1, ν_2 in the span of $\nu_1 + \nu_2$:

$$\mathfrak{h}_1 \cap \operatorname{span}(\nu_1, \nu_2) = \operatorname{span}(\nu_1 + \nu_2).$$

The subspace $\mathfrak{h}_1$ generates a torus subgroup H_1. Let $\lambda_1 \in \mathfrak{h}_1$ be the projection of some point in $\operatorname{Cone}(Q) \subset \mathfrak{h}^\vee$. Choose a splitting $\mathfrak{h} \cong \mathfrak{h}_1 \oplus \mathbb{R}$ using, for example, the standard metric on $\mathfrak{h} \cong \mathbb{R}^n$. Let

$$\Psi_1 : \operatorname{Cone}^*(Y) \to \mathfrak{h}_1^\vee$$

denote the moment map for the action of H_1 obtained by composing the moment map Ψ for H with the projection $\mathfrak{h}^\vee \to \mathfrak{h}_1^\vee$. Let

$$Y_1 = \Psi_1^{-1}(\lambda_1) = \operatorname{Cone}^*(Y) /\!\!/ H_1$$

denote the symplectic reduction of $\operatorname{Cone}(Y)^*$. For generic values λ_1, the symplectic quotient Y_1 is a toric orbifold of dimension two, and has a residual action of H/H_1 with moment polytope P_1 some slice of $\operatorname{Cone}^*(P)$. The reduction of $\operatorname{Cone}^*(Y)$ at the locus $\Psi^{-1}(\operatorname{Cone}(Q))$ is smooth. As in Delzant [18], smoothness of the symplectic quotient follows from the fact that the vectors

$$(\nu_1 + \nu_2, 0), (\nu_1, 1), (\nu_2, 1) \in \mathfrak{h}_{\mathbb{Z}} \oplus \mathbb{Z}.$$

form a lattice basis for their real span. In particular, $(\nu_1 + \nu_2, 0)$ generates a circle subgroup that acts freely on inverse image of the cone on Q. Since reductions by free actions are smooth, the claim follows. The polytope P_1 has a vertex at some point $v \in \operatorname{Cone}^*(P)$ with normal vectors $(1,0)$, $(0,1)$, and the line segment $(\lambda, \lambda), \lambda \in [0,1]$ is entirely contained in P_1. It follows that Y_1 may be viewed as a symplectic cut of $\mathbb{C}P^2$ by circle actions.

There is a natural bijection between invariant Lagrangians and Lagrangians in the symplectic quotient. An invariant Lagrangian is obtained by pulling back any Lagrangian in the four-dimensional quotient. In particular, the antidiagonal

$$L_1 := \{z_1 = -z_2\} = \{|z_1|^2 = |z_2|^2,\ z_1 z_2 \in [0, \infty)\} \subset \mathbb{C}^2$$

defines a smooth Lagrangian in $\mathbb{C}P^2$, and therefore Y_1, whose image under the moment map on Y_1 is the line segment as in Figure 2. The inverse image in $\operatorname{Cone}^*(Y)$ is a Lagrangian in $\operatorname{Cone}^*(Y)$ with the stated property. □

Example 2.29. (Harvey-Lawson via reduction) We explain how to obtain the Harvey-Lawson filling via symplectic reduction to a projective plane in projective three-space. Suppose that $Y = \mathbb{C}P^2$ is the projective plane with moment polytope the convex hull of the three standard basis vectors

$$P = \operatorname{hull}\{(1,0,0), (0,1,0), (0,0,1)\}.$$

We identify $\mathbb{R}^3$ with its dual using the standard inner product. Let

$$Q = \{(0,0,1)\}$$

be the last vertex. The normal vectors to the edge with direction $(1,0,-1)$ resp.$(0,1,-1)$ are

$$\nu_1 = (1,-2,1), \quad \nu_2 = (-2,1,1).$$

The sub-torus used for reduction has Lie algebra

$$\mathfrak{h}_1 = \operatorname{span}(\nu_1 + \nu_2), \quad \nu_1 + \nu_2 = (-1, -1, 2).$$

The symplectic reduction $\mathbb{C}P^3 /\!\!/ H_1$ may be identified with $\mathbb{C}P^2$ via projection on the first two variables. The Harvey-Lawson Lagrangian is

$$\{|z_1|^2 + \epsilon = |z_2|^2 + \epsilon = |z_3|^2, z_1 z_2 z_3 \in [0, \infty)\}$$

and is the lift of the antidiagonal $\overline{\{[z, \overline{z}, 1]\}}$ in the reduction $\mathbb{C}P^2 \cong \mathbb{C}P^3 /\!\!/ H_1$. See Figure 2.

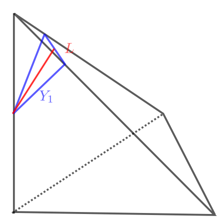

Figure 2. Obtaining the Harvey-Lawson filling via reduction

2.4. **Fillings of lifts of exotic tori.** Vianna [46] has constructed monotone Lagrangian tori in the complex projective plane corresponding to Markov triples. In this section we describe how to construct their Legendrian lifts if the Markov triple is of Fibonacci type.

The idea of Vianna's construction is to use toric degenerations. A result of Hacking and Prokhorov [32] describes toric degenerations of the complex projective plane in terms of Markov triples. Recall that a *Markov triple* is a solution $(a, b, c) \in \mathbb{Z}^3_{>0}$ to the Diophantine equation

$$a^2 + b^2 + c^2 = 3abc.$$

Given a Markov triple (a, b, c), define the corresponding weighted projective plane as the quotient

$$\mathbb{C}P(a^2, b^2, c^2) = \frac{\{a^4|z_1|^2 + b^4|z_2|^2 + c^4|z_3|^2 = 1\}}{(z_1, z_2, z_3) \sim (g^{a^2} z_1, g^{b^2} z_2, g^{c^2} z_3), \ g \in S^1}$$

be the corresponding weighted projective plane considered as a symplectic toric orbifold. Let

$$\Pi = \left\{ a^4|z_1|^2 = b^4|z_2|^2 = c^4|z_3|^2 = \frac{1}{3} \right\} \subset \mathbb{C}P(a^2, b^2, c^2)$$

denote the monotone torus fiber. Vianna [46] shows that one obtains a monotone Lagrangian torus from each of these degenerations. More precisely, by a triple of symplectic rational blow-downs at the orbifold singularities, one obtains the ordinary projective plane. Since the rational blow-downs are disjoint from the Lagrangian torus, one obtains a Lagrangian torus

$$L(a, b, c) \subset \mathbb{C}P^2$$

for each Markov triple. The Lagrangians $L(a,b,c), L(a',b',c')$ are not Hamiltonian isotopic for any two choices of Markov triples $(a,b,c),(a',b',c')$. The Lagrangians corresponding to the triples $(1,1,1)$ and $(1,1,2)$ are the Clifford and Chekanov tori respectively.

We construct fillings of the Legendrian lifts of Vianna's tori assuming that at least one of the vertices is not an orbifold singularity. Recall that $\mathbb{C}P^2(a^2,b^2,c^2)$ is the symplectic quotient of the circle action on $\mathbb{C}^3$ with moment map

$$\phi(z_1,z_2,z_3) = \frac{1}{2}(a^4|z_1|^2 + b^4|z_2|^2 + c^4|z_3|^2)$$

acting with weights a^2, b^2, c^2. In particular, one obtains a free action at the point $(1,0,0)$ if and only if $a = 1$. By a result of Luca-Srinivasan [37], such triples are of the form

$$(a,b,c) = (1, F_{2i-1}, F_{2i+1})$$

for some natural number i, where F_{2i-1}, F_{2i+1} are consecutive odd Fibonacci numbers.

Lemma 2.30. *For any Markov triple of the form $(a,b,c) = (1, F_{2i-1}, F_{2i+1})$ there exists a filling $L \subset \mathbb{C}^3 - \{0\}$ of the corresponding Legendrian lift of a Vianna torus so that the filling L is diffeomorphic to $S^1 \times \mathbb{R}^2$.*

Proof. We check that the Harvey Lawson filling associated to the orbifold projective plane survives the rational blow-downs. Suppose that $L \subset \mathbb{C}^3$ is the filling of the Clifford torus in $\mathbb{C}P(1,b^2,c^2)$ defined as in Lemma 2.28. We suppose that λ_1 is chosen so that the zero level set for the action of H_1 meets only the edge of the moment polytope for $\mathbb{C}P(1,1,b^2,c^2)$ corresponding to the face between $(0,0,0)$ and $(1,0,0)$, which does not correspond to an orbifold singularity. Explicitly

$$L = \{|z_1|^2 = 1+\epsilon, \quad |z_2|^2 = F_{2n}^2, \quad |z_3|^2 = F_{2n+1}^2 \quad z_1z_2z_3 \in [0,\infty]\} \tag{7}$$

which meets the locus $\{z_1z_2z_3 = 0\}$ in the locus $|z_1|^2 = \epsilon$. To obtain fillings in the usual complex projective plane recall the rational blow-down construction. Let $\mathbb{C}P(a^2,b^2,c^2)$ be a weighted projective space for a Markov triple. Define

$$E(a^2,b^2,c^2) := \{a^4z_1^2 + b^4z_2^2 + c^4z_3^2 = 1\} \tag{8}$$

so that

$$\mathbb{C}P(a^2,b^2,c^2) = E(a^2,b^2,c^2)/S^1.$$

We may consider $E(a^2,b^2,c^2)$ as an orbifold S^1-principal bundle with orbifold singularities over the three fixed points. That is,

$$E(a^2,b^2,c^2) \to \mathbb{C}P(a^2,b^2,c^2)$$

is a locally trivial S^1-bundle in an open neighborhood of any point after passing to a ramified cover.

The rational blow-down construction replaces an open neighborhood of an orbifold fixed point with a rational homology ball; Vianna [46] shows that this construction produces the complex projective plane. Let

$$p \in \mathbb{C}P(a^2,b^2,c^2)^{\text{fixed}}$$

be a torus fixed point, and

$$U \subset \mathbb{C}P(a^2, b^2, c^2)$$

an open neighborhood of p. Since $\mathbb{C}P(a^2, b^2, c^2)$ is an orbifold, U is necessarily homeomorphic to $\mathbb{C}^2/\Gamma$ for some finite group Γ. The boundary ∂U of U is a lens space. Choose a rational homology ball V with the same boundary $\partial U \cong \partial V$. By the symplectic rational blow-down construction in Symington [44], there is a symplectomorphism ψ from a punctured open neighborhood W of p in U to $\psi(W) \subset V$.

We wish to lift this symplectomorphism in an open neighborhood of the boundary to an isomorphism of contact five-manifolds. The rational ball is obtained as a finite quotient of some collection of blow-ups of an A_n toric singularity, as in, for example Evans [28, Section 9.2]; the bundle $E(a^2, b^2, c^2)$ is isomorphic to a quotient of the trivial bundle (with non-trivial cyclic action) and so extends, as a circle bundle, over the rational ball as a bundle E_V. Thus we have an isomorphism of circle bundlee

$$\hat{\psi} : E_U|W \to E_V|\psi(W).$$

We may identify the connections after a gauge transformation as follows. The difference between the connection one-forms is closed,

$$\mathrm{d}(\hat{\psi}^*\alpha_V - \alpha_W) = 0.$$

The fundamental group of a lens space is torsion, the first homology with real coefficients $H_1(W, \mathbb{R})$ vanishes. Hence, the difference between the connection one-forms is exact:

$$\hat{\psi}^*\alpha_V = \alpha_W + \mathrm{d}\phi$$

for some function $\phi : W \to \mathbb{R}$. The map

$$E_U|W \to E_V|\psi(W), \quad e \mapsto e^{-\phi}\hat{\psi}(e)$$

identifies $E_U|W$ and $E_V|\psi(W)$ as bundles-with-connection. Gluing in E_V produces a contact five manifold as desired.

By the rational blow-down construction above at the two strata with orbifold singularities

$$(\mathbb{R} \times S^1) \cup (\mathbb{R} \times S^1) \cup \mathbb{C}P(1, 1, b^2, c^2) - \mathbb{C}P(1, b^2, c^2) - \{(0, 0, 0)\}$$

mapping to the edges of the moment polytope connecting $(0, 0, 0, 0)$ to $(0, 0, 1/b^2, 0)$ and $(0, 0, 0, 1/c^2)$ respectively, we obtain a filling of the corresponding Legendrian torus in the symplectization of a circle bundle over $\mathbb{C}P^2$. □

3. Pseudoholomorphic buildings

In this section, we construct the moduli spaces of holomorphic buildings used for both the differential in the Chekanov-Eliasberg algebra and the chain maps used associated to Lagrangian cobordisms. Because our stable Hamiltonian manifolds are circle-fibered over symplectic manifolds and we consider only holomorphic disks rather than curves of higher genus, we may use Cieliebak-Mohnke [14] perturbations to regularize.

3.1. **Punctured surfaces.** We recall basic terminology for surfaces with strip-like ends at punctures.

Definition 3.1. A *surface with strip-like and cylindrical ends* S is obtained from a closed oriented surface-with-boundary $\overline{S}$ by removing a finite collection of boundary and interior points:

$$S = \overline{S} - \{z_{e,\circ}, e = 1, \ldots, e(\circ), \quad z_{e,\bullet}, e = 1, \ldots, e(\bullet).\}$$

We call S a *punctured surface* for short, and the removed points *punctures*.

Suppose S is equipped with a conformal structure, giving rise to an almost complex structure

$$j : TS \to TS, \quad j^2 = \mathrm{Id}_{TS}\,.$$

An open neighborhood $U_{e,\circ} \subset S$ of each puncture $z_{e,\circ}$ is assumed to be equipped with a local coordinate, and similarly for open neighborhoods $U_{e,\bullet}$ of $z_{e,\bullet}$. Each such coordinate gives holomorphic embeddings called *strip-like* resp. *cylindrical ends*

$$\kappa_{e,\circ} : U_{e,\circ} - \{0\} \cong \pm\mathbb{R}_{>0} \times [0, 1], \quad \kappa_{e,\bullet} : U_{e,\bullet} - \{0\} \cong \pm\mathbb{R}_{>0} \times S^1.$$

The set of ends is denoted $\mathcal{E}(S)$ and is equipped with a partition into incoming resp. outgoing ends

$$\mathcal{E}(S) = \mathcal{E}_-(S) \cup \mathcal{E}_+(S).$$

To keep with the terminology standard in the field, the punctures corresponding to incoming ends are called *positive* while the outgoing punctures are called *negative*.

In the Gromov compactification of the moduli space of holomorphic maps, nodal surfaces will appear as degenerations.

Definition 3.2. A *nodal surface* S is obtained from a surface with boundary $\tilde{S}$ with strip-like and cylindrical ends by gluing along disjoint pairs of points $w_\pm(e) \in \tilde{S}$, where e ranges over some index set, which may be either a pair of boundary points $w_\pm(e) \in \partial\tilde{S}$ or a pair of interior points $w_\pm(e) \in \mathrm{int}(\tilde{S})$.

A nodal surface S is a *nodal disk* if each component S_v is a disk, the boundary ∂S is connected, and there are no non-self-crossing cycles $S_{v_1}, \ldots, S_{v_k}$ of components joined by nodes.

A *marking* is a pair $\underline{z} = (\underline{z}_\bullet, \underline{z}_\circ)$ consisting of a tuple of points $\underline{z}_\circ$ on the boundary ∂S disjoint from the nodes and a tuple of points $\underline{z}_\bullet \in \mathrm{int}(S)$ in the interior, with the property that the ordering of points on the boundary $\partial S_v \cap \underline{z}_\circ$ is cyclic.

A marked nodal surface $(S, \underline{z})$ is *stable* if the group of automorphisms $\mathrm{Aut}(S)$ is finite.

A marked nodal surface $(S, \underline{z})$ has a combinatorial type $\Gamma = \Gamma(S)$ whose vertices $v \in \mathrm{Vert}(\Gamma)$ correspond to components S_v and edges $e \in \mathrm{Edge}(\Gamma)$ correspond to nodes $w_\pm(e) \in S$ or markings $z(e) \in S$; we define the subset $\mathrm{Edge}_\to(\Gamma)$ to be the set of edges corresponding to markings and call them *leaves*.

For nodal disks $(S, \underline{z})$, this combinatorial type Γ is required to be a tree.

3.2. **Pseudoholomorphic buildings.** We define pseudoholomorphic buildings in symplectic cobordisms with ends modelled on cylinders over stable Hamiltonian manifolds.

Definition 3.3. Let $Z \to Y$ be a circle-fibered stable Hamiltonian manifold as above and $X = \mathbb{R} \times Z$ be the trivial cobordism. An almost complex structure $J : TX \to TX$ is *cylindrical* if there exists an almost complex structure $\overline{J} : TY \to TY$ so that the projection $p_X : X \to Y$ is almost complex and J is the standard almost complex structure on any fiber. More precisely, let

$$\partial_s \in \operatorname{Vect}(\mathbb{R} \times Z), \quad \partial_\theta \in \operatorname{Vect}(\mathbb{R} \times Z)$$

denote the translational vector field on $\mathbb{R}$ resp. rotational vector field on Z. The almost complex structure J is determined on the vertical part of TX by

$$J\partial_s = \partial_\theta, \quad J\partial_\theta = -\partial_s \text{ in } \operatorname{Vect}(Z).$$

On the other hand, the projection to Y is required to be almost complex:

$$Dp_X J = \overline{J} Dp_X \text{ in } \operatorname{Map}(TX, TY).$$

Suppose now X is an arbitrary cobordism from Z_- to Z_+. An almost complex structure J on X is called *cylindrical* if it is the restriction of cylindrical almost complex structures on $\mathbb{R} \times Z_\pm$ on the cylindrical ends $\pm(0, \infty) \times Z_\pm \to X$. The space of cylindrical almost complex structures is denoted $\mathcal{J}_{\mathrm{cyl}}(X)$. This ends the Definition.

We recall some notions of energy for maps from a punctured surface S.

