

Graded traces and irreducible representations of $\text{Aut}(A(\Gamma))$ acting on graded $A(\Gamma)$ and $A(\Gamma)!$

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- Subalgebra of $\text{gr}A(\Gamma)$

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Goal

- ▶ Original Goal - better understand algebra Q_n related to factorizations of noncommutative polynomials.

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Goal

- ▶ Original Goal - better understand algebra Q_n related to factorizations of noncommutative polynomials.
- ▶ Accomplished - found decomposition of graded Q_n into S_n -modules
 - found decomposition for other algebras $A(\Gamma)$ associated with graphs Γ
 - techniques developed can be used on general $A(\Gamma)$
 - considered interesting subalgebras

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History

- ▶ Classic problem - express coefficients of $P(t) = t^n + a_{n-1}t^{n-1} + \dots + a_0 = (t - y_1)(t - y_2) \cdots (t - y_n)$ in terms of the right roots
- ▶ Q_n - algebra describing factorizations of $P(t)$
- ▶ study Q_n as an algebra associated to a directed, layered graph: lattice of subsets of $\{1, \dots, n\}$

Idea

- ▶ Q_n has a natural grading.
- ▶ S_n acts on each homogeneous component of graded Q_n .
- ▶ Each homogeneous component can be written as a direct sum of irreducible S_n -modules

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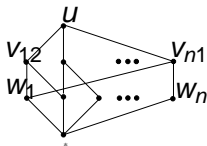
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Idea

- ▶ Q_n has a natural grading.
- ▶ S_n acts on each homogeneous component of graded Q_n .
- ▶ Each homogeneous component can be written as a direct sum of irreducible S_n -modules
- ▶ can find decomposition for more general algebras $A(\Gamma)$ associated to directed, layered graphs Γ

$A(\Gamma)$ in words

- ▶ generated by edges in graph
- ▶ relations defined by associating a polynomial to each path in Γ and requiring that polynomials for two paths connecting same pair of vertices are equal in $A(\Gamma)$



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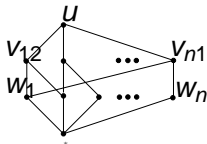
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$A(\Gamma)$ in words

- ▶ generated by edges in graph
- ▶ relations defined by associating a polynomial to each path in Γ and requiring that polynomials for two paths connecting same pair of vertices are equal in $A(\Gamma)$



- ▶ Examples: lattice of subsets of $\{1, \dots, n\}$, graphs whose automorphism group is the dihedral group on n elements, graphs whose automorphism groups are Coxeter groups, the complete layered graph, and the Hasse graph of the lattice of subspaces of a finite-dimensional vector space over a finite field

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Layered Graph Γ

- ▶ $T(W)$ - free associative algebra on set W over field k
- ▶ Let $\Gamma = (V, E)$ - directed, layered graph with vertices V and edges E
- ▶ $V = \bigcup_{i=0}^n V_i$, $E = \bigcup_{i=0}^n E_i$, edges go from level i to level $i-1$, $V_0 = \{*\}$

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Field $A(\Gamma)$

▶ path - sequence of edges $\pi = \{e_1, \dots, e_m\}$

▶ $v > w$ - there exists a path from v to w .

▶ $e(\pi, k) := \sum_{1 \leq i_1 < \dots < i_k \leq m} e_{i_1} \cdots e_{i_k}$

from $P_\pi(t) = (1 - te_1) \cdots (1 - te_m)$

▶ can choose distinguished path from v to $*$; then write $e(v, k)$

$A(\Gamma)$

- ▶ Let R be the two-sided ideal of $T(E)$ generated by $\{e(\pi_1, k) - e(\pi_2, k)\}$ such that π_1, π_2 connect same pair of vertices
- ▶ $A(\Gamma) := T(E)/R$

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$A(\Gamma)$

- ▶ Let R be the two-sided ideal of $T(E)$ generated by $\{e(\pi_1, k) - e(\pi_2, k)\}$ such that π_1, π_2 connect same pair of vertices
- ▶ $A(\Gamma) := T(E)/R$
- ▶ Let $\hat{e}(v, k)$ denote the image in $A(\Gamma)$ of $e_1 \cdots e_k$.
- ▶ say (v, k) covers (w, l) if $v > w$ and $k = |v| - |w|$, write this as $(v, k) \succ (w, l)$

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Theorem

[RSW, Thm 1] - Let $\Gamma = (V, E)$ be a layered graph,

$$V = \bigcup_{i=0}^n V_i, V_0 = \{*\}. \text{ Then } \mathcal{B}(\Gamma) := \{\hat{e}(v_1, k_1) \cdots \hat{e}(v_l, k_l) :$$

$$l \geq 0, v_1, \dots, v_l \in V_+, 1 \leq k_i \leq |v_i|, (v_i, k_i) \not\rightarrow (v_{i+1}, k_{i+1})\}$$

is a basis for $A(\Gamma)$.

