Practice problems for the final Exam, Math 251, Fall 2014

1. Evaluate \( \int f_R(x+y)e^{x^2-y^2}dA \), Where \( R \), is the rectangle enclosed by the lines \( x-y=0, x-y=2, x+y=0, x+y=3. \) (ANS: \((e^6 - 7)/4.\) Hint: use the change of variable formula)

2. Evaluate using spherical coordinates \( \int_{\frac{\pi}{2}}^{\pi} \int_0^{\sqrt{2}} \int_0^{\sqrt{2-\sin^2\theta}} f \) \( \sqrt{4-x^2-y^2} \) \( y^2 \sqrt{x^2+y^2+z^2} \) \( dz \) \( dx \) \( dy \). ANS \( 64\pi/9 \).

3. Evaluate \( \int \int \int_E y^2x^2dV \), Where E is the region bounded by the paraboloid \( x = 1 - y^2 - z^2 \), and the plane \( x=0. \) ANS \((\frac{\pi}{36})\).

4. Evaluate \( \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\tan\theta} f \) \( r^2 \) \( e^{r^2} \) \( r^2 \) \( z \) \( dV \). ANS \( e^2a - e^a - \frac{3}{8} \).

5. Find the volume bounded by the ellipsoidal paraboloids \( z = x^2 + 9y^2 \), and \( z = 18 - x^2 - 9y^2 \), ANS: \( 27\pi \).

6. Calculate the volume of a solid whose base is in a xy-plane and is bounded by the parabola \( y = 4 - x^2 \), and the straight line \( y = 3x \). While the top of the solid is in the plane \( z = x + 4. \) ANS: \( \frac{625}{12} \).

7. Evaluate \( \int_C \) \( yzdx + (xz + 1)dy + xyzdz \), where \( C \) is any path from \((1,0,0), \) to \((2,1,4)\). ANS. 9. Show that the vector field is conservative.

8. Verify the divergence theorem for \( F = 4xyz - y^2j + yzk \), taken over the cube bounded by \( x=0,x=1,y=0,y=1,z=0,z=1\).

9. Apply Green’s theorem in the plane to evaluate \( \int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy] \), where \( C \) is the boundary of the curve enclosed by the x-axis and the semi-circle \( y = (1 - x^2)^{1/2} \). ANS 4/3.

10. Evaluate by Stokes’ theorem \( \int_C (\sin xdx - \cos ydy + \sin ydz) \), where \( C \) is the boundary of the rectangle \( 0 \leq x \leq \pi, 0 \leq y \leq 1, z = 3 \). ANS: 2.

11. Apply Stokes’ theorem to prove that \( \int_C (ydx + zdy + xdz) = -2\sqrt{2}\pi \), where \( C \) is the curve given by \( x^2 + y^2 + z^2 - 2x - 2y = 0, x + y = 2 \), and begins at the point \( (2,0,0) \) and goes at first below the xy-plane.

12. Evaluate \( \int \int_S (y^2z^2i + z^2x^2j + x^2y^2k) \cdot ds \), where \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 1 \), above the xy-plane. ANS: \( \pi/24 \).

13. If \( F = (2x^2 + y^2)i + (3y - 4x)j \), evaluate \( \int F \cdot dr \), around the triangle ABC whose vertices are \( A(0,0), B(2,0), C(2,1) \). ANS: \(-14/3\).

14. Find \( \nabla \times f \), or \( \text{curl} F \), where \( i \). \( F = x^2yi - 2xzj + 2yzk \), \( ii \). \( F = (x^2 - y^2)i + 2xyj + (y^2 - 2xy)k \). ANS: \((i) < 2x + 2z, -2z - x^2 >, (ii) < 2y - 2x, 2y, 4y >\)

15. Find the equation of the tangent plane and normal line to the surface \( xyz = 4 \), at the point \((1,2,2)\). ANS: \( 2x + y + z = 6, (x - 1)/2 = (y - 2)/1 = (z - 2)/1 \).

16. Find the equation of the tangent line to the curve of intersection of \( x^2 + y^2 + z^2 = 1, x + y + z = 1 \), at \((1,0,0)\). ANS: \( y = -z, x - 1 = 0 \).

17. Calculate the maximum rate of change and the corresponding direction for the function \( \phi = x^2y^2z^4 \), at the point \( 2i + 3j - k \). ANS: \( 324\sqrt{2} \).

18. If \( \phi(xyz) = xy^2z \), and \( f = xzi - xyj + yzk \), Find \( \frac{\partial^2 \phi}{\partial x \partial z} \), at \((2,-1,1)\). ANS: \( 4i + 2j \).
19. Find and classify the critical points of \( f(x, y) = x^2 + y^2 - (x - 1)^4 - (y - 1)^4 - 4x - 4y \). ANS. (0,0), max point.

20. Find all extreme values of \( f = x^2 + 2y^2 \) on \( x^2 + y^2 = 1 \). ANS: 1. (±1, 0)

21. Use Lagrangian multiplier to find the points on \( x^2 + y^2 = 4 \) that are closest to and farthest from (3, 1). ANS: ±(3/√10, 1/√10).

22. Find the curvature of \( r(t) = < t, t^2, t^3 > \) at a general point. ANS. \( 2\sqrt{9t^4 + 9t^2 + 1} \).

23. Find the point on \( y = 3x^2 + 1 \) which has the largest curvature. ANS. (0,1).