Definition 3.4. (a) (Horizontal energy) The *horizontal energy* of a holomorphic map $u = (\phi, v) : (S, j) \to (\mathbb{R} \times Z, J)$ bounding $\mathbb{R} \times \Lambda$ is ([8, 5.3])

$$E^{\mathbb{h}}(u) = \int_S v^* \omega.$$

(b) (Vertical energy) The *vertical energy* of a holomorphic map $u = (\phi, v) : (S, j) \to (\mathbb{R} \times Z, J)$ is ([8, 5.3])

$$E^{\mathbb{v}}(u) = \sup_\zeta \int_S (\zeta \circ \phi) \mathrm{d}\phi \wedge v^* \alpha \tag{9}$$

where the supremum is taken over the set of all non-negative C^∞ functions

$$\zeta : \mathbb{R} \to \mathbb{R}, \quad \int_{\mathbb{R}} \zeta(s) \mathrm{d}s = 1$$

with compact support.

(c) (Hofer energy) The *Hofer energy* of a holomorphic map

$$u = (\phi, v) : (S, j) \to (\mathbb{R} \times Z, J)$$

is ([8, 5.3]) is the sum

$$E(u) = E^{\mathbb{h}}(u) + E^{\mathbb{v}}(u).$$

(d) (Generalization to manifolds with cylindrical ends) Suppose that X is a manifold with cylindrical ends modelled on $\pm(0,\infty) \times Z_\pm$. The vertical energy $E^{\mathrm{v}}(u)$ is defined as before in (9). The Hofer energy $E(u)$ of a map $u : S^\circ \to X^\circ$ from a surface S° with cylindrical ends to X° is defined by dividing X° into a compact piece X^{com} and cylindrical ends diffeomorphic to $\pm(0,\infty) \times Z_\pm$. Then we set

$$E(u) = E(u|X^{\mathrm{com}}) + E(u|(0,\infty) \times Z_+) + E(u| - (0,\infty) \times Z_-).$$

In symplectic field theory, the compactification of punctured holomorphic curves is given by *holomorphic buildings* in which bubbling occurs on the cylindrical ends. Let $(S, \partial S, j)$ be a bordered Riemann surface, with punctures in the interior and on the boundary. Let (X, J) be an almost complex manifold. A smooth map $u : S \to X$ is *pseudoholomorphic* if the antiholomorphic part of the derivative vanishes:

$$\overline{\partial}_J u := \frac{1}{2}(\mathrm{d}u + J\mathrm{d}uj) = 0.$$

Now suppose that X is a cobordism from Z_- to Z_+, and J is an almost complex structure on X that is cylindrical on the ends.

Definition 3.5. For integers $k_-, k_+ \geq 0$ define topological spaces $\mathbb{X}[k_-, k_+]$ by adding on k_- resp. k_+ incoming resp. outgoing neck pieces:

$$\begin{aligned}(10) \quad \mathbb{X}[k_-, k_+] := (\overline{\mathbb{R} \times Z_-}) \cup_Y \ldots \cup_Y (\overline{\mathbb{R} \times Z_-}) \cup_Y \overline{X} \\ \cup_Y (\overline{\mathbb{R} \times Z_+}) \cup_Y \ldots \cup_Y (\overline{\mathbb{R} \times Z_+}).\end{aligned}$$

Denote by $\overline{\mathbb{X}}[k_-, k_+]_k$ the k-th topological space in the union above, and $\mathbb{X}[k_-, k_+]_k$ the complement of the copies of Y. Thus $\mathbb{X}[k_-, k_+]_k$ is either $\mathbb{R} \times Z_\pm$ if $k \neq 0$ or X if $k = 0$. Let L be a Lagrangian cobordism from Λ_- to Λ_+. Let

$$\begin{aligned}(11) \quad \overline{L}[k_-, k_+] := \overline{\mathbb{R} \times \Lambda_-} \cup_Y \ldots \cup_\Pi \overline{\mathbb{R} \times \Lambda_-} \cup_\Pi \overline{L} \\ \cup_\Pi \overline{\mathbb{R} \times \Lambda_+} \cup_Y \ldots \cup_\Pi \overline{\mathbb{R} \times \Lambda_+}\end{aligned}$$

be the union of the closure $\overline{L}$ with k_-, k_+ cylindrical pieces attached to the ends $\Pi_\pm$.

A *holomorphic* (k_-, k_+)*-building* in X bounding L is a continuous map of a nodal curve S to $\mathbb{X}[k_-, k_+]$ with the following properties: For each irreducible component S_v of V, let S_v° denote the complement of the nodes. Then the restriction u_v of u to S_v is a holomorphic map to some component $\mathbb{X}[k_-, k_+]_{k(v)}$

$$u_v : S_v \to \mathbb{X}[k_-, k_+]_{k(v)}, \ v \in \mathrm{Vert}(\Gamma)$$

with finite Hofer energy, and for each node w of S connecting components S_{v_-} and S_{v_+} and mapping to some copy of Y, the limiting Reeb chord or orbit of u_{v_-} at w is equal to limiting Reeb chord or orbit of u_{v_+} at w. We write $u : S \to \mathbb{X}$ for short.

An *isomorphism* between buildings $u_0, u_1 : S_0, S_1 \to \mathbb{X}[k_-, k_+]$ is an isomorphism of nodal curves $\phi : S_0 \to S_1$ and a combination of translations $\psi : \mathbb{X}[k_-, k_+]$ on the end pieces so that $u_1 \circ \phi := \psi \circ u_0$.

A *trivial cylinder or strip* is a map $u : S \to \mathbb{R} \times Z$ whose projection $u_Y = p_Y \circ u$ is not a stable map to Y. Any trivial cylinder u is contained in a fiber of p_Y and has exactly two punctures. Indeed, the angle changes at the incoming and outgoing punctures are equal as in Lemma 3.13.

A holomorphic building u is *stable* if u has finitely many automorphisms, or equivalently the map u_0 to the component $\mathbb{X}[k_-, k_+]_0 \cong \overline{X}$ is stable in the usual sense, and the map $u_j, j \neq 0$ to each component $\mathbb{X}[k_-, k_+]_j = \overline{\mathbb{R} \times Z_\pm}$ has at least one connected component u_v that is not a trivial cylinder or strip.

3.3. **Treed pseudoholomorphic buildings.** Treed disks, also known as cluster configurations, were introduced in the context of Lagrangian Floer homology by Cornea-Lalonde [17]. The domains for treed disks are combinations of disks with strip-like or cylindrical ends and line segments.

Definition 3.6. A *treed disk* C is obtained from a nodal disk S by replacing each node $w_\pm(e) \in \partial S$ or marking $w(e) \in \partial S$ with an oriented segment T_e (connected compact one-manifold with boundary) equipped with a metric, corresponding to choice of length $\ell(e)$, taken to be infinite in the case of an edge e corresponding to a marking.

The *combinatorial type* Γ of any treed disk is the graph

$$(\mathrm{Vert}(\Gamma) = \mathrm{Vert}_\circ(\Gamma) \cup \mathrm{Vert}_\bullet(\Gamma), \quad \mathrm{Edge}(\Gamma) = \mathrm{Edge}_\circ(\Gamma) \cup \mathrm{Edge}_\bullet(\Gamma))$$

whose vertices correspond to disk and sphere components, respectively, and whose edges correspond to boundary resp. interior edges. The set of semi-infinite edges is denoted

$$\mathrm{Edge}_\to(\Gamma) \subset \mathrm{Edge}(\Gamma).$$

The type Γ has the additional data of subsets

$$\mathrm{Edge}_0(\Gamma), \mathrm{Edge}_\infty(\Gamma) \subset \mathrm{Edge}(\Gamma) - \mathrm{Edge}_\to(\Gamma)$$

describing which edges have zero or infinity length.

Any treed disk C is the union of disks and spheres C_v for the vertices $v \in \mathrm{Vert}(\Gamma)$ and interior and boundary edges T_e for the edges $e \in \mathrm{Edge}(\Gamma)$.

The moduli space of treed disks with a fixed number of punctures has the structure of a compact Hausdorff space as in Charest-Woodward [12]. Let $\mathcal{M}_\Gamma$ denote the moduli space of treed bordered surfaces of type Γ and

$$\overline{\mathcal{M}} = \bigcup_\Gamma \mathcal{M}_\Gamma$$

the union over types Γ, possibly disconnected. If Γ is a type with an edge $e \in \mathrm{Edge}_0(\Gamma)$, we declare Γ to be equivalent to the type Γ' obtained by removing the edge e and identifying the adjacent vertices $v_\pm \in \mathrm{Vert}(\Gamma)$. The corresponding moduli spaces $\mathcal{M}_\Gamma$ and $\mathcal{M}_{\Gamma'}$ are identified in the obvious way. Denote by $\mathcal{M} \subset \overline{\mathcal{M}}$ the subset of strata of top dimension, consisting of configurations C so that each edge T_e has non-zero length $\ell(e)$ in $(0, \infty)$. The set of types has a natural partial order with $\Gamma' \preceq \Gamma$ if and only if $\mathcal{M}_{\Gamma'}$ is contained in the closure of $\mathcal{M}_\Gamma$.

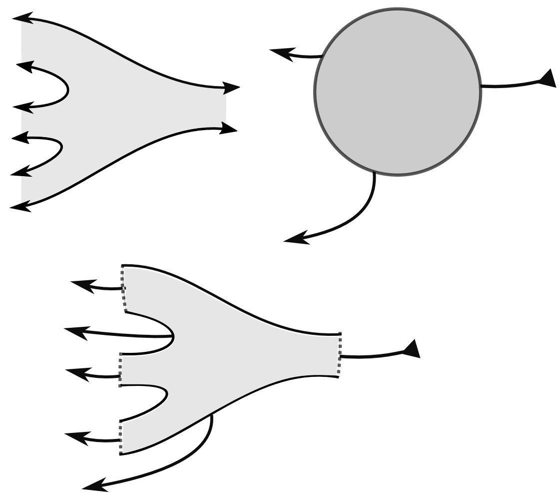

FIGURE 3. A punctured disk and two treed punctured disks

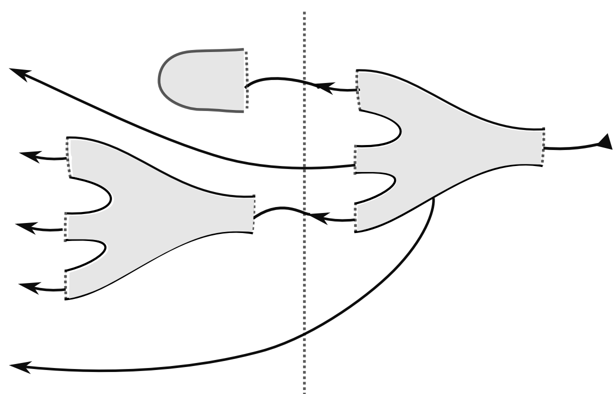

FIGURE 4. Treed building

We will also consider domains with components labelled by integers, describing which *level* they map to the language of symplectic field theory.

Definition 3.7. A genus zero *building* is a nodal disk S equipped with a partition

$$S = S_1 \cup \ldots \cup S_k \tag{12}$$

into (possibly disconnected) components $S_1, \ldots, S_k \subset S$ called *levels* so that intersection between two levels $S_i \cap S_j$ empty unless $i = j \pm 1$. Removing the intersections $S_i \cap S_{i\pm1}$ gives a surface with cylindrical and strip-like ends

A *treed building* is a treed disk C equipped with a decomposition

$$C = C_1 \cup \ldots \cup C_k$$

so that C_i, C_j intersect only if $j \in \{i-1, i, i+1\}$. See Figure 4.

The *combinatorial type* of a treed building is a graph

$$\Gamma = (\mathrm{Vert}(\Gamma), \mathrm{Edge}(\Gamma))$$

equipped with a decomposition into not-necessarily-connected subgraphs

$$\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_k$$

so that $\Gamma_i \cap \Gamma_j$ is empty unless $j \in \{i-1, i, i+1\}$.

Treed holomorphic buildings in symplectic cobordisms are obtained by adding trajectories of the following kind. Let Z be a fibered contact or stable Hamiltonian manifold, and $\Lambda \subset Z$ a Legendrian. We assume that a metric is fixed on Λ, hence on $\mathcal{R}(\Lambda)$ using the identification of components of $\mathcal{R}(\Lambda)$ with Λ in Lemma 2.8.

Definition 3.8. A *Morse datum* for (Z, Λ) consists of a pair of vector fields on the space of Reeb chords and on the Legendrian

$$\zeta_\circ \in \mathrm{Vect}(\mathcal{R}(\Lambda)), \quad \zeta_\bullet \in \mathrm{Vect}(\mathbb{R} \times \Lambda)^{\mathbb{R}}$$

arising as follows:

(a) There exists a Morse function on the space of Reeb chords

$$f_\circ : \mathcal{R}(\Lambda) \to \mathbb{R};$$

so that $\zeta_\circ$ is the gradient vector field:

$$\zeta_\circ := \mathrm{grad}(f_\circ) \in \mathrm{Vect}(\mathcal{R}(\Lambda)). \tag{13}$$

(b) There exists a Morse function

$$f_\bullet : \Lambda \to \mathbb{R};$$

with gradient vector field

$$\mathrm{grad}(f_\bullet) \in \mathrm{Vect}(\Lambda)$$

so that $\zeta_\bullet$ is a translation-invariant lift of $\mathrm{grad}(f_\bullet)$.

Since each component of the space of Reeb chords is isomorphic to the Legendrian, we could take the vector fields to be equal under the identification of the various components; however, this assumption is not necessary.

Definition 3.9. A vector field $\zeta_\bullet \in \mathrm{Vect}(\mathbb{R} \times \Lambda)$ is *positive* if in coordinates (s, λ) on $\mathbb{R} \times \Lambda$ there exists a function

$$a : \Lambda \to \mathbb{R}_{>0}$$

so that

$$\zeta_\bullet = a(\lambda)\partial_s + p^* \,\mathrm{grad}(f_\bullet) \tag{14}$$

where

$$p^* : \mathrm{Vect}(\Lambda) \to \mathrm{Vect}(\mathbb{R} \times \Lambda)^{\mathbb{R}}$$

is the obvious identification of translationally-invariant vector fields trivial in the $\mathbb{R}$-direction with vector fields on Λ.

The limit of any Morse trajectory along any infinite length trajectory is a zero of the gradient vector field. We introduce labels for the possible limits of the trajectories above as follows. Denote by

$$\underline{\mathbb{R}} \cong T\mathbb{R} \times \Lambda$$

the translational factor in $T(\mathbb{R}\times\Lambda) = T\mathbb{R} \oplus T\Lambda$. The zeroes of the vector field $p_*(\zeta_\bullet)$ correspond to tangencies of $\zeta_\bullet$ with the translational factor:

$$p_*(\zeta_\bullet)^{-1}(0) = \zeta_\bullet^{-1}(\underline{\mathbb{R}}) \subset \mathbb{R} \times \Lambda.$$

Let

$$\mathcal{I}(\Lambda) := \mathcal{I}_\circ(\Lambda) \cup \mathcal{I}_\bullet(\Lambda), \quad \mathcal{I}_\circ(\Lambda) := \zeta_\circ^{-1}(0), \quad \mathcal{I}_\circ(\Lambda) := \zeta_\bullet^{-1}(0) \tag{15}$$

be the set of zeroes of these vector fields; these will be the generators of our Chekanov-Eliashberg algebras. The inclusion of the generators $\mathcal{I}_\bullet(\Lambda)$ is similar to the inclusion of the chains on the Lagrangian in the definition of immersed Lagrangian Floer theory, while the generators $\mathcal{I}_\circ(\Lambda)$ correspond to the self-intersections.

Treed holomorphic disks in cobordisms are combinations of pseudoholomorphic maps and trajectories of the vector fields above. Let (X, L) be a cobordism from (Z_-, Λ_-) to (Z_+, Λ_+). Suppose $\mathbb{R}\times Z_\pm$ are equipped with cylindrical almost complex structures $J_\pm$ and $\mathbb{R}\times\Lambda_\pm$ are equipped with vector fields $\zeta_\pm \in \mathrm{Vect}(\mathbb{R}\times\Lambda_\pm)$ as above.

Definition 3.10. An almost complex structure

$$J : TX \to TX$$

is *cylindrical-near-infinity* if J restricts to cylindrical almost complex structure $J_\pm$ on the ends

$$\kappa_\pm(\pm(0,\infty) \times Z_\pm) \subset X.$$

A function resp. gradient vector field

$$f_L : L \to \mathbb{R}, \quad \zeta_\bullet = \mathrm{grad}(f_L) \in \mathrm{Vect}(L)$$

is *cylindrical near infinity* if $\mathrm{grad}(f_L)$ if the restriction of the vector field to the cylindrical ends is given by

$$\zeta_\bullet|_{\kappa_\pm(\pm(0,\infty)\times Z_\pm)} = \zeta_{\bullet,\pm}$$

for some vector fields $\zeta_{\bullet,\pm}$ of the form (14).

A *decorated cobordism* is a cobordism (X, L) together with a pair (J, f_L) as above.

Definition 3.11. Let (X, L) be a decorated cobordism from (Z_-, Λ_-) to (Z_+, Λ_+) as above, so that the Lagrangians $\Pi_\pm = p_\pm(\Lambda_\pm)$ are embedded. Let $D \subset \overline{X}$ be a Donaldson hypersurface so that $\overline{L}$ is exact in the complement of D.