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$$l \geq 0, v_1, \dots, v_l \in V_+, 1 \leq k_i \leq |v_i|, (v_i, k_i) \not\rightarrow (v_{i+1}, k_{i+1})\} \text{ is}$$

a basis for $A(\Gamma)$.

- ▶ write $e(v, k)$ for $\hat{e}(v, k)$ from now on

Graded $A(\Gamma)$

- ▶ 1st grading: $T(E) = \sum_i T(E)_{[i]}$ is given by degree in the tensor algebra
- ▶ 2nd grading: $T(E) = \sum_{i \geq 0} T(E)_i$ where
$$T(E)_i = \text{span}\{e_1 \cdots e_r : r \geq 0, e_j \in E_{l_j}, l_1 + \cdots + l_r = i\}$$
- ▶ 2nd induces filtration
- ▶ So, $T(E)$ can be identified with associated graded algebra
- ▶ $gr(T(E)/R) \cong T(E)/(grR)$
- ▶ If $(v, k) \succ (u, l)$, $e(v, k+l) - e(v, k)e(u, l)$ in grR

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$A(\Gamma)^!$

- ▶ dual of graded algebra - $A(\Gamma)^! := T(E^*)/(grR)^\perp$
- ▶ presentation - generators: $\{e(v, 1)^*\}$
relations:
 $\{e(v, 1)^* e(u, 1)^* : v \not\triangleright u\} \cup \{e(v, 1)^* \sum_{v \triangleright u} e(u, 1)^*\}$
- ▶ take fixed generators to get a subalgebra of the dual

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$A(\Gamma^\sigma)$

- ▶ σ - automorphism of graph that preserves layers
- ▶ $\Gamma^\sigma := (V_\sigma, E_\sigma)$
 - $V_\sigma = \{v \in V : \sigma(v) = v\}$
 - E_σ - edges that connect vertices minimally

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- ▶ σ - automorphism of graph that preserves layers
- ▶ $\Gamma^\sigma := (V_\sigma, E_\sigma)$
 - $V_\sigma = \{v \in V : \sigma(v) = v\}$
 - E_σ - edges that connect vertices minimally
- ▶ $A(\Gamma^\sigma) := \text{span}\{e(v_1, k_1) \cdots e(v_l, k_l) : l \geq 0, v_1, \dots, v_l \in V_\sigma \setminus *, 1 \leq k_i \leq |v_i|, (v_i, k_i) \not\rightarrow (v_{i+1}, k_{i+1})\}$
- ▶ Theorem - $A(\Gamma^\sigma)$ is a subalgebra of graded $A(\Gamma)$

$A(\Gamma^\sigma)$

- ▶ presentation - generators:

$$\{e(v, k) : v \in V_\sigma, 1 \leq k \leq |v|\}$$

relations:

$$\{e(v, k+l) - e(v, k)e(u, l) : v > u \in V_\sigma, k = |v| - |u|\}.$$

- ▶ can form dual of subalgebra
- ▶ $A(\Gamma^\sigma)$ are examples of algebras associated to generalized layered graphs

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Automorphism Group

Lemma

$$\text{Aut}(A(\Gamma)) \supseteq k^* \times \text{Aut}(\Gamma)$$

Main Theorem

If Γ satisfies i) more than two vertices in level 1, ii) no two vertices are above same set of vertices, and iii) there are either zero or two paths between any two vertices which are two levels apart in Γ , then $\text{Aut}(A(\Gamma)) = k^ \times \text{Aut}(\Gamma)$, k the base field.*

Proof.

(idea) The relations in the algebra must be preserved. \square

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Definition of Graded Trace

- ▶ Hilbert series of graded algebra A -
$$H(A, t) = \sum \dim(A(\Gamma)_{[k]})t^k$$
- ▶ Let σ be an automorphism, then the graded trace of σ on $A(\Gamma)$ is $Tr_\sigma(A(\Gamma), t) = \sum (Tr \sigma|_{A(\Gamma)_{[k]}})t^k$
- ▶ Because the basis of $A(\Gamma)$ is invariant under σ , the trace of σ on $grA(\Gamma)$ is the number of fixed basis elements.
- ▶ $Tr_\sigma(A(\Gamma), t)$ is the Hilbert series of $A(\Gamma^\sigma)$

