1. Use Lagrange multipliers to find the maximum value of \( f(x, y, z) = xyz \) subject to the condition \( x + 2y + 3z = 9 \).

**Solution:** The question is misworded. There is no maximum value. To see this, we will intersect the constraint (a plane) with another plane: \( y = z \). Substituting, \( y = z \) in the equation of the plane we find \( x + 5y = 9 \) and so \( x = 9 - 5y \). So we can parameterize this line as \( (9-5y, y, y) \). Now we can evaluate \( f \) on this line, \( f(9-5y, y, y) = y^3(9-5y) = -5y^3 + 9y^2 \). Now note that \( \lim_{y \to -\infty} -5y^3 + 9y^2 = \infty \). So \( f \) increases without bound along this line which satisfies the constraint. So \( f \) has no maximum.

Now, we can use Lagrange multipliers to try to find a local extremum. So we let \( g(x, y, z) = x + 2y + 3z - 9 \) and seek a point such that \( \nabla f = \lambda \nabla g \). Computing the partial derivatives we have \( (yz, xz, xy) = \lambda (1, 2, 3) \). So we have the equations

\[
\begin{align*}
yz &= \lambda \\
xz &= 2\lambda \\
xy &= 3\lambda.
\end{align*}
\]

Now we consider two cases. Case one is that \( \lambda = 0 \). In this case we see from the first equation that \( yz = 0 \) and so at any such point \( f(x, y, z) = xyz = 0 \). The second case is that \( \lambda \neq 0 \). Then we can divide the second equation by the first above to obtain \( x/y = 2 \) and thus \( x = 2y \). Dividing the third equation by the second we find \( y/z = 3/2 \) and thus \( z = \frac{2}{3} y \). Having solved for \( x \) and \( z \) both in terms of \( y \) we can substitute into our constraint and obtain \( 2y + 2y + 3 \cdot \frac{2}{3} y = 9 \), thus \( y = 3/2 \). Then \( x = 3 \) and \( z = 1 \). So we obtain the point \( (3, 3/2, 1) \). The value of \( f \) at this point is \( f(3, 3/2, 1) = 3 \cdot \frac{3}{2} \cdot 1 = 9/2 \). If we had add the further constraint that \( x, y \) and \( z \) are all non-negative then this is the maximum value of \( f \). Think about why!