A *treed holomorphic disk* from C to X is a collection of holomorphic maps and trajectories

$$\begin{aligned} u_v : \quad & S_v \to X \quad v \in \mathrm{Vert}(\Gamma) \\ u_e : \quad & T_e \to \begin{cases} L & e \in \mathrm{Edge}_L(\Gamma) \\ \Lambda & e \in \mathrm{Edge}_\bullet(\Gamma) \\ \mathcal{R}(\Lambda) & e \in \mathrm{Edge}_\circ(\Gamma) \\ D & e \in \mathrm{Edge}_D(\Gamma) \end{cases} \end{aligned}$$

for some partition of the edges $\mathrm{Edge}(\Gamma)$ into subsets

$$\mathrm{Edge}_D(\Gamma), \mathrm{Edge}_L(\Gamma), \mathrm{Edge}_\bullet(\Gamma), \mathrm{Edge}_\circ(\Gamma) \subset \mathrm{Edge}(\Gamma)$$

so that the boundary conditions

$$u_v(\partial S_v) \subset \mathbb{R} \times \Lambda, \quad \forall v \in \mathrm{Vert}_\circ(\Gamma)$$

are satisfied and matching conditions hold at the end of each edge according to

$$\mathrm{ev}_z(u_v) = \mathrm{ev}_z(u_e), \quad \forall z \in T_e \cap \overline{S}_v, \quad \forall e \in \mathrm{Edge}(\Gamma),\ v \in \mathrm{Vert}(\Gamma).$$

We may also allow the Lagrangians $\Pi_\pm = p_\pm(\Lambda_\pm)$ to be immersed, so that some of the components of $\mathcal{R}(\Lambda)$ are finite covers rather than diffeomorphic to Λ. The edges mapping to such components are then required to be trajectories of the vector field $\partial_s \in \mathrm{Vect}(\mathbb{R} \times \Lambda_\pm)$.

A *treed holomorphic building* in X is a treed disk equipped with a decomposition $C = C_{k_-} \cup \ldots \cup C_{k_+}$ together with treed disks

$$\begin{aligned} u_j : \quad & C_j \to \mathbb{X}[k_-, k_+]_j := \mathbb{R} \times Z_- \quad j = k_-, \ldots, -1 \\ u_0 : \quad & C_0 \to \mathbb{X}[k_-, k_+]_0 := X \\ u_j : \quad & C_j \to \mathbb{X}[k_-, k_+]_j := \mathbb{R} \times Z_+ \quad j = 0, \ldots, k_+ \end{aligned}$$

satisfying

(a) (Balancing conditions) For any two adjacent $j, j+1$ there exists a number λ_j such that the following holds: Suppose T_e is an edge connecting C_j and C_{j+1} and $u|_{T_e} : T_e \to \mathcal{R}(\Lambda)_\pm$ is a trajectory on the space of Reeb chords. Then the length of T_e is

$$\ell(T_e) = \lambda_j, \quad \forall T_e \text{ connecting } C_j, C_{j+1}.$$

It follows from the definitions that the length of any edge T_e connecting C_j and C_{j+1} representing a trajectory on Λ has infinite length.

(b) (Matching conditions) If u_i, u_{i+1} are adjacent levels joined by an edge $T_e = T_{e,i} \cup T_{e,i+1}$ with coordinate s on $T_{e,i}, T_{e,i+1}$ then

$$\lim_{s \to \infty} u_i(s) = \lim_{s \to -\infty} u_{i+1}(s).$$

This limit should be interpreted as an element of the space $\mathcal{R}(\Lambda_\pm)$ of Reeb chords, if the trajectory represents a trajectory on $\mathcal{R}(\Lambda_\pm)$, or an element of $\Lambda_\pm$ if the trajectory represents a trajectory on $\mathcal{R}(\Lambda_\pm)$.

An *isomorphism* of treed buildings u', u'' with k levels is an isomorphism of domains $\phi : C' \to C''$ (holomorphic on the surface parts and length-preserving on the segments) together with translations

$$\tau_j : \mathbb{X}[k_-, k_+]_j \to \mathbb{X}[k_-, k_+]_j, j = 1, \ldots, k$$

so that

$$u''_v \circ \phi = \tau_j \circ u'_v$$

for each vertex $v \in \mathrm{Vert}(\Gamma_j)$.

We introduce the following notation for moduli spaces with fixed limits along the leaves. Denote by $\mathcal{M}_{\mathbb{\Gamma}}(\Lambda)$ the moduli space of stable treed buildings bounding $\mathbb{R} \times \Lambda$ of type $\mathbb{\Gamma}$. For each type $\mathbb{\Gamma}$ (possibly disconnected) let $\mathcal{M}_{\mathbb{\Gamma}}(\Lambda)$ denote the moduli space of isomorphism classes of finite energy holomorphic maps $u : C \to X$ bounding L of type $\mathbb{\Gamma}$. Denote

$$\mathcal{I}(L) = \zeta_\bullet^{-1}(0) = \mathrm{crit}(f_L)$$

denote the zeroes of $\zeta_\bullet$. Taking the limit of any trajectory along a semi-infinite edge e defines an *evaluation map*

$$\mathrm{ev}_e : \mathcal{M}_{\mathbb{\Gamma}}(L) \to \mathcal{I}(\Lambda_-)^{d_-} \cup \mathcal{I}(L)^d \cup \mathcal{I}(\Lambda_+)^{d_+}.$$

For any given collection

$$\underline{\gamma} = (\underline{\gamma}_-, \underline{\gamma}_\bullet, \underline{\gamma}_+) \in \mathcal{I}(\Lambda_-)^{d_-} \times \mathcal{I}(L)^d \times \mathcal{I}(\Lambda_+)^{d_+}$$

of generators $\mathcal{I}(\Lambda_\pm)$ and critical points $\underline{\gamma}_\bullet$ in the interior, let

$$\overline{\mathcal{M}}(L, \underline{\gamma}) \subset \overline{\mathcal{M}}(L)$$

the moduli space of treed disks with limits along the leaves given by $\underline{\gamma}$. If the domain C is disconnected, then we assume the set of components $C_1, \ldots, C_k$ of the domain is equipped with an ordering and the incoming and outgoing labels $\underline{\gamma}_\pm$ are compatible with the ordering in the sense that the incoming labels for each component are in cyclic order around the boundary and the labels for the various components are in the same order; similarly for the outgoing labels. Also, the incoming labels are ordered before the outgoing labels.

The *map type* of a treed building u is the decorated graph

$$\mathbb{\Gamma} = (\Gamma, h, l)$$

where

$$h : \mathrm{Vert}(\Gamma) \to \pi_2(\overline{X}, \overline{L})$$

(with $\overline{X}$ resp. $\overline{L}$ the compactification of $X = \mathbb{R} \times Z$ resp. $L = \mathbb{R} \times \Lambda$) is the collection of homology classes of maps associated to the vertices and

$$l : \mathrm{Edge}_\to(\Gamma) \to \mathcal{I}(\Lambda) \cup \{D\}$$

is the collection of labels on the leaves; that is, the constraints $l(e) \in \mathcal{I}(\Pi)$ on constrained edges; or D for the edge $T_e, e \in \mathrm{Edge}_D(\Gamma)$ constrained to map to D.

3.4. **Monotonicity assumptions.** We make several assumptions on the cobordism to allow us to restrict to Chekanov-Eliashberg differentials counting rational curves with one incoming boundary puncture. These assumptions guarantee that spherical components do not bubble off in one-dimensional components of the moduli space; such bubbling would force us to include Reeb orbits in the set of generators of the Chekanov-Eliashberg complex.

Our assumptions also force each component of the domain of a rigid treed disk to have at least one incoming puncture; otherwise, we would have to consider curves with multiple incoming punctures and the resulting differential would no longer respect the algebra structure given by concatenation of words. Notably, we do not assume any monotonicity assumption on the Lagrangian projection of the Legendrian, or on the Lagrangian cobordisms we consider.

Definition 3.12. A fibered contact manifold (Z, α) with base $(Y, -\operatorname{curv}(\alpha))$ is *tamed* if the symplectic form on the base $\omega_Y = -\operatorname{curv}(\alpha)$ is integral,

$$[\omega_Y] \in H^2(Y, \mathbb{Z})$$

and monotone in the sense that there exists a *monotonicity constant* $\tau > 1$ so that

$$c_1(Y) = \tau[\omega_Y] \in H^2(Y, \mathbb{R}).$$

Lemma 3.13. *Let (Z, α) be a fibered contact/stable-Hamiltonian manifold and $\Lambda \subset Z$ a Legendrian. Let $u : S \to \mathbb{R} \times Z$ be a punctured surface with boundary mapping to $\mathbb{R} \times \Lambda$ and limiting to Reeb chords and orbits γ_e on the ends $e \in \mathcal{E}(S)$ with angle changes θ_e. Then the incoming and outgoing angles are related by*

$$\sum_{e \in \mathcal{E}_+(S)} \theta_e - \sum_{e \in \mathcal{E}_-(S)} \theta_e = 2\pi \int_S u^* \mathrm{d}\alpha = -2\pi \int_S u_Y^* \omega_Y.$$

Proof. Let $\hat{S}$ denote the compactification of S obtained by adding in intervals along each strip-like end. Denote by $u^*\alpha$ the pull-back connection form on the manifold with corners $\hat{S}$. By Stokes' formula, the difference in incoming and outgoing angles is given by

$$\sum_{e \in \mathcal{E}_+(S)} \theta_e - \sum_{e \in \mathcal{E}_-(S)} \theta_e = \int_{\partial S} u^* \alpha = \int_S \mathrm{d}u^* \alpha = -\int_{\partial S} u_Y^* \omega_Y.$$

Here, the integral over ∂S vanishes since α restricts to zero on Λ. □

The following corollary is used to show that the count of curves with a single input defines a differential.

Corollary 3.14. *Suppose that $Z \to Y$ is equipped with a contact form α and J is a cylindrical almost complex structure on $\mathbb{R} \times Z$. Let $u : S \to \mathbb{R} \times Z$ be a punctured J-holomorphic curve with boundary on $\mathbb{R} \times \Lambda$. The sum of the angles of Reeb chords at outgoing punctures is smaller than the sum of the angles at the incoming punctures:*

$$\sum_{e \in \mathcal{E}_+(S)} \theta_e \leq \sum_{e \in \mathcal{E}_-(S)} \theta_e \tag{16}$$

with equality only if the map u projects to a constant map u_Y to Y. In particular, any non-constant punctured holomorphic curve $u : S \to \mathbb{R} \times Z$ has at least one incoming puncture.

Proof. The claim (16) follows from Lemma 3.13. To prove the last claim, let u be a map as in the statement of the Lemma. Suppose that u projects to a constant map u_Y but is non-constant in $\mathbb{R} \times Z$. The equation (16) implies that there are both positive and negative punctures. On the other hand, if u_Y is non-constant then (16) implies that there is at least one incoming puncture, that is, a point in $\overline{u}^{-1}(Y_-)$. □

We assume the following conditions on our cobordisms to guarantee the chain maps are well-defined. Recall that $(Z_\pm, \alpha_\pm)$ are fibered contact/stable-Hamiltonian manifolds with base the symplectic manifolds $(Y_\pm, -\mathrm{d}\alpha_\pm)$. Let (X, L) be a cobordism from (Z_-, Λ_-) to (Z_+, Λ_+). The normal bundle to $Y_\pm$ in $\overline{X}$ is the line bundle associated to $Z_\pm$, and so has a well-defined *relative Thom class*

$$[Y_\pm]^\vee \in H^2(\overline{X}, \overline{L});$$

see Definition 4.18 for construction of such classes. The *logarithmic Chern class* of $\overline{X} - Y_\pm$ is

$$c_1^{\log}(\overline{X} - Y_\mp) := c_1(\overline{X} - Y_\mp) - [Y_\pm]^\vee \in H^2(\overline{X} - Y_\pm)$$

Definition 3.15. The pair (X, L) is an *tamed cobordism* from (Z_-, Λ_-) to (Z_+, Λ_+) if and only if

- **(P1)** The symplectic classes $[\omega_Y] \in H^2(Y)$ and $[\overline{\omega}] \in H^2(\overline{X})$ are integral; this is a technical assumption for our regularization scheme which presumably could be removed and the Lagrangians are rational in the sense of Definition 4.3 below;
- **(P2)** The logarithmic first Chern class of $\overline{X} - Y_+$ is a very positive multiple of the symplectic class on the complement of Y_+ in the sense that there exists a constant $\lambda_1 > 0$ so that
$$c_1^{\log}(\overline{X} - Y_+) = (1 + \lambda_1)[\overline{\omega}] \in H^2(\overline{X} - Y_+).$$
- **(P3)** If the outgoing end Λ_+ of the cobordisms L is non-empty then the relative Thom class of the outgoing end $[Y_+]^\vee \in H^2(\overline{X} - Y_-, \overline{L} - Y_-)$ is non-positive in the sense that there exists a constant $\lambda_2 \geq 0$ so that
$$[Y_+]^\vee = -\lambda_2[\overline{\omega}] \in H^2(\overline{X} - Y_-, \overline{L} - Y_-).$$

Example 3.16. In the case of the cobordism $X = \mathbb{C}^n - \{0\}$ from S^{2n-1} to itself, $\overline{X}$ is the blow-up of $\mathbb{C}P^n$ at 0 with $Y_\pm \cong \mathbb{C}P^{n-1}$, the logarithmic first Chern class is

$$c_1^{\log}(\overline{X} - Y_+) = n[Y_-]^\vee$$

and the outgoing divisor has class

$$[Y_+]^\vee = c_1(\mathcal{O}(-1)) = -[\overline{\omega}].$$

Therefore the constants are

$$\lambda_1 = n - 1, \quad \lambda_2 = 1,$$

hence the cobordism is tamed.

We now discuss our motivating example given by the Harvey-Lawson filling. According to Lemma 2.24, the Harvey-Lawson filling defines a cylindrical-near-infinity filling $L \subset X$ of a Legendrian Λ_ϵ with respect to some stable Hamiltonian triple $(Z, \alpha_\epsilon, \omega_Z)$ with $Z \cong S^{2n-1}$ and Λ_ϵ a lift of a perturbation of the Clifford torus. Although the form α_ϵ for which Λ_ϵ is Legendrian, is contact, the symplectic form ω on X is an extension of α_0 so that $\mathrm{d}\alpha_0 = -\omega_Z$, rather than the perturbed form α_ϵ. From now on, by the *Harvey-Lawson filling* we mean the filling L of $\Lambda_\epsilon \subset S^{2n-1}$ produced by Lemma 2.24 applied to the Harvey-Lawson Lagrangian (2).

Lemma 3.17. *The Harvey-Lawson filling L of $\Lambda_\epsilon \subset S^{2n-1}$ is tamed.*

Proof. The compactified filling is $\overline{X} = \mathbb{C}P^n$ which has Chern class

$$c_1(\overline{X}) = (n+1)[\mathbb{C}P^{n-1}]^\vee = (n+1)[Y_-]^\vee.$$

Hence

$$c_1(\overline{X}) - [Y_-]^\vee = n[Y_-]^\vee = (1+\lambda_1)[\overline{\omega}]$$

for $n \geq 2$. The outgoing end Λ_+ is empty, so the condition **(P3)** is vacuous. Hence (X, L) is tamed. □

We now give an example of a cobordism that is not tamed. Let X be the blow-up of the unit ball $B_1(0)$ in $\mathbb{C}^n$. Since the Harvey-Lawson filling L of Λ_ϵ is disjoint from $0 \in B_1(0)$, it defines a Lagrangian filling in X as well.

Lemma 3.18. *The Harvey-Lawson filling $L \subset X$, where X is the blow-up of $B_1(0)$ at 0, is not tamed as a filling.*

Proof. We check that the condition **(P2)** fails. The compactification $\overline{X}$ is the blow-up $\mathrm{Bl}_0 \mathbb{C}P^n$ of $\mathbb{C}P^n$ at $0 \in \mathbb{C}^n \subset \mathbb{C}P^n$. The space $\overline{X}$ fibers over $\mathbb{C}P^{n-1}$ via a map

$$\pi : \mathrm{Bl}_0 \mathbb{C}P^n \to \mathbb{C}P^{n-1}.$$

The blow-up $\mathrm{Bl}_0 \mathbb{C}P^n \cong \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(1))$ is isomorphic to the projectivization of the sum of the hyperplane and trivial bundles $\mathcal{O}(1), \mathcal{O}(0)$. The Chern numbers of the fibers of the projection are

$$c_1(p^{-1}(\ell)) = 2, \quad \forall \ell \in \mathbb{C}P^{n-1}.$$

The evaluation of the logarithmic Chern class on a fiber is

$$\langle c_1^{\log}(\overline{X} - Y_+), [p^{-1}(\ell)] \rangle = \langle c_1(\overline{X}) - [Y_-], [p^{-1}(\ell)] \rangle = 2 - 1 = 1$$

which is less than the required bound in **(P2)**. □

The moduli spaces of holomorphic disks in the example in Lemma 3.18 have boundary components where a sphere representing a fiber has bubbled off. Thus, for a well-defined theory of Legendrian contact homology in which such fillings define augmentations, one would have to include the Reeb orbits in the set of generators.

Lemma 3.19. *Let Λ be a Legendrian in a fibered contact manifold Z. Suppose that $X = \mathbb{R} \times Z$ is the symplectization of Z and*

$$L = (\mathbb{R} \times \Lambda)_{[\sigma_1, \sigma_2]} \cong \Pi \cup ((\sigma_1, \sigma_2) \times \Lambda) \cup \Pi$$

the symplectic cut at σ_1, σ_2. For choices of $\sigma_1, \sigma_2 \in \mathbb{R}$ so that $[e^{\sigma_1}\omega_Y], [e^{\sigma_2}\omega_Y] \in H_2(Y)$ are integral and $e^{\sigma_2} - e^{\sigma_1} \in \mathbb{Z}$, L is a tamed cobordism.

Proof. We check the integrality of the symplectic class. The homology of $\overline{X}$ is generated by classes in $\overline{Y}$ together with classes in the fibers of $\pi : \overline{X} \to \overline{Y}$, by the Leray spectral sequence. The fiber $\pi^{-1}(y)$ over any $y \in \overline{Y}$ has area $e^{\sigma_2} - e^{\sigma_1}$ which is an integer by assumption. On the other hand, the degree two homology classes in $\overline{Y}$ have integral areas by assumption. Hence, **(P1)** is satisfied.