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Use of Graded Traces

- ▶ consider associated graded algebra, $\text{gr}A(\Gamma)$
- ▶ ϕ_1, \dots, ϕ_l - all of the distinct irreducible representations of $\text{Aut}A(\Gamma)$
- ▶ χ_j - character afforded by ϕ_j
- ▶ $\text{Aut}A(\Gamma)$ acts on each $A(\Gamma)_{[i]}$,
- ▶ so the completely reducible $\text{Aut}A(\Gamma)$ -module $A(\Gamma)_{[i]}$

may be written as $\bigoplus_{j=1}^l m_{ij} \phi_j$

- ▶ $\vec{m} = (C^T)^{-1} \vec{Tr}(t)$, C is character table

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Calculating Graded Trace - Method 1

Let $|\cdot|$ denote the graded trace in this theorem.

$$W(\Gamma^\sigma) := \text{span}\{e(v, k) : l \geq 0, v \in V_\sigma, 1 \leq k \leq |v|\}$$

$$\tilde{R}(\Gamma^\sigma) := \text{span}\{e(v, k)e(u, l) : v > u \in V_\sigma, k = |v| - |u|\}$$

Theorem

$$|A(\Gamma^\sigma)| = \frac{1}{1 - |W(\Gamma^\sigma)| + |\tilde{R}(\Gamma^\sigma)| - |\tilde{R}(\Gamma^\sigma)W(\Gamma^\sigma) \cap W(\Gamma^\sigma)\tilde{R}(\Gamma^\sigma)| + \dots}$$

Calculating Graded Trace - Method 2

Main Theorem

$$Tr_{\sigma}(A(\Gamma), t) = \frac{1-t}{1-t \sum_{\substack{v_1 > \dots > v_l \geq * \\ v_1, \dots, v_l \in \bar{V}_{\sigma}}} (-1)^{l-1} t^{|\bar{v}_1| - |\bar{v}_l|}}$$

Proof.

(idea) Write linear recurrences in matrix form involving subsets of the basis and of products of generators. We get a Möbius-type matrix. □

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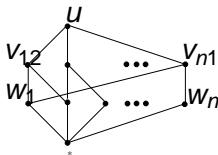
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$A(\Gamma_{D_n})$

Hasse graph of n-gon.



Theorem

a) If $n \geq 3$, $\text{Aut}(A(\Gamma_{D_n})) = k^* \times D_n$, k the base field

b) If $n = 2$, $\text{Aut}(A(\Gamma_{D_2})) \cong \{M \in \text{GL}(3, k) : M =$

$$\begin{bmatrix} c_1^1 + c_2^1 & c_1^2 - c_2^1 & c_2^1 - c_1^2 \\ 0 & c_1^1 & c_2^1 \\ 0 & c_2^2 & c_1^1 + c_2^1 - c_1^2 \end{bmatrix}, c_i^j \in k \forall i, j\}$$

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Example using Method 1:

- ▶ $W(\Gamma_{D_n}^{id})$ has basis $\{e(u, 3), e(u, 2), e(u, 1), e(v_{ii+1}, 2), e(v_{ii+1}, 1), e(w_i, 1), 1 \leq i \leq n\}$

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Example using Method 1:

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- ▶ Reducible words of degree 2: $e(u, 2)e(w_i, 1), e(u, 1)e(v_{ii+1}, 2), e(u, 1)e(v_{ii+1}, 1), e(v_{ii+1}, 1)e(w_i, 1), e(v_{ii+1}, 1)e(w_{i+1}, 1)$

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Example using Method 1:

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- ▶ Reducible words of degree 2: $e(u, 2)e(w_i, 1), e(u, 1)e(v_{ii+1}, 2), e(u, 1)e(v_{ii+1}, 1), e(v_{ii+1}, 1)e(w_i, 1), e(v_{ii+1}, 1)e(w_{i+1}, 1)$
- ▶ Overlaps of reducible words:
 $e(u, 1)e(v_{ii+1}, 1)e(w_i, 1), e(u, 1)e(v_{ii+1}, 1)e(w_{i+1}, 1)$

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Example using Method 1:

- ▶ $W(\Gamma_{D_n}^{id})$ has basis $\{e(u, 3), e(u, 2), e(u, 1), e(v_{ii+1}, 2), e(v_{ii+1}, 1), e(w_i, 1), 1 \leq i \leq n\}$
- ▶ Reducible words of degree 2: $e(u, 2)e(w_i, 1), e(u, 1)e(v_{ii+1}, 2), e(u, 1)e(v_{ii+1}, 1), e(v_{ii+1}, 1)e(w_i, 1), e(v_{ii+1}, 1)e(w_{i+1}, 1)$
- ▶ Overlaps of reducible words:
 $e(u, 1)e(v_{ii+1}, 1)e(w_i, 1), e(u, 1)e(v_{ii+1}, 1)e(w_{i+1})$
- ▶ Thus, $Tr_{id}(A(\Gamma_{D_n}), t) =$

$$\frac{1}{1 - \frac{1}{((1+n+n)t - ((1+n) - (n+n+n))t^2 + (1 - (n+n) + (n+n))t^3)}} = \frac{1}{1 - ((2n+1)t - (2n-1)t^2 + t^3)}$$

