Amalgams and the Coset Graph

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Motivation & Introduction

Informally, an amalgam is a group 'sitting inside' two other groups and the coset graph is a way to look at how the cosets of subgroups interact.

These are combined in the amalgam method — a way of classifying amalgams satisfying certain conditions.

The amalgam method was used in the classifications of finite simple groups, and is discussed in *Classification of Finite Simple Groups no. 2* by Gorenstein, Lyons and Solomon.

Amalgams

Definition. An **amalgam** $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$ has A, B, C groups and

$$\varphi_1 : C \longrightarrow A \qquad \varphi_2 : C \longrightarrow B$$

injective homomorphisms.

$$A \xleftarrow{\varphi_1} C \xrightarrow{\varphi_2} B$$

Figure 1: Amalgam

We define the **index** of an amalgam to be the ordered pair $(|A : \varphi_1(C)|, |B : \varphi_2(C)|)$.

Definition. A **completion** of $\mathcal{A}$ is (G, ψ_1, ψ_2) with G a group,

$$\psi_1 : A \longrightarrow G \qquad \psi_2 : B \longrightarrow G$$

homomorphisms such that

$$\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2.$$

That is, the following diagram commutes:

The completion is said to be **faithful** if ψ_1 and ψ_2 are both injective.

Theorem. *Given an amalgam $\mathcal{A} = (A, B, C, \varphi_1, \varphi_2)$ there exists a faithful completion. Moreover, if A and B are finite, then there is a faithful completion with G finite.*

It is obvious that every amalgam has a completion (for example taking G to be trivial) but it is not that simple to see there exists a faithful one.

The faithful completion is constructed using amalgamated products of cartesian products of A and B with the transversals of $\varphi_i(C)$.

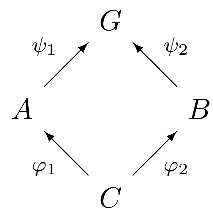

Figure 2: Completion of an Amalgam

The Coset Graph

Definition. The **right coset graph** $\Gamma = \Gamma(G, P_1, P_2, C)$ for $P_1, P_2 \leq G$ and $C \leq P_1 \cap P_2$ is a graph with vertex set

$$V(\Gamma) = \{P_i g \mid g \in G\}$$

and edge set

$$E(\Gamma) = \{Ck \mid k \in G\}$$

where vertices are connected with an edge when

$$P_i g \sim P_j h \iff \exists k \in G \text{ such that } Ck \subseteq P_i g \cap P_j h$$

Example. We can have $G = \mathfrak{S}_4$, $P_1 = A_4$, $P_2 = V_4$ and $C = \langle (12)(34) \rangle$. Then the right coset graph is where $u_1 = A_4$, $u_2 = A_4(12)$ and $v_1 = V_4$, $v_2 = V_4(123)$, $v_3 = V_4(132)$, $v_4 = V_4(12)$, $v_5 = V_4(23)$ and

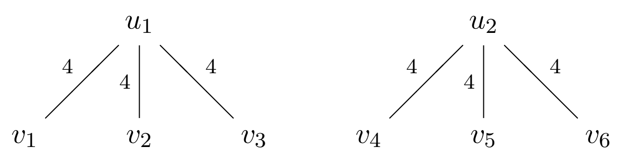

Figure 3: Example coset graph

$v_6 = V_4(13)$. The number of edges has been written above each edge for simplicity.

Lemma. *If $|P_1 \cap P_2 : C| = n$ then each pair of adjacent vertices has exactly n edges between them.*

Γ is also bipartite, with vertex sets $\{P_i g\}_{g \in G}$ for $i = 1, 2$.

Action of a Group on its Coset Graph

Definition. G acts on $\Gamma(G, P_1, P_2, C)$ by

$$(P_i g) \cdot x = P_i(gx) \qquad x \in G$$

and this preserves the graph's structure as

$$Ckg \subseteq P_1 h_1 g \cap P_2 h_2 g.$$

Lemma.
- *$\alpha = P_i g \in V(\Gamma)$ then the stabilizer G_α is P_i^g.*
- *G_α is transitive on $\Gamma(\alpha) = \{\beta \mid \beta \sim \alpha\}$.*
- *The kernel of the action of G on Γ is the largest normal subgroup of G contained in C.*

Theorem. *The coset graph* $\Gamma(G, P_1, P_2, C)$ *is connected if and only if* $G = \langle P_1, P_2 \rangle$.

Proof. Assume $G = \langle P_1, P_2 \rangle$. We know that $C \subseteq P_1 \cap P_2$ so let $\Gamma_\star$ be the connected component of Γ containing the edge C. Also $P_1, P_2 \in \Gamma_\star$ and connected components are disjoint so

$$\Gamma_\star = \Gamma_\star^{\langle P_1, P_2 \rangle} = \Gamma_\star^G.$$

So $\Gamma_\star$ is invariant under the stabilizers of all the vertices so all vertices must be in $\Gamma_\star$, so $\Gamma_\star = \Gamma$.