To check the positivity of the logarithmic first Chern class, note that the complement $\overline{X} - Y_+$ contracts to Y_-, compatible with the Lagrangian boundary condition, so it suffices to check **(P2)** on spheres in Y_-. The class $c_1(\overline{X} - Y_+)$ restricts to $(\tau + 1)c_1(Z)$ on Y_-, where it is equal to the restriction of $(\tau + 1)[\omega_Y]$. So

$$c_1^{\log}(\overline{X} - Y_+) = c_1(\overline{X} - Y_+) - [Y_-]^\vee = p^* c_1(Y) = \tau[\omega_Y].$$

So the condition **(P2)** holds with proportionality constant $\lambda_1 = \tau - 1$.

Finally, , we check the negativity of the outgoing end. We compute

$$[Y_+] = p^* c_1(Z) = -p^*[\omega_Y] = -[\overline{\omega}] \in H_2(\overline{X} - Y_-, \overline{L} - Y_-).$$

So **(P3)** holds with proportionality constant $\lambda_2 = 1$. □

Remark 3.20. By a similar computation, one can show that an exact cobordism with vanishing first Chern class between two fibered contact manifolds is tamed.

The conditions in Definition 3.12 could presumably be relaxed by dealing with more complicated curve counts; for example, including Reeb orbits in the set of generators would allow relaxation of the second inequality while presumably allowing all-genus counts would allow relaxation of the first. However, allowing these possibilities introduces further technical difficulties.

Lemma 3.21. *If X is a cobordism satisfying **(P3)** then any holomorphic punctured disk $u : S \to X$ bounding L with an outgoing strip-like end has at least one incoming end.*

Proof. Suppose otherwise that $u : S \to X$ is a punctured disk bounding L with only outgoing ends. The intersection number of the map u with the outgoing divisor Y_+ is

$$[u] \cdot [Y_+] = [u] \cdot \lambda_2[\omega] = -\lambda_2 A(u) \leq 0.$$

Since u is holomorphic with non-positive area, u must be constant. Since by definition u maps to the interior X, the map u has no outgoing ends, which is a contradiction. □

Lemma 3.22. *Suppose X is a tamed cobordism. There are no rigid holomorphic spheres $u : S \to X$ with a single puncture.*

Proof. Consider a punctured sphere $u : S \to X$ of type $\mathbb{\Gamma}$ whose closure $\overline{u}$ meets the incoming divisor Y_- but not the outgoing divisor Y_+, and $\overline{u}$ its extension to a map to $\overline{X}$. The dimension of the moduli space $\mathcal{M}_{\mathbb{\Gamma}}(X)$ of spheres constrained to intersect Y_- at a single puncture with multiplicity

$$m = (\overline{u}_*[\mathbb{P}^1], [Y_-])$$

is

$$\begin{aligned}\dim \mathcal{M}_{\Gamma}(X) &= 2(c_1(\overline{X}), \overline{u}_*[\mathbb{P}^1]) - 2 - 2m \\ &= 2((1+\lambda_1)[\overline{\omega}], \overline{u}_*[\mathbb{P}^1]) - 2 > 0.\end{aligned}$$

Here we used that X satisfies **(P2)**. Hence such spheres cannot be rigid. On the other hand, if $\overline{u}$ has domain a sphere S with a single outgoing puncture and no incoming punctures then by **(P3)**

$$0 > -[Y_+].[\overline{u}] = \lambda_2[\overline{\omega}].[\overline{u}].$$

This is a contradiction. □

3.5. **Homological invariants.** In the definition of contact homology we will work with coefficient rings that are completed group rings for various homology groups. The standard in the field is to use a group ring on first homology. The standard approach is to assign homology classes to punctured disks by closing up the Reeb chords at infinity. Unfortunately, knowledge of these boundary classes are not enough in general to determine the energy of the corresponding holomorphic disk which is necessary for the completion of the group ring needed for the definition of the contact homology. Therefore, we discuss various other possibilities for coefficients (that is, Novikov rings). Denote by

$$\Lambda_1, \ldots, \Lambda_k \subset \Lambda$$

the connected components of the Legendrian Λ.

Definition 3.23. For each connected component Λ_i of Λ choose a base point

$$\lambda_i \in \Lambda_i, \quad i = 1, \ldots, k.$$

For each Reeb chord $\gamma : [0,1] \to Z$ choose *capping paths*

$$\hat{\gamma}_b : [0,1] \to \Lambda, \quad \hat{\gamma}_b(0) = \gamma(b), b \in \{0,1\}, \quad \hat{\gamma}_b(1) = \lambda_i. \tag{17}$$

Let $u : S \to \mathbb{R} \times Z$ be a holomorphic map bounding $\mathbb{R} \times \Lambda$ with limits on the strip-like ends given by the collection of Reeb chords

$$\underline{\gamma} = (\gamma_0, \ldots, \gamma_k) \in \mathcal{R}(\Lambda)^{k+1}$$

The boundary ∂S is divided into components

$$\partial S = (\partial S)_0 \cup \ldots \cup (\partial S)_k$$

by the strip-like ends. The restriction $(\partial u)_m$ of u after projection to Λ concatenates with the capping paths for chords in $\underline{\gamma}$ to produce a sequence of loops

$$\overline{\partial u}_m := \hat{\gamma}_{m+1,0} \circ (\partial u)_m \circ \hat{\gamma}_{m,1}^{-1} : S^1 \to \Lambda_{i(m)}, m = 0, \ldots, k$$

in the components $\Lambda_{i(m)}$. The total homology class of the combined loop is denoted

$$[\partial u] = \sum_k [\overline{\partial u}_j] \in H_1(\Lambda). \tag{18}$$

Similarly, in the case that the Legendrian is connected and the contact manifold is simply-connected, second homology classes are associated to punctured disks by a capping procedure. For each Reeb chord, the loop obtained by concatenating

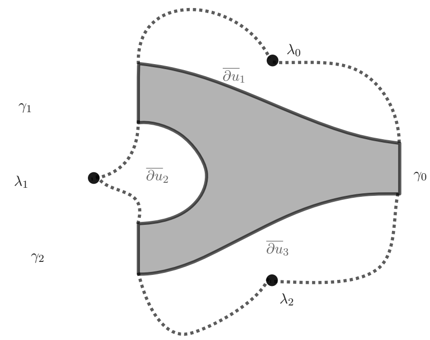

FIGURE 5. Capping paths to make a cycle

$\gamma, \hat{\gamma}_0, \hat{\gamma}_1$ is the boundary of some continuous map $v_\gamma : S_\gamma \to Z$ from a disk S_γ to Z by the assumption that Z is simply-connected. The homology class obtained from a punctured disk $u : S \to Z$ by gluing in the disk v_γ at each Reeb chord or orbit is denoted

$$[u] = \left[u_*(S) + \sum_\gamma v_{\gamma,*} S_\gamma \right] \in H_2(Z, \Lambda) \tag{19}$$

Finally, we define homology classes of disks in cobordisms. Let $\tilde{L}$ denote the closure of L in the manifold with boundary $\tilde{X}$. For each component $\tilde{L}_i$ of a cobordism $\tilde{L}$ choose a choice of base point λ_i in each component and a collection of paths $\hat{\gamma}_b, b \in \{0, 1\}$ from the base point to the start and end points of any Reeb chord γ. Let $u : S \to X$ bounding L punctured holomorphic curve. By concatenating with the disks v_{γ_b} and paths $\hat{\gamma}_b$ for each strip-like end of S one obtains homology classes denoted

$$[\partial u] \in H_1(\tilde{L}) \cong H_1(L).$$

If L is connected and X is simply connected, one obtains second homology classes

$$[u] \in H_2(\tilde{X}, \tilde{L}) \cong H_2(X, L)$$

by a choice of capping disk at each Reeb chord. If $u = (u_1, u_2, \ldots, u_k)$ is a holomorphic building with levels $u_1, u_2, \ldots, u_k$ then define

$$[\partial u] := [\partial u_1] + [\partial u_2] + \ldots + [\partial u_k], \quad [u] := [u_1] + [u_2] + \ldots + [u_k].$$

The *boundary homology class* of a treed holomorphic disk

$$u : C \to \mathbb{R} \times Z \quad \text{or} \quad u : C \to X$$

is defined as follows: In the first case of maps to $\mathbb{R} \times Z$, the boundary components $(\partial S)_i, i = 1, \ldots, l(i)$ of each disk component $u_v : S_v \to X$ define paths

$$(\partial u_v)_i : (\partial S)_i \to \Lambda_{k(i)}, \quad i = 1, \ldots, l(i)$$

where $\Lambda_{k(i)} \subset \Lambda$ is some connected component. Each trajectory $u_e : T_e \to \mathcal{R}(\Lambda)$ defines by composition a pair of maps

$$u_{e,b} : T_e \to \Lambda_{k_b(e)}, \quad b \in \{0, 1\}$$

by composing u_e with evaluation $\mathcal{R}(\Lambda) \to \Lambda$ at the start or endpoint of the path. The boundary homology class $[\partial u] \in H_1(\Lambda)$ is defined as the class of the sum of the chains above. In the case of a treed disk in X bounding a cobordism L, one obtains a class in $H_1(\tilde{L}) \cong H_1(L)$ by a similar construction.

The *boundary class* of a treed building $u : C \to \mathbb{X}$ is the element $[\partial u] \in H_1(L)$ obtained by concatenating the restrictions ∂u_v, the compositions of the trajectories $u_e : T_e \to \mathcal{R}(\Lambda)$ with the projections $\mathcal{R}(\Lambda) \to \Lambda$ given by evaluation at the end point or starting point, and the capping paths to the base points λ_i in the corresponding components Λ_i of Λ. This ends the Definition.

Lemma 3.24. *For any disk $u : C \to X$ bounding L, there exist capping paths $\hat{\gamma}_b$ for each $\gamma = \gamma_e$ at the strip-like ends of u so that the homology classes $[\partial u] \in H_1(L)$ are trivial; similarly in the case that Z is simply connected there exist capping disks so that $[u] \in H_2(\overline{X}, \overline{L})$ is trivial.*

Proof. Suppose u is a rigid holomorphic map from a once-punctured surface S to $\mathbb{R} \times Z$ with a single strip-like end $e \in \mathrm{Edge}_\circ(\Gamma)$ on which u is asymptotic to a Reeb chord $\gamma \in \mathcal{R}(\Lambda)$. Let $\hat{\gamma}_1$ to be the inverse homotopy class of the path (∂u) composed with the capping path $\hat{\gamma}_0$. The homology class $[\partial u] \in H_1(\Lambda)$ is then trivial. Similarly, take the capping disk v_γ to be the map u. □

4. Regularization

In this section, we discuss the necessary Fredholm theory, compactness results, and regularization techniques to make the moduli spaces of treed buildings of dimension at most one compact manifolds with boundary. The compactness results in symplectic field theory are similar to those covered in [8], [1]. We take as a regularization scheme the stabilizing divisors scheme of Cieliebak-Mohnke [14]; the adaptation to the case at hand requires a small extension of the results in Pascaleff-Tonkonog [42] to the case of clean intersection. The reader could presumably substitute their favorite regularization scheme.

4.1. **Fredholm theory.** We begin with the Fredholm theory, in which we show that the moduli space is cut out locally by a Fredholm map. We state our results in the situation of a Lagrangian cobordism in a symplectic cobordism as in Section 2.2. Recall that $(\overline{X}, \omega)$ is a closed symplectic manifold, $Y_\pm$ are two disjoint symplectic hypersurfaces in $\overline{X}$ and $X = \overline{X} - Y_\pm$. Moreover $\overline{L}$ is a Lagrangian in $\overline{X}$ such that $L_\pm = \overline{L} \cap Y_\pm$ are Lagrangians in $Y_\pm$. Then $L = \overline{L} - L_\pm$ is a Lagrangian cobordism in X. Our Sobolev maps are perturbations of a class model maps defined as follows. Let S be a punctured disk with ends $\mathcal{E}(S)$.

A *model map* is a smooth map $u : (S, \partial S) \to (X, L)$ such that with coordinates given by the cylindrical/strip-like ends $\kappa_e : \pm(0, \infty) \times [0, 1] \to X$ or $\kappa_e : \mathbb{R} \times S^1 \to X$ the map u either is linear in the sense that

$$u(\kappa_e(s, t)) = (\mu s, \gamma(\mu t)) \in \mathbb{R} \times Z, \quad \forall (s, t) \in \pm(0, \infty) \times [0, 1] \tag{20}$$

where γ is a Reeb chord or a Reeb orbit with length μ. The set of model maps is defined as

$$\mathrm{Map}_0(S, X, L) = \{u : (S, \partial S) \to (X, L) \mid \ (20) \text{ holds for all } \ e \in \mathcal{E}(S)\}$$

Let β be a *good bump function* on $\mathbb{R}$ such that

(a) β is non-decreasing;
(b) $\beta(s) = 1$ when $s \geq 1$;
(c) $\beta(s) = 0$ when $s \leq 0$.

For any *Sobolev decay constant* $\delta > 0$ we define a *Sobolev weight function* κ^δ on S such that

(a) $\kappa^\delta = 0$ outside cylindrical/strip-like ends;
(b) on each cylindrical/strip-like end, there is a constant $M > 0$ with $\kappa^\delta(s,t) = \delta\beta(|s| - M)$.

Pick a Riemannian metric on X which is cylindrical near infinity and a metric connection ∇. For Sobolev constants k, p with $kp > 2$ and $u \in \mathrm{Map}_0(S, X, L)$, consider the *weighted Sobolev space*

$$W^{k,p,\lambda}(u^*TX, \partial u^*TL) := \{\eta \in W^{k,p,\lambda}_{\mathrm{loc}}(S, u^*TX) \mid ||\eta||_{k,p,\lambda} < +\infty\}.$$

Here the Sobolev (k, p, λ)-norm is defined as

$$||\eta||_{k,p,\lambda} := \int_S (|\eta|^p + |\nabla\eta|^p + \cdots + |\nabla^k\eta|^p)e^{p\kappa^\lambda}.$$

Let

$$\exp : TX \to X$$

denote the geodesic exponentiation map defined using the above metric on X.

Define the *weighted Sobolev space of maps*

$$\mathrm{Map}_{k,p,\lambda}(S, X, L) := \{\exp_u \xi \mid u \in \mathrm{Map}_0(S, X, L), \xi \in W^{k,p,\lambda}(u^*TX, \partial u^*TL)\}.$$

Charts for a Banach manifold structure on $\mathrm{Map}_{k,p,\lambda}(S, X, L)$ are induced by geodesic exponentiation with respect to the metric on X which is cylindrical near infinity.

The moduli space is cut out locally over the moduli space of domains by a Fredholm map. Recall that $\mathcal{M}_\Gamma$ is the moduli space of treed disks $C = S \cup T$ of type Γ. Let $\mathcal{M}^i_\Gamma, i = 1, \ldots, m$ be an open cover of $\mathcal{M}_\Gamma$ over which the universal treed disk

$$\mathcal{U}_\Gamma = \{(C, z \in C)\} \to \mathcal{M}_\Gamma$$

is trivial as a stratified bundle. That is, smoothly every treed disk $C' \in \mathcal{M}^i_\Gamma$ may be identified with C, and we view C' as the smooth space C equipped with an almost complex structure $j(C') \in \mathcal{J}(S)$ and metric $g(C') \in \mathcal{G}(T)$ depending on C'. We define

$$\mathcal{B}^{i,k,p,\lambda}_\Gamma := \mathcal{M}^i_\Gamma \times \mathrm{Map}_{k,p,\lambda}(C, X, L).$$

The fiber of the vector bundle $\mathcal{E}^i_\Gamma$ over any map u is the bundle of one-forms of Sobolev differentiability class one less:

$$\mathcal{E}^{k-1,p,\lambda}_{\Gamma,u} := \Omega^{0,1}(S, (\overline{u}|S)^* T\overline{X})_{k-1,p,\lambda} \oplus \Omega^1(T, (u|T)^* T\overline{L})_{k-1,p,\lambda} \tag{21}$$

where

$$\Omega^1(T, (u|T)^* T\overline{L}) := \Omega^1(T_\bullet, (\overline{u}|T_\bullet)^* T\overline{L}) \oplus \Omega^1(T_\circ, (\overline{u}|T_\circ)^* T\mathcal{R}(\Lambda)). \tag{22}$$

We remark that in general, the bundles $\mathcal{E}^{k-1,p,\lambda}_{\Gamma,u}$ do not form a smooth Banach vector bundle over $\mathcal{B}^{k,p,\lambda}_\Gamma$, because the transition maps between local trivializations involve reparametrizations of the domain, whose derivatives lower Sobolev classes. Define

(23)

$$\mathcal{F}^i_\Gamma : \mathcal{B}^{i,k,p,\lambda}_\Gamma \to \mathcal{E}^{k-1,p,\lambda}_{\Gamma,u}, \quad (C', \xi) \mapsto (\mathcal{T}^\xi_u)^{-1} \left(\overline{\partial}_{j(C')} u'|S, \ \frac{d}{dt} u'|_T - \operatorname{grad}(\zeta(u') \right)$$

where $\mathcal{T}^\xi_u$ denotes parallel transport along the geodesic $\exp_u(\rho\xi)$ and ζ denotes either $\zeta_\circ$ or $\zeta_\bullet$ depending on the type of edge. The moduli space of maps over $\mathcal{M}^i_\Gamma$ is

$$\mathcal{M}^i_\Gamma(\underline{L}) = (\mathcal{F}^i_\Gamma)^{-1}(\underline{0}),$$

that is, the inverse image of the zero section $\underline{0} \subset \mathcal{E}^{k-1,p,\lambda}_\Gamma$over $\mathcal{M}^i_\Gamma$. The global moduli space $\mathcal{M}_\Gamma(\underline{L})$ is obtained by patching together the local manifolds $\mathcal{M}^i_\Gamma(\underline{L})$; once one passes to solutions to elliptic equations then the transition maps become smooth.

We introduce the following terminology. The *linearized operator* at any solution u is denoted

$$\tilde{D}_u : T\mathcal{B}_\Gamma \to \mathcal{E}_{\Gamma,u}, \quad \xi \mapsto \frac{\partial}{\partial \rho}|_{\rho=0} \mathcal{F}^i_\Gamma(\exp_u(\rho\xi)) \tag{24}$$

and is independent of the choice of trivialization. The restriction of $\tilde{D}_u$ to variations of the map $\xi \in \Omega^0(S, u^*TX)$ is the linearized operator studied in McDuff-Salamon [39].

A map u is *regular* if $\tilde{D}_u$ is surjective, in which case $\mathcal{M}_\Gamma(\Lambda)$ is a smooth manifold of dimension $\operatorname{Ind}(\tilde{D}_u)$ in an open neighborhood of u.

The *virtual dimension* of the moduli space $\mathcal{M}_\Gamma(\Lambda)$ at u is the index of the linearized operator

$$\operatorname{vdim}(\mathcal{M}_{\mathbb{\Gamma}}(L)) := \operatorname{Ind}(\tilde{D}_u)$$

which is the honest dimension if every element is regular. For any integer $d \geq 0$, denote

$$\overline{\mathcal{M}}(L)_d = \bigcup_{\mathbb{\Gamma}} \mathcal{M}_{\mathbb{\Gamma}}(L) \tag{25}$$

the union of types with

$$\operatorname{vdim} \mathcal{M}_{\mathbb{\Gamma}}(L) + \operatorname{codim}(\mathcal{M}_\Gamma) = d. \tag{26}$$

That is, $\mathcal{M}_{\mathbb{\Gamma}}(L)$ is in the locus of expected dimension d, allowing for deformations of domain which change the combinatorial type.

The locus of *rigid* configurations $\mathcal{M}(L)_0$ is the locus of maps $u : C \to X$ for which the expected dimension $\operatorname{Ind}(\tilde{D}_u)$ vanishes and $\mathcal{M}_\Gamma$ represents a stratum of top dimension in the moduli space of domains. The last condition is equivalent to requiring that the lengths $\ell(e)$ of all edges $e \in \operatorname{Edge}(\Gamma)$ are finite and non-zero.

The case of maps in the symplectization is a special case of maps in the cobordism above. For a Legendrian Λ in Z let $\mathcal{M}(\Lambda)$ be the moduli space $\mathcal{M}(\mathbb{R} \times \Lambda)$ up to the equivalence given by the translations in the $\mathbb{R}$-direction. Thus $\mathcal{M}(\Lambda)_0$ is the locus in $\mathcal{M}(\Lambda)$ that are rigid (after modding out by translation) and whose domains $C = S \cup T$ are in a top-dimensional strata $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}$ of treed disks.

4.2. **Donaldson hypersurfaces.** To achieve manifold structures on the moduli spaces of treed holomorphic maps we use Cieliebak-Mohnke perturbations [14]. Since the compactification of our Lagrangians are only cleanly-intersecting, we extend the construction to that case, generalizing the construction in Pascaleff-Tonkonog [42].

A *Bertini hypersurface* in a complex projective subvariety $X \subset \mathbb{C}P^{n-1}$ is the intersection of X with a generic hyperplane. By Bertini's theorem, such a hypersurface is a smooth subvariety.

A *Donaldson hypersurface* in a symplectic manifold X with symplectic form ω is a codimension two symplectic submanifold $D \subset X$ such that the dual cohomology class $[D]^\vee$ satisfies

$$[D]^\vee = k[\omega] \in H^2(X)$$

for some positive $k > 0$ called the *degree* of D.

Results of Cieliebak-Mohnke [14] and Charest-Woodward [11] imply that there exist Donaldson hypersurfaces which intersect holomorphic disks and spheres in only finitely many points:

Proposition 4.1. *There exists a $k_0 \in \mathbb{Z}_+$ so that if $D \subset \overline{X}$ is a Donaldson hypersurface D of degree at least k_0 then there exists an tamed almost complex structure on $\overline{X}$ so that D contains no rational curves $u : S^2 \to \overline{X}$. Furthermore, for any such J_0 and energy bound E, there exists an open neighborhood,*

$$\mathcal{J}^E(\overline{X}) \subset \mathcal{J}(\overline{X}),$$

in the space of tamed almost complex structures on $\overline{X}$ with the property that if $J_1 \in \mathcal{J}^E(\overline{X})$ then any non-constant J_1-holomorphic sphere $u : S^2 \to \overline{X}$ of energy at most E meets D in at least three but finitely many points $u^{-1}(D)$.

In the case that X is a complex projective variety, a result of Clemens [16] implies that we may take D to be a Bertini hypersurface, and leave the complex structure unperturbed.

We wish to find Donaldson hypersurfaces that intersect each non-constant punctured holomorphic disk. Let $\overline{L} \subset \overline{X}$ be the closure of a Lagrangian cobordism, so that $\overline{L}$ is the union of a smooth Lagrangian L and Lagrangians $\Pi_\pm$ in $Y_\pm$, with angles between the branches at the self-intersections bounded from below.

Proposition 4.2. *Let (X, L) be a cobordism from (Z_-, Λ_-) to (Z_+, Λ_+). There exists a manifold with boundary $\overline{L}$ and an immersion $\phi : \overline{L} \to \overline{X}$ with clean self-intersection so that $\phi(\phi^{-1}(X)) = L$.*

Proof. Suppose that Λ intersects some fiber $Z_y \cong S^1$ over $y \in Y$ in a subset $\Lambda \cap Z_y = \{\theta_1, \ldots, \theta_k\}$. Then the subset

$$\mathbb{R}e^{i\theta_1} \cup \ldots \cup \mathbb{R}e^{i\theta_k} \subset \mathbb{C} \cong Z_y \times_{\mathbb{C}^\times} \mathbb{C}$$

is a cleanly-intersecting subset of the fiber of the normal bundle of Y in $\overline{X}$. The union of such fibers over Π is locally a cleanly-intersecting Lagrangian in the normal bundle N_Y. Via the identification of an open neighborhood U of Y with an open neighborhood of the zero section in N_Y, denote the union of these fibers with L by $\hat{L}$. Then $\hat{L}$ is a non-compact, cleanly-self-intersecting Lagrangian containing $\overline{L} \cap U$ described locally as

$$L \cap U \cong \mathbb{R}_+ e^{i\theta_1} \cup \ldots \cup \mathbb{R}_+ e^{i\theta_k}.$$

□

In particular, Proposition 4.2 implies that $\overline{L}$ is a closed subset of a cleanly-self-intersecting Lagrangian $\tilde{L} \to \overline{X}$.

Definition 4.3. A cleanly-self-intersecting Lagrangian $\overline{L} \subset \overline{X}$ is *rational* if there exists a line-bundle-with-connection

$$\hat{X} \to \overline{X}$$

whose curvature is $k\omega \in \Omega^2(\overline{X})$ for some integer $k > 0$ such that the pull-back of $\hat{X}$ to the Lagrangian $\overline{L}$ has a flat section.

The Lagrangian L is *exact* on a subset $U \subset X$ with $\overline{U} \subset \overline{X}$ open if there exists a one-form $\alpha \in \Omega^1(\overline{U})$ with $\mathrm{d}\alpha = \omega|_{\overline{U}}$ and a continuous function $f : \overline{L} \to \mathbb{R}$ so that $\mathrm{d}f|_L = \alpha|_L$.

Lemma 4.4. *Suppose that $\overline{X}$ is simply connected and every disk $u : S \to \overline{X}$ bounding $\overline{L}$ has rational area $\langle [u], [\overline{\omega}] \rangle \in \mathbb{Q}$. Then $\overline{L}$ is rational.*

Proof. Let $\hat{X}$ be a line bundle-with-connection with curvature $k\omega$. After replacing k with a sufficiently large integer multiple, every disk bounding $\overline{L}$ has integer $k\omega$-area. It follows from the assumption that X is simply connected that the holonomy around any loop in $\overline{L}$ is trivial, as in the computation in the proof of Proposition 2.4. Parallel transport of an arbitrary point in $\hat{X}$ along paths in $\overline{L}$ then defines the desired flat section $s : \overline{L} \to \hat{X}^k | \overline{L}$. □

Definition 4.5. Let $\overline{X}$ be a compact symplectic manifold. A compact subset $\overline{L} \subset \overline{X}$ has a *branch point* at $p \in \overline{L}$ if there exists a Darboux chart $\phi : U \to (\mathbb{C}^n, \omega_0)$ where U is a neighborhood of p and ϕ is a symplectomorphism sending p to 0, such that $\phi(L)$ is the union of of a collection of disjoint open subsets of a finite collection

$$W_1, \ldots, W_k \subset \mathbb{C}^n$$

of Lagrangian subspaces of $(\mathbb{C}^n, \omega_0)$ and an isotropic subspace $W_\cap \subset \mathbb{C}^n$. A subset $\overline{L} \subset \overline{X}$ is a *branched Lagrangian* for any $p \in \overline{L}$, either $\overline{L}$ is a smooth Lagrangian in an open neighborhood of p or p is a branch point of $\overline{L}$ as defined above.

We have in mind the case that the Lagrangian is locally homeomorphic to the product of a smooth (lower dimensional) Lagrangian with a union of half-lines ending at the origin in the complex plane, as in Figure 6. If $\overline{L} \subset \overline{X}$ is a branched Lagrangian

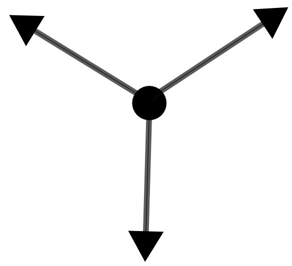

FIGURE 6. Local model for the Lagrangian $\overline{L}$ near Π

then the set of branch points $L_\cap$ is an isotropic submanifold, being locally modelled on the subspace $W_\cap$ near any branch point.

The idea of creating symplectic divisors in rational symplectic manifolds as introduced in [20] was to take zero locus of an "almost" holomorphic section of the line bundle $\hat{X}^{\otimes k}$. We recall the construction of Donaldson divisor in the complement of an isotropic submanifold as done in [5]. The idea is to find a sequence of almost holomorphic sections over tensor powers of the line bundle $\hat{X}^{\otimes k}$ which is concentrated away from zero on the isotropic submanifold. Thus, the zero-locus of such a section will not intersect the isotropic submanifold. In [42], Theorem 3.3 proves that one can chose a Donaldson divisor on the complement of two isotropic which are cleanly intersecting. We extend the ideas introduced there to construct Donaldson divisors on the complement of a branched rational Lagrangian.

Proposition 4.6. *Let $\overline{L}$ be a branched rational Lagrangian submanifold then for large k. There exists a section $\sigma_{k,L}$ of $\hat{X}^{\otimes k}$ which is concentrated over $\overline{L}$ in the sense of [5].*

In preparation for the proof, we cover the singular locus by Darboux balls as follows. Let $\overline{L}$ be a branched rational Lagrangian. Choose Darboux charts at $p \in L_\cap$ so that the intersection with L is the union of Lagrangian subspaces. We will use these charts to construct peak sections $\sigma_{k,p}$ of $\widehat{X}^{\otimes k}$ as done in Auroux-Mohsen-Gay [5]. Define

$$d_k : \overline{L} \times \overline{L} \to \mathbb{R}_{\geq 0}$$

to be the distance with respect to the scaled metric kg where g is the compatible metric on M induced from J. Since $\overline{L}$ is a rational Lagrangian, there exists a flat section

$$\tau_k : \overline{L} \to \widehat{X}^{\otimes k}|_{\overline{L}}.$$

We construct an approximately holomorphic section $\sigma_{k,L_\cap}$ by the process described in [5]. Near any $p \in L_\cap$, there is the symplectomorphism ϕ from Definition 4.5, under which $\overline{L}$ is a union of k Lagrangian subspaces in the local model. Cover $L_\cap$ with such Darboux balls

$$B_1, \ldots, B_N \subset L_\cap$$

of radius 1 with respect to d_k which are centered $\frac{2}{3}$ distance away from each other. Let

$$P_k \subset L_\cap$$

denote the center of the balls. Define $\sigma_{k,L_\cap}$ as follows:

$$\sigma_{k,L_\cap} = \sum_{p \in P_k} \frac{\tau_k(p)}{\sigma_{k,p}(p)} \sigma_{k,p} : X \to \hat{X}^k \tag{27}$$

Lemma 4.7. *For large enough integers k, if $d_k(x, L_\cap) < k^{\frac{1}{30}}$ for $x \in \overline{L}$, then*

$$|\sigma_{k,L_\cap}(x)| > \frac{1}{2} e^{-2(d_k^2(x,L_\cap)+d_k(x,L_\cap))}.$$

Proof. We run through the construction of asymptotically holomorphic sections again, paying careful attention to the differences in phases. Fix some $x \in \overline{L}$ such that $d_k(x, L_\cap) < k^{\frac{1}{30}}$. By the proof of Lemma 4 of [5] verbatim, when $p \in L_\cap$ and $x \in \overline{L}$ are at most $k^{1/10}$ distance apart, their phases are close in the sense that

$$\left| \arg\left(\frac{\sigma_{k,p}(x)}{\tau_k(x)}\right) - \arg\left(\frac{\sigma_{k,p}(p)}{\tau_k(p)}\right) \right| \leq \frac{\pi}{4}. \tag{28}$$

Let $p_f \in P_k$ be the center of the ball which contains the closest point in $\mathcal{L}_\cap$ to x. For a large enough k, the distance from x to the center p_f satisfies the bound

$$d_k(p_f, x) < k^{\frac{1}{20}}.$$

By the proximity of the phases $\arg(\frac{\sigma_{k,p}}{\tau_k})$ to each other in (28),

$$\left| \sum_{p \in P_k | d_k(x,p) < k^{\frac{1}{20}}} \frac{\tau_k(p)}{\sigma_{k,p}(p)} \sigma_p(x) \right| \geq \left| \sigma_{k,p_f}(x) \right|. \tag{29}$$

The cutoff function to define the section σ_{k,p_f} is equal to one at x. From the definition of p_f and the triangle inequality

$$d_k(x, L_\cap) + 1 > d_k(x, p_f).$$

Thus for any $x \in \overline{L}$ with $d_k(x, L_\cap) < k^{\frac{1}{30}}$,

$$|\sigma_{k,p_f}(x)| \geq e^{-d_k^2(x,p_f)} \geq e^{-(d_k(x,L_\cap)+1)^2}. \tag{30}$$

Lemma 3 from [5] implies that there exists $\widetilde{c}, \widetilde{r} > 0$ independent of k, P_k such that

$$\left| \sum_{p \in P_k | d_k(x,p) > k^{\frac{1}{20}}} \frac{\tau_k(p)}{\sigma_{k,p}(p)} \sigma_p(x) \right| \leq \widetilde{c} e^{-\widetilde{r} k^{\frac{1}{20}}} \ \forall x \in X. \tag{31}$$

From the inequalities (30) and (31), we get that for large k,

$$|\sigma_{k,L_\cap}(x)| = \left| \sum_{p\in P_k | d_k(x,p)<k^{\frac{1}{20}}} \frac{\tau_k(p)}{\sigma_{k,p}(p)}\sigma_p(x) + \sum_{p\in P_k | d_k(x,p)>k^{\frac{1}{20}}} \frac{\tau_k(p)}{\sigma_{k,p}(p)}\sigma_p(x) \right|$$

$$\text{so } |\sigma_{k,L_\cap}(x)| \geq \left| \sum_{p\in P_k | d_k(x,p)<k^{\frac{1}{20}}} \frac{\tau_k(p)}{\sigma_{k,p}(p)}\sigma_p(x) \right| - \left| \sum_{p\in P_k | d_k(x,p)>k^{\frac{1}{20}}} \frac{\tau_k(p)}{\sigma_{k,p}(p)}\sigma_p(x) \right|$$

$$\text{so } |\sigma_{k,L_\cap}(x)| \geq e^{-(d_k(x,L_\cap)+1)^2} - \widetilde{c}e^{-\widetilde{r}k^{1/10}}$$

$$\text{so } |\sigma_{k,L_\cap}(x)| \geq \frac{1}{2}e^{-2(d_k^2(x,L_\cap)+d_k(x,L_\cap))}.$$

The last inequality follows from the chain of inequalities

$$\frac{1}{2}e^{-d_k^2(x,L_\cap)-d_k(x,L_\cap)} > \frac{1}{2}e^{-k^{1/15}-k^{\frac{1}{30}}} > \widetilde{c}e^{-\widetilde{r}k^{1/10}}$$

for large k. This gives the desired inequality

$$|\sigma_{k,L_\cap}(x)| \geq \frac{1}{2}e^{-2(d_k^2(x,L_\cap)+d_k(x,L_\cap))}.$$

□

Proof of Proposition 4.6. We aim to construct an approximately holomorphic section concentrated on the Lagrangian by summing the locally Gaussian sections $\sigma_{k,L_\cap}$ as defined in Equation (27) and Lemma 4.7. We define σ_{k,L_i} by summing peak sections on the m Lagrangian branches L_i, while including the sections on the points P_k on $L_\cap$. We start with balls centered around P_k and extend it to a finite cover of L_i by radius $\frac{1}{3}$ balls. Denote the set of centers of these balls as

$$P_k^{\text{pre}} \subset \overline{L}.$$

Iterated removal of points from P_k^{pre} ensures that any two points in P_k^{pre} are at least $\frac{2}{3}$ distance apart. That is, if $p, q \in P_k^{\text{pre}}$ are closer than $\frac{2}{3}$ we drop either p or q from the set P_k^{pre}, without remove points from P_k while doing the iterated removal. The proof of Lemma 4 of [5] goes over verbatim to show that for large k, close points we have that their phases are close: That is, if

$$d_k(p,x) \leq k^{1/10}, \quad p, x \in \overline{L}$$

then

$$\left| \arg\left(\frac{\sigma_{k,p}(x)}{\tau_k(x)}\right) - \arg\left(\frac{\sigma_{k,p}(p)}{\tau_k(p)}\right) \right| \leq \frac{\pi}{4}.$$

We imitate the argument of Theorem 3.3 in Pascaleff-Tonkonog [42]. To match with the notation in [42] do the following renaming: call σ_{k,L_i} as s_i and $\sigma_{k,L_\cap}$ as $s_\cap$. We will drop k from the subscript in the notation since the following bounds for some fixed parameters c_l, c_M, r_l, r_M do not depend on k:

- $|s_\cap(x)| \geq c_l \exp(-r_l(d_k^2(x,L_\cap) + d_k(x,L_\cap)))$ for $x \in \overline{L}$, with $d_k(x,L_\cap) < k^{\frac{1}{3}0}$ from Lemma 4.7,

- $c_l \leq |s_i| < 1$ for all i,
- $|s_i(x)| \leq c_M \exp(-r_M(d_k^2(x,L)))$.