Now assume Γ is connected. Let $G_0 = \langle P_1, P_2 \rangle$ and assume $G_0 \gneqq G$. Let $\alpha = P_1$ and choose $\beta = P_i x$ not in G_0 such that the distance from α to β, $d(\alpha, \beta)$, is minimal. Choose a shortest path $\gamma_0, \dots, \gamma_n$ such that $\gamma_0 = \alpha$, $\gamma_n = \beta$ and $\gamma_i \sim \gamma_{i+1}$ for all i.

Now let $\gamma_{n-1} = P_i g$ and $\gamma_{n-2} = P_j h$. By minimality of n we must have $g, h \in G_0$. But $G_{\gamma_{n-1}} = P_i^g$ is transitive on $\Gamma(\gamma_{n-1})$ so there exists $k \in P_i^g \leq G_0$ so that

$$\gamma_{n-2} \cdot k = \beta.$$

But then $\beta \in G_0$ which is the desired contradiction.

□

The Amalgam Method

Let G be such that the connected component of a Sylow 2-subgroup is unipotent and non-trivial. Let $T \in Syl_2(G)$ and let N_T be the normalizer of T.

Either:

- N_T is in a unique maximal 2-local subgroup of G; or
- this subgroup is not unique.

The first case is dealt with using near components and is discussed in Gorenstein, Lyons and Solomon.

In the second case we call these subgroups $M_1, \dots M_n$ and form the amalgam

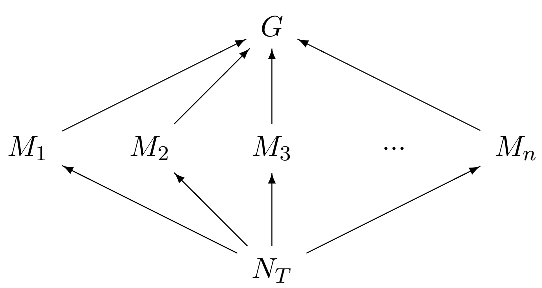

Figure 4: Amalgam formed from n maximal 2-local subgroups

The amalgam method uses assumptions on N_T and M_i to deduce what G is. We use the coset graph to assist in this process.

This is done by considering two groups assigned to each vertex:

$$\begin{aligned} Q_\alpha &:= O_2(G_\alpha) \\ Z_\alpha &:= \langle \Omega(Z(T)) \mid T \in Syl_2(G_\alpha) \rangle \end{aligned}$$

where $O_2(G_\alpha)$ is the largest normal 2-subgroup of G_α and

$$\Omega(Z(T)) = \langle x \in Z(T) \mid x^2 = 1 \rangle.$$

Definition. We say α and β are a **critical pair** if

$$Z_\alpha \nleq Q_\alpha$$

and

$$d(\alpha,\beta) = \min_{\gamma,\delta\in V(\Gamma)} \{d(\gamma,\delta) \mid Z_\gamma \nleq Q_\delta\}.$$

The amalgam method reduces to showing that for a critical pair α and β we actually have

$$d(\alpha,\beta) = 1.$$

Example

Let G be a group, $P_1, P_2 \leq G$ and $T := P_1 \cap P_2$. Assume that for $i = 1, 2$:

$\mathcal{G}_1$ $C_{P_i}(O_2(P_i)) \leq O_2(P_i)$;

$\mathcal{G}_2$ $T \in \mathrm{Syl}_2 P_i$;

$\mathcal{G}_3$ $T_G = 1$;

$\mathcal{G}_4$ $P_i/O_2(P_i) \cong S_3$; and

$\mathcal{G}_5$ $[\Omega(Z(T)), P_i] \neq 1$.

Then by applying the amalgam method we get either

- $P_1 \cong P_2 \cong S_4$; or
- $P_1 \cong P_2 \cong C_2 \times S_4$.

Example (Case 1)**.** We can see an example of Case 1 when $G = \mathbf{GL}_3(2)$ and

$$P_1 := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \mid a,b,c,d,e,f,g \in \mathbb{F}_2 \right\},$$

$$P_2 := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} \mid a,b,c,d,e,f,g \in \mathbb{F}_2 \right\}.$$

Example (Case 2)**.** We can see an example of Case 2 when $G = \mathfrak{S}_6$ and

$$P_1 = C_G((12)) \qquad P_2 = C_G((12)(34)(56)).$$

Generalizations

Definition. A partially ordered set D is a **connected partially ordered set** if for all $a, b \in D$ there exists $x_0, \ldots, x_n$ such that $x_0 = a$, $x_n = b$ and for all i we can compare x_i and x_{i+1} with the partial order.