Define $c_\theta \in \mathbb{R}$ such that for $p \in L_i, q \in L_j$ with $i \neq j$ which are at a distance d from $L_\cap$, then $d_k(p,q) \geq c_\theta d$. Such a constant c_θ exists because the angles between L_i, L_j have a lower bound, by assumption. Let $\delta > 0$ be large enough so that

$$c_M \exp(-r_M c_\theta^2 \delta^2) < \frac{c_l}{10m^2}.$$

This choice of δ does not depend on k. Set

$$c := \frac{c_l \exp(-r_l(\delta^2 + \delta))}{10m^2}. \tag{32}$$

and

$$s := s_\cap + c \sum s_i : \overline{X} \to \hat{X}^k.$$

For k large, we have $\delta < k^{\frac{1}{30}}$. Thus, in a δ neighborhood of $L_\cap$, from Lemma (4.7), we have that

$$|s_\cap(x)| > c_l \exp(-r_l(\delta^2 + \delta)) \; \forall x \in \overline{L}$$

on L_i. By (32) the sections

$$|cs_j| < \frac{1}{10m^2} c_l \exp(-r_l(\delta^2 + \delta)).$$

Thus, their sum is bounded above in a δ neighborhood of $L_\cap$ as follows:

$$\left| \sum_{j=1}^m cs_j \right| \leq m \frac{c_l \exp(-r_l(\delta^2 + \delta))}{10m^2} = \frac{c_l \exp(-r_l(\delta^2 + \delta))}{10m}.$$

Thus we have in a δ neighborhood of $L_\cap$,

$$|s| \geq |s_\cap| - \left| \sum_{j=1}^m cs_j \right| \geq \left(1 - \frac{1}{10m}\right) c_l \exp(-r_l(\delta^2 + \delta)).$$

When $x \in L_i$ and outside an open neighborhood of δ distance from $L_\cap$ we have that $s_\cap + cs_i$ is bounded below by cc_l on L_i whereas we have an upper bound

$$|cs_j| \; < cc_M \exp(-r_M c_\theta \delta^2) < \frac{cc_l}{10m^2}.$$

Thus, after adding the $m-1$ sections cs_j, for $j \neq i$, to $s_\cap + cs_i$ we have the following lower bound for any point on L_i:

$$\left| s_\cap + cs_i + \sum_{i \neq j} cs_j \right| \geq |s_\cap + cs_i| - \sum_{i \neq j} |cs_j| \geq cc_l - cc_l \frac{m-1}{10m^2} > \frac{cc_l}{10m^2}.$$

We get a global lower bound of s on any point in $\overline{L}$:

$$|s(x)| \geq \min \left\{ \frac{cc_l}{10m^2}, c_l \exp(-r_l(\delta^2 + \delta)) \right\}.$$

$\square$

Corollary 4.8. *If L is a rational Lagrangian cobordism then there exists a Donaldson hypersurface D of arbitrarily large degree in $\overline{X}$ with the property that $\overline{L}$ is exact in the complement of D. In particular, any holomorphic disk in $\overline{X}$ bounding $\overline{L}$ with positive area intersects D in at least one (interior) point.*

Proof. Since $\overline{L}$ is a branched Lagrangian in $\overline{X}$, the claim follows from Lemma 4.6 and the perturbation technique described in Donaldson [20] and Auroux-Gayet-Mohsen [5]. $\square$

4.3. **Transversality.** We consider the moduli space of treed disks with interior edges mapping to a Donaldson hypersurface as constructed in the previous section. As in Cieliebak-Mohnke [14], for adapted maps some of the interior edges correspond to the intersections with the Donaldson hypersurface.

Definition 4.9. An *adapted treed building* is a treed building $u : C \to X$ with the property that each component of $u^{-1}(D)$ contains an edge $T_e, e \in \text{Edge}_D(\mathbb{\Gamma})$ and each such edge T_e is contained in $u^{-1}(D)$.

Thus the domains of stable adapted treed buildings are automatically stable, except for trivial cylinders: For a non-constant sphere component $u_v : S_v \to X$ there must be at least three edges $T_e \subset C, e \in \text{Edge}_D(\mathbb{\Gamma})$ incident to S_v. Any holomorphic disk $u : S \to X$ not contained in a fiber of the projection to Y meets D in at least one interior point, by the exactness of Π in $X - D$.

Domain-dependent perturbations are defined as maps from the universal curve. Suppose that Γ is a stable type of domain and denote by $\mathcal{M}_\Gamma$ the moduli space of disks of type Γ. Over $\mathcal{M}_\Gamma$ we have a universal curve $\mathcal{U}_\Gamma$ which decomposes into tree and surface parts

$$\mathcal{U}_\Gamma = \{[C, z], z \in C\}, \quad \mathcal{U}_\Gamma = \mathcal{S}_\Gamma \cup \mathcal{T}_\Gamma$$

depending on whether the point z lies in the surface S or tree T part of C.

Definition 4.10. A *domain-dependent perturbation* is a collection of maps

$$\begin{array}{rcccl} J_\Gamma & : & \mathcal{S}_\Gamma & \to & \mathcal{J}_{\text{cyl}}(X) \\ \zeta_{\bullet,\Gamma} & : & \mathcal{T}_{\bullet,\Gamma} & \to & \text{Vect}(\Lambda \times \mathbb{R})^{\mathbb{R}} \\ \zeta_{\circ,\Gamma} & : & \mathcal{T}_{\circ,\Gamma} & \to & \text{Vect}(\mathcal{R}(\Lambda)) \end{array}$$

consist of a domain-dependent cylindrical almost complex structure J_Γ and domain-dependent vector fields $\mathcal{T}_{\bullet,\Gamma}$ and $\mathcal{T}_{\circ,\Gamma}$ so that the corresponding maps

$$\begin{array}{rcl} \mathcal{S}_\Gamma \times TX & \to & TX \\ \mathcal{T}_{\bullet,\Gamma} \times L & \to & TL \\ \mathcal{T}_{\circ,\Gamma} \times \mathcal{R}(\Lambda) & \to & T\mathcal{R}(\Lambda) \end{array}$$

are smooth, $\zeta_\Gamma(z)$ is of the form in (b), (13) on each edge depending on the label of the edge. Later, we will assume that the perturbations vanish on an open subset containing the zeros of the vector fields and the intersections $\mathcal{S}_\Gamma \cap \mathcal{T}_\Gamma \subset \mathcal{U}_\Gamma$.

One may also allow Hamiltonian perturbations. These are not necessary for regularization of the moduli spaces in the case of Legendrians with embedded projection but is necessary for those with immersed Lagrangian projections. The use of Hamiltonian perturbations is also useful for our later localization computations. Let

$$\mathrm{Vect}_h(X,U) \subset \mathrm{Vect}(X)$$

denote the space of $\mathbb{R}$-invariant Hamiltonian vector fields vanishing on an open subset U (which later will be taken to include the Donaldson hypersurface used for regularization). A domain-dependent Hamiltonian perturbation is a one-form

$$H_\Gamma \in \Omega^1(\mathcal{S}_\Gamma, \mathrm{Vect}_h(X,U)^{\mathbb{R}}).$$

A *domain-dependent perturbation* is a tuple

$$P_\Gamma = (J_\Gamma, \zeta_{\circ,\Gamma}, \zeta_{\bullet,\Gamma}, H_\Gamma)$$

as above.

Domain-dependent perturbations help us obtain transversality of moduli spaces by breaking the symmetry of the auxiliary choices such as the almost complex structure and Morse functions. Although beneficial for obtaining transversality, it comes with a cost: We lose expected symmetry such as the divisor axiom for curve counts. To overcome this difficulty we use multi-valued domain dependent perturbation which allows us to restore part of the expected symmetry back while still preserving transversality.

Definition 4.11. A *multi-valued domain-dependent perturbation* is a formal sum of domain-dependent perturbations with real coefficients

$$\widetilde{P}_\Gamma := \sum_{i=1}^{k_\Gamma} c_i P_{\Gamma,i} \tag{33}$$

where

$$\sum_{i=1}^{k_\Gamma} c_i = 1, \quad c_i > 0.$$

We further impose the condition that $k_\Gamma = 1$ for all Γ with number of outgoing edges of the combinatorial type Γ at least 1. We call the terms $P_{\Gamma,i}$ *sheets* of the multi-valued perturbation and the terms c_i are the *weights* of the corresponding sheets. Two multi-valued perturbations are considered equivalent if they are related by replacements of the form

$$c_i P_{\Gamma,i} + c_j P_{\Gamma,j} = (c_i + c_j) P_{\Gamma_i} = (c_i + c_j) P_{\Gamma,j} \tag{34}$$

whenever $P_{\Gamma,i} = P_{\Gamma,j}$. The number of sheets is the minimum of the number k_Γ over all representatives, that is, the number of distinct $P_{\Gamma,j}$.

Our perturbations of the almost complex structure are required to be small In the following, sense.

Definition 4.12. Define a (discontinuous) function recording the number of interior edges meeting the surface component containing the given point:

$$n_D : \mathcal{S}_\Gamma \to \mathbb{Z}_{\geq 0}, \quad z \mapsto \max_{z \in S_v} \#\{e \in \mathrm{Edge}_D(\Gamma), T_e \cap S_v \neq \emptyset\}.$$

Suppose that $[D] = k[\omega_Y]$ has degree k. A domain-dependent almost complex structure $J_\Gamma : \mathcal{S}_\Gamma \to \mathcal{J}(X)$ is *stabilizing* if

$$J_\Gamma(z) \in \mathcal{J}^{<n_D/k}(Y), \quad \forall z \in \mathcal{S}_\Gamma.$$

The set of perturbations satisfying the stabilizing condition contains a non-empty open subset of domain-dependent almost complex structures, by definition.

We will require that the perturbations for each type agree with those already chosen on the boundary. The following morphisms of graphs $\Gamma_2 \to \Gamma_1$ correspond to inclusions of $\overline{\mathcal{M}}_{\Gamma_1}^{<E}(\Lambda)$ in the formal boundary of $\overline{\mathcal{M}}_{\Gamma_2}^{<E}(\Lambda)$.

Definition 4.13. An *elementary morphism* of domain types $\Gamma_1 \to \Gamma_2$ is one of the following:

(a) (Collapsing or breaking edges) An isomorphism $\Gamma_1 \to \Gamma_2$ where Γ_2 is obtained from Γ_1 by removing an edge e from a set $\mathrm{Edge}_0(\Gamma)$ or $\mathrm{Edge}_\infty(\Gamma)$ and placing it in $\mathrm{Edge}(\Gamma)$; corresponding to a degeneration of C in which the length $\ell(e)$ of $e \in \mathrm{Edge}(\Gamma)$ becomes infinite or zero.

(b) (Cutting edges) A morphism $\Gamma_1 \to \Gamma_2$ where Γ_2 is obtained from Γ_1 by identifying two vertices v_-, v_+ joined by an edge e which is then collapsed, corresponding to a degeneration of C in which a component $S_v \subset S$ develops a node, either in the interior or boundary depending on the collapsed edge. Thus S_v becomes the union of two components S_{v_1}, S_{v_2}.

We apply a Sard-Smale argument to show that the moduli space of expected dimension at most one is cut out transversally for generic perturbations. For this, it is convenient to adopt a space of perturbations introduced by Floer.

Definition 4.14. Let $\underline{\epsilon} = (\epsilon_0, \epsilon_1, \epsilon_2, \ldots)$ be a sequence of positive real numbers. For a smooth function $f : \mathbb{R} \to \mathbb{R}$ define the $\underline{\epsilon}$-norm

$$\|f\|_{\underline{\epsilon}} = \sum_{j \geq 0} \epsilon_j \sup \|f\|_{C^j}$$

denote the (possibly infinite) *Floer norm* of f, as in [48, Appendix B].

For sequences $\underline{\epsilon}$ which converge to zero sufficiently fast, the space $C^{\underline{\epsilon}}$ of functions with finite Floer norm contains bump functions supported in arbitrarily small neighborhoods of any point. Similar spaces are defined for functions on any manifolds, sections of vector bundles and so on. Let

$$\mathcal{P}_\Gamma^{\underline{\epsilon}} = \{P_\Gamma \in C^{\underline{\epsilon}}\}$$

denote the space of perturbations for type Γ of finite $\underline{\epsilon}$-norm. Let

$$\mathcal{B}^{\mathrm{univ}}_{k,p,\underline{\epsilon},\lambda} = \mathcal{M}_\Gamma \times \mathrm{Map}(C, X)_{k,p,\lambda} \times \mathcal{P}^{\underline{\epsilon}}$$

the space of triples (C, u, P) where $P = (J, \zeta)$ is a perturbation, and the subscript k, p, λ indicates that the restriction of u to each component T_e, S_v is in the weighted

Sobolev class $W^{k,p,\lambda}$. Denote by

$$\begin{aligned}\mathcal{E}^{\mathrm{univ}}_{k-1,p,\underline{\epsilon},\lambda} &= \bigcup_{(C,u,P)\in\mathcal{B}^{\mathrm{univ}}_{k,p,\underline{\epsilon},\lambda}} \Omega^{0,1}(C,(u|S)^*TX)_{k-1,p,\lambda} \\ &:= \bigcup_{(C,u,P)\in\mathcal{B}^{\mathrm{univ}}} \Omega^{0,1}(S,u^*TX)_{k-1,p,\lambda}\oplus\Omega^1(T,(u|T)^*TL)_{k-1,p}\end{aligned}$$

where

$$\Omega^1(T,(u|T)^*TL)_{k-1,p} = \oplus\Omega^1(T_\circ,(u|T_\circ)^*TL)_{k-1,p}\oplus\Omega^1(T_\bullet,(u|T_\bullet)^*T\mathcal{R}(\Lambda))_{k-1,p}.$$

The space $\mathcal{E}^{\mathrm{univ}}_{k-1,p,\epsilon,\lambda}$ is a vector bundle whose fibers are the space of pairs $0,1$-forms η_S on the surface part and η_T on the tree part, with values in the pullbacks of TX, $T\mathcal{R}(\Lambda), TL$ respectively. Define a section

$$\begin{aligned}(35)\quad & \mathcal{F}^{\mathrm{univ},i}_\Gamma : \mathcal{B}^{\mathrm{univ},i}\to\mathcal{E}^{\mathrm{univ},}, \\ & (C',u,P_\Gamma)\mapsto\left(\overline{\partial}_{j(C'),J_\Gamma,H_\Gamma}u_S,\left(\frac{d}{ds}+\zeta_{C',\Gamma,\circ}(u_{T_\circ})\right),\left(\frac{d}{ds}+\zeta_{C',\Gamma,\bullet}(u_{T_\bullet})\right)\right)\end{aligned}$$

where $\overline{\partial}_{j(C'),J_\Gamma}$ is the Cauchy-Riemann operator defined by the choice of curve C' in the domain, and $\zeta_{C',\Gamma,\cdot}$ is the perturbed vector field determined by the choice of domain C'.

Theorem 4.15. *The map $\mathcal{F}^{\mathrm{univ},i}_\Gamma$ is transverse to the zero section at a map $u : C\to X$ as long as there are no constant spheres $S_v, v\in\mathrm{Vert}(\Gamma)$ with more than one edge $e\in\mathrm{Edge}_D(\Gamma)$ attached.*

Proof. We treat transversality for constant, non-constant, and trivial strip components separately. Let η be an element of the cokernel $\mathrm{coker}(\tilde{D}_u)$ of the linearized operator of (24), as usual identified with an element of the kernel $\ker(\tilde{D}^*_u)$ of the adjoint

$$\eta\in\Omega^{0,1}(S,(u|T)^*TX)_{1-k,p,-\lambda}\oplus\Omega^1(T_\bullet,(u|T_\bullet)^*TL)_{1-k,p}\oplus\Omega^1(T_\circ,(u|T_\circ)^*T\Lambda)_{1-k,p}.$$

We write the restriction of η to the sphere or disk components resp. edges as

$$\eta_v\in\Omega^{0,1}(S_v,u_v^*TX), v\in\mathrm{Vert}(\Gamma)$$

and

$$\eta_e\in\Omega^1(T_e,u_e^*TL),\ e\in\mathrm{Edge}_\bullet(\Gamma),\quad \eta_e\in\Omega^1(T_e,u_e^*T\Lambda),\ e\in\mathrm{Edge}_\circ(\Gamma).$$

We wish to show that all of the components η_v and η_e vanish.

First we show that the one-form vanishes on components on which the map is non-constant. Suppose $v\in\mathrm{Vert}(\Gamma)$ with map $u_v := u|S_v$ mapping to X such that u_v is non-constant. Then $D^*_{u_v}\eta_v = 0$ weakly and η_v is perpendicular to variations $K_\Gamma\in T_{J_\Gamma}\mathcal{J}_\Gamma(X)$ which are domain-dependent. Standard techniques imply that η_v vanishes in an open neighborhood of any point $z\in S_v$ on which $\mathrm{d}u_v$ is non-zero. Thus η_v vanishes by the unique continuation principle in [39, Section 2.3].