Definition. A D**-amalgam** is a collection of groups $\{X_a\}_{a\in D}$ and homomorphisms $\{\delta_{ab}\}_{\substack{a,b\in D\\ a\leq b}}$ such that

$$\delta_{ab} : X_a \longrightarrow X_b$$

and

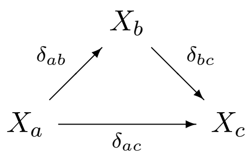

Figure 5: Commutative Diagram for a D-amalgam

- if $a \leq b \leq c$ then $\delta_{bc} \circ \delta_{ab} = \delta_{ac}$ — so the diagram below commutes; and
- for all $a \in D$, δ_{aa} is the identity on X_a.

If D is a totally ordered set then a D-amalgam can be visualized as a tower of subgroups with maps from each level to all those 'above' it.

Definition. A **completion** of a D-amalgam is (H, η_a) with H a group and homomorphisms

$$\eta_a : X_a \longrightarrow H$$

such that

- for all $a \leq b$ we have $\eta_a = \eta_b \circ \delta_{ab}$ — so the diagram below commutes; and
- $H = \langle \eta_a(X_a) \mid a \in D \rangle$.

Definition. For a group X an X**-amalgam** is a D-amalgam with $X_a \leq X$ and

- $X = \langle X_a \mid a \in D \rangle$; and
- $X_a \leq X_b$ whenever $a \leq b$.

Universal & Defining Amalgams

Definition. A **morphism of completions** is a group homomorphism $\Psi : H \longrightarrow \widehat{H}$ such that for all $a \in D$ we have $\widehat{\eta}_a = \Psi \circ \eta_a$.

Lemma. *Ψ a morphism of completions from H to $\widehat{H}$. Then Ψ is onto and $\widehat{H}$ is contained in $\Psi(H)$.*

Definition. A completion (H, η_a) is **universal** if for any completion $(\widehat{H}, \widehat{\eta}_a)$ there is a unique morphism of completions from (H, η_a) to $(\widehat{H}, \widehat{\eta}_a)$.

Lemma. *For any amalgam there is a unique (up to isomorphism) universal completion.*

Definition. An X-amalgam $\mathcal{D}$ is **defining** if the injections of X_a into X form a universal completion of $\mathcal{D}$.

Coset Geometry & Complex

Definition. For an X-amalgam $\mathcal{D} = \{X_a, \delta_{ab}\}$ the **coset geometry** Γ has

$$V(\Gamma) = \{X_a g \mid a \in D, g \in X\}$$

and $X_a g$ and $X_b h$ are connected if and only if

- $a < b$ and $X_a g \subseteq X_b h$; or
- $b < a$ and $X_b h \subseteq X_a g$.

Definition. For an X-amalgam $\mathcal{D}$, the **coset complex** $\mathcal{C}$ is the simplicial complex with vertices $V(\Gamma)$ and whose simplices are the subsets of $V(\Gamma)$ in which any two objects are incident (the **flag complex**).

Definition. $\mathcal{C}$ is simply connected if it is connected simplicially and whenever $\widehat{\mathcal{C}}$ is another connected simplicial complex and χ a covering from $\widehat{\mathcal{C}}$ to $\mathcal{C}$ then χ is an isomorphism.

Properties of the Coset Complex

Lemma. *$\mathcal{D}$ an X-amalgam:*

- *$\Gamma(\mathcal{D}, X)$ is a connected graph;*
- *X acts on Γ and $\mathcal{C}$ by right translation transitively;*
- *the stabilizer of a vertex $X_a g$ is conjugate to X_a; and*
- *the kernel of the action of Γ on $\mathcal{C}$ is*

$$\bigcap_{\substack{g \in X \\ a \in D}} X_a^g.$$

Theorem. *$\mathcal{D}$ an X-amalgam. If $\mathcal{C}$ is simply connected, then $\mathcal{D}$ is a defining X-amalgam.*

The Major Theorem

We finally give the major theorem used in the classification of finite simple groups.

Theorem. *Suppose that we have*

- *G a group and $\mathcal{D} = \{X_a\}$ a G-amalgam;*
- *$\widehat{G}$ a group and $\widehat{\mathcal{D}} = \{\widehat{X}_a\}$ a* defining *$\widehat{G}$-amalgam;*
- *$\mathcal{D}$ and $\widehat{\mathcal{D}}$ based on the same connected partially ordered set D; and*
- *For all $a \in D$, there exists a surjective homomorphism*

$$\varphi_a : \widehat{X}_a \longrightarrow X_a.$$

Then there exists a surjective homomorphism

$$\Psi : \widehat{G} \longrightarrow G.$$

References

For more information, proofs and references, see *Amalgams and the Coset Graph* by David Wilson at `http://math.rutgers.edu/~davidjwi/NamedAmalgams.pdf`