Next we show that the one-form vanishes on a component mapping to the neck on which the projection to the base is non-constant. Suppose $v\in\mathrm{Vert}(\Gamma)$ with map $u_v := u|S_v$ such that $\overline{u}_v := p\circ u_v : S_v\to Y$ non-constant. The restriction η_v is a weak solution to $D^*_{u_v}\eta_v = 0$ and η_v is perpendicular to one-forms produced by variations K_Γ in J_Γ. Such variations satisfy $J_\Gamma K_\Gamma = -K_\Gamma J_\Gamma$ and vanish on the first factor in the decomposition

$$u^*TX \cong u^* \ker(Dp) \oplus \overline{u}^*TY.$$

Supposet $\eta_v(z) \neq 0$. Choose a (J_Γ, j)-antilinear map from T_zS to $T_{u(z)}X$ pairing non-trivially with $\eta_v(z)$, which we may take to be of the form $K_\Gamma(z)\mathrm{d}_z uj(z)$ for some $K_\Gamma(z)$ since the second component of $\mathrm{d}_z u$ is non-vanishing. Using a cutoff function, we obtain a one-form $K_\Gamma \mathrm{d}uj$ that pairs non-trivially with η, which is a contradiction. Thus η_v vanishes.

The case of components mapping to fibers in the neck piece, that is, to the images of Reeb chords or orbits, requires special consideration and is typically the hardest part of the transversality argument in symplectic field theory. Any disk $u_v : S_v \to X$ contained in a fiber

$$p^{-1}(y) \cong \mathbb{CP}^1, y \in Y$$

is by assumption a disk bounding

$$L \cap p^{-1}(y) \cong e^{i\theta_1}\mathbb{R}P^1 \cup \ldots e^{i\theta_k}\mathbb{R}P^1.$$

The linearized operator D_{u_v} is a direct sum

$$D_{u_v} \cong D^{\mathbb{v}}_{u_v} \oplus D^{\mathbb{h}}_{u_v}$$

of a trivial horizontal part $D^{\mathbb{h}}_{u_v} \cong \overline{\partial}$ taking values in $T_{\overline{u}_v}Y$ and a vertical part $D^{\mathbb{v}}_{u_v}$ taking values in $\mathbb{C}$ which has boundary condition $\mathbb{R}$. We examine the kernel and the cokernel separately. The operator D_{u_v} splits into a sum of the standard operator $\overline{\partial}$ on the one-dimensional factor, c.f. Ekholm [21, Lemma 6.8]. The kernel of $D^{\mathbb{h}}_{u_v} \cong \overline{\partial}$ on S_v consists of bounded holomorphic functions $\xi : S_v \to \mathbb{C}$ with real boundary values. Therefore $\ker(D^{\mathbb{h}}_{u_v})$ is the one-dimensional real vector space space of constant functions with values in the reals. On the other hand, the cokernel $\operatorname{coker}(D^{\mathbb{h}}_{u_v})$ may be identified with holomorphic $0,1$-forms η of Sobolev class $-k, p, \lambda$ enforcing exponential decay (since constant functions near infinity lie in the domain). In the local coordinate on $\overline{S}_v$ near each puncture such a holomorphic $0,1$-form is vanishing at the puncture. Since there are no such holomorphic $0,1$-forms η on a disk S_v, the cokernel of $D^{\mathbb{h}}_{u_v}$ vanishes. As in Oh [40], any rank one Riemann-Hilbert problem on $\overline{S}$ has either vanishing kernel or vanishing cokernel.

Tranversality of the gradient flow equation on both types of edges follows from the fundamental theorem of ordinary differential equations. For any edge $e \in \mathrm{Edge}_\circ(\Gamma)$ the cokernel of the operator D_{u_e} on zero-forms on T_e is identified, via Hodge star, with the space of solutions to the adjoint equation

$$D^*_{u_e}\xi_e = \nabla\xi_e - \nabla \operatorname{grad}(\zeta(u_e))\mathrm{d}s$$

with vanishing at the endpoints; there are no such solutions. In the case without endpoints $T_e \cong \mathbb{R}$ that may occur in a broken trajectory, transversality follows from the Morse-Smale assumption on the Morse function $f_\circ$. On the other hand, D_{u_e} has kernel isomorphic to $T_{u_e(z)}L$ for any point $z \in T_e$, via evaluation.

It remains to show that the matching conditions are transversally cut out. Consider a tree $\Gamma_0 \subset \Gamma$ such that for each vertex $v \in \mathrm{Vert}(\Gamma_0)$, the component $u_v := u|S_v$ is vertical. The argument of the previous section shows that such configurations are transversally cut out if each S_v is a disk or sphere and Γ_0 is a tree: Let u_0 denote the restriction of u to S_0 and $\ker(D_{u_0})'$ the kernel of the linearized operator acting on sections without matching conditions enforced at the nodes of S_0. By the previous paragraph $\ker(D_{u_0})'$ is a sum of factors of $\mathbb{C} \oplus T_{p \circ u(C)} Y$ for each sphere component and $\mathbb{R} \oplus T_{p\circ u(C)}\Pi$ for each disk component. Transversality is equivalent to the condition that the map

$$\ker(D_{u'})' \to \bigoplus_{e \in \mathrm{Edge}_{\bullet,\mathrm{fin}}(\Gamma)} T_{p\circ u(C)} Y \oplus \bigoplus_{e \in \mathrm{Edge}_{\circ,\mathrm{fin}}(\Gamma)} T_{p\circ u(C)} \Pi$$

is surjective. This follows from an induction starting with the nodes contained in terminal edges of Γ_0.

Putting everything together, we obtain that the operator D_u' whose domain consists of sections without matching conditions is surjective, and the kernel of D'_{u_v} surjects onto any intersection point $TL|S_v \cap T$ or $T\Pi|S_v \cap T$. Similarly the kernel of D'_{u_e} surjects onto any intersection point $TL|T_e \cap S$. It follows that the matching conditions at the nodes are cut out transversally. □

Perturbations are constructed by an induction on the dimension of the source moduli space. Suppose that single-valued perturbations $P_{\Gamma'}$ have been chosen for all types $\Gamma' \prec \Gamma$. A natural gluing procedure gives perturbations P_Γ in an open neighborhood U_Γ of the boundary of $\overline{\mathcal{M}}_\Gamma$. The gluing theorem for linearized operators (see, for example, Wehrheim-Woodward [47, Theorem 2.4.5] for the closed case) implies that for U_Γ sufficiently small, the perturbation P_Γ is regular over U_Γ if all the perturbations $P_{\Gamma'}$ are regular. The case of multi-valued perturbations is the same, with the following caveat: It could be the boundary of $\mathcal{M}_\Gamma$ is disconnected, so that there are for examples two types $\Gamma_1, \Gamma_2 \preceq \Gamma$ so that P_{Γ_1} and P_{Γ_2} have different number of sheets, say k_1 and k_2 respectively. In this case there is an obvious replication process which $P_{\Gamma_1}, P_{\Gamma_2}$ are each replaced by multi-valued perturbations with $k_1 k_2$ sheets, with each of the original sheets in P_{Γ_1} repeated k_2 times, and similarly for P_{Γ_1}. The gluing construction then proceeds as before.

Similarly, given a type Γ obtained from Γ_1, Γ_2 by gluing along an edge, any multivalued perturbations $P_{\Gamma_1}, P_{\Gamma_2}$ for Γ_1, Γ_2 with k_1, k_2 sheets respectively induces a multivalued perturbation for type Γ with $k_1 k_2$ sheets.

Theorem 4.16. *Suppose that regular perturbations $P_{\Gamma'}$ have been chosen for all types $\Gamma' \prec \Gamma$. Then there exists a comeager set $\mathcal{P}_\Gamma^{\mathrm{reg}} \subset \mathcal{P}_\Gamma$ of perturbations P_Γ agreeing with the glued perturbations arising from $P_{\Gamma'}$ in an open neighborhood U_Γ of the boundary of $\overline{\mathcal{M}}_\Gamma$ such that any $P_\Gamma \in \mathcal{P}_\Gamma^{\mathrm{reg}}$ is regular.*

Proof. This is a standard Sard-Smale argument. By Theorem 4.15 the universal moduli space (locally over the moduli space) is transversally cut out. Consider the forgetful map

$$f_{\mathbb{\Gamma}}^i : \mathcal{M}^{\mathrm{univ},i}_{\mathbb{\Gamma},k,p,\underline{\epsilon},\lambda}(L) \to \mathcal{P}^{\underline{\epsilon}} \quad (u : C \to X, P_\Gamma) \mapsto P_\Gamma.$$

For map types $\mathbb{\Gamma}$ of expected dimension at most one, the index of $f^i_{\mathbb{\Gamma}}$ is at most one and by Sard-Smale the set of regular values of $f^i_{\mathbb{\Gamma}}$ is comeager. Since the space of map types $\mathbb{\Gamma}$ with underlying domain type Γ is countable, there exists a comeager set $\mathcal{P}^{\mathrm{reg}}_{\Gamma} \subset \mathcal{P}_{\Gamma}$ of perturbations P_{Γ} extending the given perturbations on the boundary so that $\mathcal{M}^{\mathrm{univ},i}_{\mathbb{\Gamma},k,p,\mathbb{E},\lambda}(L)$ is regular for every map type $\mathbb{\Gamma}$ of expected dimension at most one. $\square$

Theorem 4.17. *For any regular perturbation system $\underline{P} = (P_{\Gamma})$, for any map type $\mathbb{\Gamma}$ decorating Γ of expected dimension at most one the moduli space $\mathcal{M}_{\mathbb{\Gamma}}(L)$ is transversally cut out and for any pair $\mathbb{\Gamma}_1 \prec \mathbb{\Gamma}_2$ with $\mathbb{\Gamma}_1$ resp $\mathbb{\Gamma}_2$ having expected dimension zero resp. one. there is a natural gluing map $\mathcal{M}_{\mathbb{\Gamma}_1}(L) \times [0,\epsilon) \to \mathcal{M}_{\mathbb{\Gamma}_2}$.*

The Theorem, for which we do not give a proof, is a consequence of gluing results similar to those in Ekholm-Etnyre-Sullivan [22, Chapter 10] and Palmer-Woodward [41]. Given a regular stratum-wise rigid treed holomorphic building $u : C \to \mathbb{X}$ of type Γ_1 representing a codimension one stratum $\mathcal{M}_{\Gamma_1}$ in the boundary of a top-dimensional stratum $\mathcal{M}_{\Gamma_2}$ there exists a unique family $u_\rho : C_\rho \to X_\rho$ of treed holomorphic curves of type Γ_2 converging to u. In the case of collapsing edges or making edges finite, each such sequence u_ρ is defined by first constructing an *approximate solution*: One removes small balls around a node and uses cutoff functions and geodesic exponential to construct the approximate solutions while in the case of trajectories one removes a small ball around the intermediate critical point and patches together using cutoff function. On the other hand, the case of making zero-length edges positive length follows immediately from the implicit function theorem.

4.4. **Intersection multiplicities.** We develop a notion of fractional intersection multiplicity with the divisors at infinity, related to the sum of angles of Reeb chords.

Definition 4.18. A *Thom form* for the inclusion $Y_\pm \to \overline{X}$ is a two-form on $\overline{X}$ given as follows. Let $\rho : [0,\infty) \to [0,1]$ be a compactly supported bump function equal to 1 in an open neighborhood of 0. The corresponding Thom form is

$$\tau_{Y_\pm} = -\frac{1}{2\pi}\mathrm{d}(\rho\alpha_\pm) \in \Omega^2(\overline{X},\overline{L})$$

where $\Omega^2(\overline{X},\overline{L})$ denotes the space of two-forms on X vanishing on each stratum of $\overline{L}$. The Thom form is closed and the *Thom* class $[\tau_{Y_\pm}] \in H^2(\overline{X},\overline{L})$ is independent of the choice of bump function ρ.

Fractional intersection multiplicities with the divisors at infinity are defined by integrating the Thom class. Let $u : (S,\partial S) \to (\overline{X},\overline{L})$ be a punctured surface bounding. The *intersection number* of u with $Y_\pm$ is the pairing of $[u]$ with $[\tau_{Y_\pm}]$. As usual, the intersection number may be computed as a sum of (now fractional) local intersection numbers, proportional to the sum of angles at the intersection points:

Lemma 4.19. *Suppose that $u : (S,\partial S) \to (X,L)$ is a punctured disk asymptotic to a collection of Reeb orbits and chords $\gamma_e, e \in \mathcal{E}(S)$. The intersection number $[\overline{u}]\cdot[Y_\pm]$*

is equal to the sum of the angles

$$[\overline{u}] \cdot [Y_\pm] = \frac{1}{2\pi} \sum_{e \in \mathcal{E}_\pm(S)} \theta_e$$

of the Reeb chords and orbits limiting to $Y_\pm$.

Proof. The proof is an argument using Stokes' formula. The cutoff-function ρ is equal to 1 near the zero section. Hence

$$\begin{aligned} \lim_{s \to \infty} \int_{S - \cup_{e \in \mathcal{E}_\pm(S)} \kappa_e^{-1}(\pm(s,\infty) \times Z_\pm)} \mathrm{d}(\rho \alpha_\pm) &= \lim_{s \to \infty} \int_{\partial(S - \cup_{e \in \mathcal{E}_\pm(S)} \kappa_e^{-1}(\pm(s,\infty) \times Z_\pm))} \alpha_\pm \\ &= \sum_{e \in \mathcal{E}_\pm(S)} \theta_e \end{aligned}$$

as claimed. □

4.5. **Compactness.** We will need a version of Gromov compactness for buildings in cobordisms with Lagrangian boundary conditions, adapted to the case of domain-dependent perturbations. In order for compactness to hold we must assume that the perturbations satisfy coherence properties.

We first note the following construction. Given a type Γ denote by $\Gamma_\circ$ the type obtained by collapsing all sphere components $S_v \subset S$ to interior leaves, and for any vertex $v \in \mathrm{Vert}(\Gamma)$ let

$$\Gamma(v) = (\{v\}, \{e \in \mathrm{Edge}(\Gamma) | e \ni v\}$$

denote the union of edges meeting v. Let

$$\mathcal{U}_{\Gamma,v} = \{(C, z) \in \mathcal{U}_\Gamma | z \in S_v\}$$

denote the union of points lying on the components $S_v \subset S$.

Definition 4.20 (Multivalued pull-back)**.** Let Γ be a combinatorial type of treed disc which has an edge of infinite length, and thus obtained from gluing Γ_1 and Γ_2. Let

$$\widetilde{P}_{\Gamma_i} = \sum_{j=1}^{k_i} c_{k,i} P_{\Gamma_i,k}$$

be multivalued perturbations for $i \in \{1,2\}$. The multivalued pull-back, $\widetilde{P}_\Gamma$, of $\widetilde{P}_{\Gamma_1}$ and $\widetilde{P}_{\Gamma_2}$ is defined as

$$\widetilde{P}_\Gamma = \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} c_{j_1,1} c_{j_2,2} P_{\Gamma,j_1,j_2},$$

where P_{Γ,j_1,j_2} is defined as the pull-back of P_{Γ_1,j_1} and P_{Γ_2,j_2} under the isomorphism

$$\overline{\mathcal{U}}_\Gamma \to \pi_1^* \overline{\mathcal{U}}_{\Gamma_1} \times \pi_2^* \overline{\mathcal{U}}_{\Gamma_2}.$$

Here

$$\pi_b : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\Gamma_b}, b \in \{1,2\}$$

are the natural projections.

Definition 4.21. A collection of multi-valued perturbations

$$\underline{P} = (\widetilde{P}_\Gamma \in \mathcal{P}_\Gamma)$$

for all types of domain Γ is *coherent* if the following holds:

(a) (Collapsing Edges) Whenever $\Gamma_1 \to \Gamma_2$, so that Γ_2 is obtained from Γ_1 by collapsing edges, the number of sheets of $\widetilde{P}_{\Gamma_1}$ is equal to the number of sheets of $\widetilde{P}_{\Gamma_2}$ and each sheet $P_{\Gamma_2,i}$ restricts to $P_{\Gamma_1,i}$ on $\mathcal{U}_{\Gamma_2}$ and their corresponding weights match $c_i^{\Gamma_1} = c_i^{\Gamma_2}$. That is,

$$J_{\Gamma_2,i} : \mathcal{S}_{\Gamma_2} \to \mathcal{J}(X)$$

restricts to $J_{\Gamma_1,i}$ on $\mathcal{S}_{\Gamma_1} \subset \overline{\mathcal{S}}_{\Gamma_2}$, the vector field

$$\zeta_{\Gamma_2,\bullet,i} : \mathcal{T}_{\Gamma_2,\bullet,i} \to \operatorname{Vect}(X)$$

restricts to $\zeta_{\Gamma_1,\bullet,i}$ on $\mathcal{T}_{\Gamma_1,\bullet,i}$ and similarly for the vector fields $\zeta_{\Gamma_2,\circ,i}, \zeta_{\Gamma_1,\circ,i}$.

(b) (Breaking Edges) Whenever Γ has an edge of infinite length, and so obtained from gluing Γ_1 and Γ_2, then P_Γ restricts to the multivalued pull-backs of $\widetilde{P}_{\Gamma_1}$ and $\widetilde{P}_{\Gamma_2}$ under the isomorphism

$$\overline{\mathcal{U}}_\Gamma \to \pi_1^* \overline{\mathcal{U}}_{\Gamma_1} \times \pi_2^* \overline{\mathcal{U}}_{\Gamma_2}$$

where $\pi_b : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\Gamma_b}, b \in \{1,2\}$ are the natural projections.

(c) (Locality) Let $C_\circ \subset C$ be the subset consisting only of disk components and boundary edges. We have a natural map

$$f_{\Gamma,v} : \mathcal{U}_{\Gamma,v} \to \mathcal{U}_{\Gamma(v)} \times \mathcal{M}_{C_\circ}$$

which sends the pair $(C, z \in S_v)$ to the pair $((S_v, z \in S_v), C_\circ)$. We require that the restriction of the perturbation J_Γ to S_v is pulled back under $f_{\Gamma,v}$. In particular, if Γ' is another component obtained from Γ by removing an interior leaf T_e on some sphere component v then P_Γ induces a perturbation datum $P_{\Gamma'}$ for the type Γ' by taking $J_{\Gamma'}$ to be constant on S'_v and equal to J_Γ on the other components. Any P_Γ holomorphic map from a curve C of type Γ to X induces a $P_{\Gamma'}$-holomorphic map from the curve C' to X, where C' is obtained from C by forgetting T_e.

We have the following version of sequential Gromov compactness.

Theorem 4.22. *Any sequence of adapted stable buildings $u_\nu : C_\nu \to \mathbb{X}$ with boundary in $\mathbb{R} \times \Lambda$ with bounded area $A(u_\nu)$ and bounded number of leaves $n(u_\nu)$ has a subsequence Gromov-converging to a stable building $u : C \to \mathbb{X}$.*

Proof. The limits of the surface parts and tree parts of the sequence may be constructed separately. For convergence on the curve parts, see Chanda [9] or, for the stable Hamiltonian case, Venugopalan-Woodward [45] (the latter without Lagrangian boundary condition). For convergence on the tree parts, let $u_\nu : C_\nu \to \mathbb{X}$ be a sequence as in the statement of the Theorem. Each edge $e \in \operatorname{Edge}(\Gamma)$ in a level mapping to $\mathbb{R} \times \Lambda$ gives rise to a sequence of trajectories by restriction

$$u_{e,\nu} : [-T_\nu, T_\nu] \to \mathbb{R} \times \Lambda.$$

By compactness of gradient trajectories up to breaking, the projections

$$\overline{u}_{e,\nu} := p \circ u_\nu : [0, T_\nu] \to \Pi$$

converge after passing to a subsequence to a (possibly broken) trajectory. That is, that the exist sequences t_ν so that the trajectories $\overline{u}_{e,\nu}(t + t_\nu)$ converge in all derivatives on compact sets to some limit

$$\overline{u}_{e,i} : [-T_{e,i}, T_{e,i}] \to \Lambda$$

which is a trajectory of of $\operatorname{grad}(f_\circ)$, for some $T_{e,i} \in [0, \infty]$; see for example Audin-Damian [4]. Choose a sequence $T_\nu \in \mathbb{R}$ so that the point $T_\nu u_{e,\nu}(0)$ (that is, the translation of $u_{e,\nu}(0)$ in the $\mathbb{R}$-direction by T_ν) converges to some limiting point $u_e(0)$; such as a sequence T_ν exists since Λ is compact. By the fundamental theorem of ordinary differential equations, the sequence $u_{e,\nu}$ converges in all derivatives on compact sets to a trajectory of ζ_ν. Since the projections $\overline{u}_{e,i}$ form a broken trajectory, the lifts $u_{e,i}$ form a broken trajectory. □

We analyze the boundary of the one-dimensional moduli spaces $\overline{\mathcal{M}}(\Lambda)_1$ resp. $\overline{\mathcal{M}}(L)_1$ from (25), assuming that every map is regular.

Definition 4.23. (a) Strata $\mathcal{M}_{\mathbb{\Gamma}}(\Lambda)$ of configurations $u : C \to \mathbb{X}$ with a single edge $T_e \subset C$ of length zero and of expected dimension $\operatorname{vdim} \mathcal{M}_{\mathbb{\Gamma}}(\Lambda) = 0$ will be called *fake boundary components* of $\overline{\mathcal{M}}(\Lambda)_1$. If $\mathcal{M}_{\mathbb{\Gamma}}(\Lambda)$ is regular then there are exactly two one-dimensional strata $\mathcal{M}_{\mathbb{\Gamma}'}(\Lambda), \mathcal{M}_{\mathbb{\Gamma}''}(\Lambda)$ containing $\mathcal{M}_{\mathbb{\Gamma}}(\Lambda)$ in their closure. These strata consist of configurations $u : C \to \mathbb{X}$ with an edge e of positive length $\ell(e)$, resp. the two adjacent disk components S_{v_-}, S_{v_+} have been glued to form a single disk component.

(b) Strata with an edge of infinite length $\ell(e) = \infty$ and of expected dimension $\operatorname{vdim} \mathcal{M}_{\mathbb{\Gamma}}(\Lambda) = 0$ are *true boundary components* of $\overline{\mathcal{M}}(\Lambda)_1$. There are two types of such edges e, depending on what type of edge has acquired infinite length:
   (i) edges $e \in \operatorname{Edge}_\circ(\Gamma)$, which if length $\ell(e)$ zero corresponds to nodes $w \in \partial S$ separating components S_{v_+}, S_{v_-} and
   (ii) edges $e \in \operatorname{Edge}_{\circ}(\Gamma)$ joining different levels via Reeb chords $\gamma \in \mathcal{R}(\Lambda)$.

The definitions of fake and true boundary strata of $\overline{\mathcal{M}}(L)_1$ are similar.

We have the following description of the boundary strata of the one-dimensional moduli spaces of buildings in cobordisms, stated in two separate theorems:

Theorem 4.24. *Suppose that $\underline{P} = (P_\Gamma)$ is a regular, coherent collection of perturbation data for a Legendrian Λ. The boundary of the one-dimensional stratum $\overline{\mathcal{M}}(\Lambda)_1$ is a union of strata $\mathcal{M}_{\mathbb{\Gamma}}(\Lambda)$ where $\mathbb{\Gamma} = (\mathbb{\Gamma}_1, \mathbb{\Gamma}_2)$ is a type of building $u : C \to \mathbb{R} \times Z$ with exactly two levels $u_1 : C_1 \to \mathbb{R} \times Z, u_2 : C_2 \to \mathbb{R} \times Z$.*

Theorem 4.25. *Suppose that $\underline{P} = (P_\Gamma)$ is a regular, coherent collection of perturbation data for a tamed Lagrangian cobordism L from Λ_- to Λ_+. The boundary of $\overline{\mathcal{M}}_{\mathbb{\Gamma}}(L)_1$ is a union of strata $\mathcal{M}_{\mathbb{\Gamma}}(L)$ where $\mathbb{\Gamma} = (\mathbb{\Gamma}_1, \mathbb{\Gamma}_2)$ a type corresponding to a building that is either*

(a) *a treed building $u : C \to X$ with exactly two levels $u_1 : C_1 \to X, u_2 : \mathbb{R} \times Z_\pm$, one of which maps to X and the other to $\mathbb{R} \times Z_\pm$, or*
(b) *a building $u : C \to X$ with a single level mapping to X consisting of two components $u_- : C_- \to X, u_+ : C_+ \to X$ glued together at an infinite length trajectory $u_e : T_e \to X$ at a critical point of the Morse function $f_\bullet$.*

Proof of Theorems 4.24 and 4.25. We prove the claim for the moduli space $\overline{\mathcal{M}}(L)_1$ of buildings in a cobordism L from Λ_- to Λ_+; the case of buildings in $\mathbb{R} \times \Lambda$ is easier.

We start with ruling out the formation of interior nodes. There are two possibilities, depending on whether the node is in a sub-level, and so maps to the interior of a cobordism, or separates levels, in which case the node maps to a Reeb orbit. In the first case, let $u : C \to \mathbb{X}$ be a building of type $\mathbb{\Gamma}$ occurring in the boundary of $\overline{\mathcal{M}}(\Lambda)_1$. The surface part of the domain S consists of two components S_{v_1}, S_{v_2} with intersection $w \in S_{v_1} \cap S_{v_2}$ lying in the interior of both components and S_{v_1}, S_{v_2} in the same level is a codimension two phenomenon in the space of domains and does not occur in zero and one-dimensional moduli spaces cut out transversally.

In the second case, the absence of nodes forming at Reeb orbits is a feature of our tameness assumptions in Definition 3.12: Suppose S_{v_1}, S_{v_2} are connected by a node w mapping to a Reeb orbit; that is, S_{v_1} and S_{v_2} each have a cylindrical end corresponding to an edge $e \in \mathrm{Edge}_\circ(\Gamma)$. The disk components S_{v_1}, S_{v_2} are parts of different levels in the building. Let $\mathbb{\Gamma} = \mathbb{\Gamma}_1 \cup \mathbb{\Gamma}_2$ denote the decomposition of $\mathbb{\Gamma}$ into subgraphs obtained by dividing at e. Since the subgraph $\Gamma_\circ$ of disk components is connected, either Γ_1 or Γ_2 consists entirely of spherical vertices $v \in \mathrm{Vert}_\bullet(\Gamma)$. Suppose $C = C_1 \cup C_2$ is the corresponding decomposition of domains, so that C_2 is a treed sphere and let $u_2 = u|_{C_2}$. Since the combinatorial type is a tree, we may assume that C_2 has incoming punctures but no outgoing punctures or vice versa, and in particular, u_2 is not a cover of a trivial cylinder. If u_2 is a level mapping to X with a single incoming puncture and no outgoing punctures then as in 3.22, with $e(\bullet)$ the number of interior punctures,

$$\begin{aligned} \dim \mathcal{M}_{\mathbb{\Gamma}_2}(L) &= \dim(X) + 2c_1(\overline{X}).[u_2] + 2e(\bullet) - 2([Y_-].[u_2] - 1) - 6 \\ &\geq \dim(X) + 2(1 + \lambda_1)[\overline{\omega}].[u_2] - 2 > \dim(X). \end{aligned}$$

Therefore, the moduli space cannot be made rigid by adding constraints at the puncture. On the other hand, if u_2 is a level with a single outgoing puncture and no incoming punctures then from **(P3)** we have,

$$0 > -[Y_+].[u_2] = \lambda_2[\omega].[u_2],$$

which is a contradiction. Similarly, if u_2 is a level mapping to $\mathbb{R} \times Z_+$ with one incoming puncture and no outgoing punctures then using $\dim(X) = \dim(Z) + 1$

$$\begin{aligned} \dim \mathcal{M}_{\mathbb{\Gamma}_2}(\Lambda) &= \dim(X) + 2c_1(\mathbb{P}(N_+ \oplus \mathbb{C})).[u_2] + 2e(\bullet) - 2([Y_+].[u_2] - 1) - 6 - 1 \\ &\geq \dim(X) + 2c_1(\overline{Y_+}).[\overline{u}_2] - 5 \geq \dim(X) - 1 \geq \dim(Y_+) + 1. \end{aligned}$$

THe last inequality uses the fact that $\overline{Y}$ has minimal Chern number at least two, by the monotonicity assumption in (3.12). Thus, the moduli space cannot be rigid

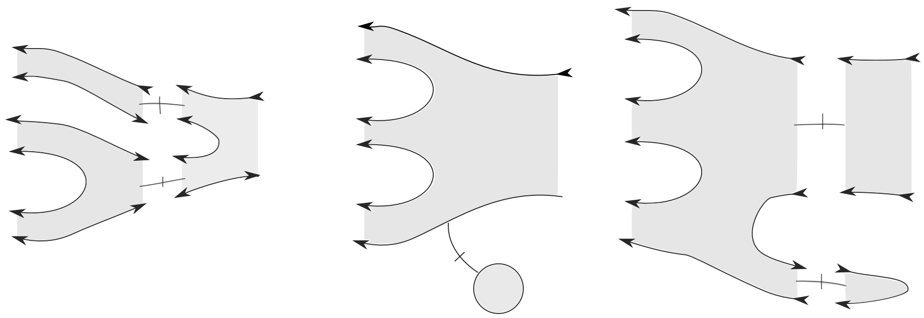

FIGURE 7. Three types of boundary configurations (a) two levels with one incoming puncture on each component (b) one level with a broken trajectory (c) two levels with zero incoming puncture on some component

by the addition of a constraint. The possibility that u_2 has one outgoing puncture and no incoming punctures violates the angle equality in Lemma 4.19.

We now describe the boundary configurations for the one-dimensional moduli spaces. Let $u : C \to \mathbb{X}$ be a configuration in the boundary of a one-dimensional component of $\mathcal{M}(\mathcal{L})$. By the previous paragraphs, C has a single boundary edge T_e with zero or infinite length. In the case of zero length, u is contained in a fake boundary component in the sense of Definition 4.23, and is not a topological boundary point. Thus the length of T_e is infinite, and either T_e separates two levels or two components of the same level. These are the possibilities listed in the statement of the Theorem. □

4.6. **Orientations.** Orientations of the moduli spaces in the case that the Legendrians resp. Lagrangians are equipped with relative spin structures are constructed by Ekholm-Etnyre-Sullivan [23], based on early work of Fukaya-Oh-Ohta-Ono [29]. See also Wehrheim-Woodward [47]. A *spin structure* on an oriented manifold L of dimension $n \geq 3$ is an equivalence class of pairs consisting of a principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(L) \to L$ and an isomorphism of the associated vector bundle

$$\mathrm{Spin}(L) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \to TL$$

for the standard representation $\mathbb{R}^n$ with the tangent bundle L.

Spin structures are used to distinguish trivializations of the pull-back of the tangent bundle to any curve. Since $SO(n)$ is connected, the pull-back γ^*TL of TL to any curve $\gamma : S^1 \to L$ is trivializable in two non-homotopic ways for $n > 2$ related by a loop $\epsilon : S^1 \to SO(n)$ with non-trivial homotopy class. Since $\mathrm{Spin}(n)$ is simply connected for $n \geq 3$, a spin structure gives in particular a choice of such trivialization up to homotopy. For $n = 2$ a spin structure gives a homotopy class of such trivializations after stabilization.

In our study of fillings we will be particularly interested in spin structures on the boundary of a filling that extend over the interior. Let L be a manifold with boundary. Any spin structure on L induces a spin structure on the boundary ∂L using the trivialization $TL_{\partial L} \cong T\partial L \oplus \mathbb{R}$. For example, consider the unit circle $\partial L = S^1$ in the unit disk $L = B^2$. On the one hand S^1 has a unique *trivial spin structure* corresponding to the unique trivialization of the tangent bundle. On the other hand, the unique spin structure on B^2 induces a *non-trivial spin structure* on the boundary S^1, since the two corresponding (stable) trivializations are related by a map $S^1 \to SO(2)$ which generates $\pi_1(SO(2))$. A *relative spin structure* is defined similarly, but allowing the the transition maps for TL to lift to a collection of transition maps taking values in $\mathrm{Spin}(n)$ that satisfy the cocycle condition up to the pull-back of a Čech two-chain on M with values in $\mathbb{Z}_2$.

Orientations on the moduli spaces are constructed by the following deformation argument. Each tangent space at a regular element $u \in \mathcal{M}(L)$ is isomorphic to the kernel $\tilde{D}_u$ of the linearized operator from (24):

$$T_u\mathcal{M}(L) := \ker(\tilde{D}_u).$$

The determinant line of $\tilde{D}_u$ is denoted

$$\mathbb{D}_u = \Lambda^{\mathrm{top}}(\ker(\tilde{D}_u)) \cong \Lambda^{\mathrm{top}}(\ker(\tilde{D}_u)) \otimes \Lambda^{\mathrm{top}}(\mathrm{coker}(\tilde{D}_u)^*).$$

An orientation of $\mathcal{M}(L)$ is a non-zero element o_u of the determinant line, modulo multiplication by a positive scalar.

Any path of Fredholm operators induces an identification of determinant lines up to isotopy, and orientations are induced by a particular equivalence class of deformations induced by the relative spin structure. In particular, the operator $\tilde{D}_u$ is isotopic to a Fredholm operator $\tilde{D}_u^v$ obtained by gluing together Cauchy-Riemann operators on collection of disks $\overline{S}_v$ without punctures with the operators $\tilde{D}_u^{e,\pm}$ over punctured surfaces S_e obtained whose boundary condition is given by a path κ in $T_{\gamma(t)}(\mathbb{R} \times Z)$ connecting $T_{\gamma(0)}(\mathbb{R} \times \Lambda)$ with $T_{\gamma(1)}(\mathbb{R} \times \Lambda)$ in the family of isomorphic $T_{\gamma(t)}\mathbb{R} \times Z$; see for example [47, Section 2.4]. Let $\Sigma^{\pm}_{\mathrm{ev}_e(u)}$ denote the stable resp. unstable manifolds associated to the critical point $\mathrm{ev}_e(u)$ at the end of the edge e. Define

$$\mathbb{D}_e^{\pm} := \det(\Sigma^{\pm}_{\mathrm{ev}_e(u)}) \otimes \mathbb{D}_{\kappa(e)}$$

where $\mathbb{D}_{\kappa(e)}$ is the determinant of the index of the Cauchy-Riemann operator $\tilde{D}_u^{e,\pm}$ associated to the capping path κ, tensored with the determinant line $\det(T_{\gamma(0)}\Lambda \cap T_{\gamma(1)}\Lambda)$ (using the identifications of $T_{\gamma(t)}(\mathbb{R} \times Z)$). For each surface component S_v we have an isomorphism of $\mathbb{D}_u$ with a product of determinant lines

$$\mathbb{D}_u \cong \det(T_{[S_v]}\mathcal{M}_\Gamma) \otimes \bigotimes_{e \in \mathcal{E}_-(\Gamma_v)} \mathbb{D}_e \otimes \mathbb{D}_u^v \otimes \bigotimes_{e \in \mathcal{E}_+(\Gamma_v)} \mathbb{D}_e^+. \tag{36}$$

Now the relative spin structure on L gives a deformation of $\mathbb{D}_u^v$ to a combination of Cauchy-Riemann operators on spheres and operators on disks with trivial bundles and boundary conditions, as in [29] and [47, Section 4]. For the subset $\mathcal{M}(L)_0$ of rigid maps denote by

$$\epsilon : \mathcal{M}(L)_0 \to \{\pm 1\}$$

the sign obtained by comparing the given orientation to the orientation of a point. In the case $L = \mathbb{R} \times \Lambda$, we obtain orientations on $\mathcal{M}(\Lambda) = \mathcal{M}(L)/\mathbb{R}$ as well, and in particular a sign map $\epsilon : \mathcal{M}(\Lambda)_0 \to \{\pm 1\}$.

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Department of Mathematics, MIT, 182 Memorial Drive, Cambridge, MA 02139, U.S.A.
Email address: kblakey@mit.edu

Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A.
Email address: sc1929@math.rutgers.edu

Department of Mathematics, Imperial College London, South Kensington, London, SW7 2AZ, U.K.
Email address: yuhan.sun@imperial.ac.uk

Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A.
Email address: ctw@math.rutgers.edu