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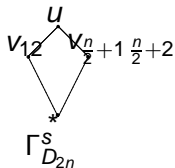
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Example Using Method 2:

- ▶ $s = (2n)(3n - 1)(4n - 2) \cdots$ (n even) fixes the top vertex, the minimal vertex, and two vertices on level two



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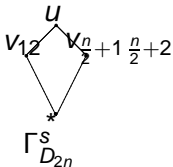
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Example Using Method 2:

- ▶ $s = (2n)(3n - 1)(4n - 2) \cdots$ (n even) fixes the top vertex, the minimal vertex, and two vertices on level two
- ▶ coefficient of t^0 : 4 (each vertex)
- ▶ of t^1 : $u > v_{12}$, $u > v_{\frac{n}{2}+1 \frac{n}{2}+2}$
- ▶ of t^2 : $v_{12} > *$, $v_{\frac{n}{2}+1 \frac{n}{2}+2} > *$
- ▶ of t^3 : $u > *$, $u > v_{12} > *$, and $u > v_{\frac{n}{2}+1 \frac{n}{2}+2} > *$.
- ▶ $Tr_s(t) = \frac{1-t}{1-t(4-2t-2t^2+t^3)} = \frac{1-t}{1-t(2-t)(2-t^2)}$



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$A(\Gamma_{D_n})$ - multiplicities

- ▶ Graded trace generating functions:

$$a = \text{Tr}_{id}(A(\Gamma_{D_n}), t) = \frac{1-t}{1-(2n+2)t+4nt^2-2nt^3+t^4}$$

$$b = \text{Tr}_{(12\dots n)^i}(A(\Gamma_{D_n}), t) = \frac{1-t}{1-t(2-t^3)}$$

$$c = \text{Tr}_s(A(\Gamma_{D_n}), t) = \text{Tr}_{(12\dots n)_s}(t) = \frac{1-t}{1-t(2-t)(2-t^2)}$$

- ▶ Let n be even. Then, $\vec{m}(t) =$

$$\begin{bmatrix} \frac{1}{2n}a + \frac{n-1}{2n}b + \frac{1}{2}c \\ \frac{1}{2n}a + \frac{n-1}{2n}b - \frac{1}{2}c \\ \frac{1}{2n}(a-b) \\ \frac{1}{2n}(a-b) \\ \frac{1}{n}(a-b) \\ \vdots \\ \frac{1}{n}(a-b) \end{bmatrix}$$

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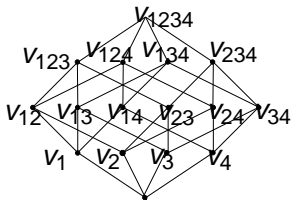
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Q_n
Algebra associated with the lattice of subsets of $\{1, \dots, n\}$



Theorem

If $n \geq 3$, $\text{Aut}(Q_n) = k^* \times S_n$.

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Q_n - Main Theorem

Theorem

Let $\sigma \in S_n$ and $\sigma = \sigma_1 \cdots \sigma_m$ be its cycle decomposition.
Denote the length of σ_j by i_j . Then

$$\text{Tr}_\sigma(Q_n, t) = \frac{1 - t}{1 - t \prod_{j=1}^m (2 - t^{i_j})}$$

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Proof: If $w \subseteq \{1, \dots, n\}$ is σ -invariant, let $\|w\|$ be the number of σ -orbits in w . Also, let \mathcal{O}_j denote the non-trivial orbit of σ_j .

By some lemmas,

$$\begin{aligned} & \sum_{v_1 \supset \dots \supset v_l \supseteq \emptyset} (-1)^l t^{|v_1| - |v_l| + 1} \\ &= \sum_{\{1, \dots, n\} \supseteq v_1 \supseteq v_l \supseteq \emptyset} (-1)^{\|v_1\| - \|v_l\| + 1} t^{|v_1| - |v_l| + 1} \\ &= \sum_{\substack{w, v_l \\ w \cap v_l = \emptyset}} (-1)^{\|w\| + 1} t^{\|w\| + 1} = \sum_w 2^{m - \|w\|} (-1)^{\|w\| + 1} t^{\|w\| + 1} \end{aligned}$$

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Proof continued:

The σ -invariant sets w are unions of σ -orbits. Write $a_j = 1$ if \mathcal{O}_j is contained in w and $a_j = 0$ if not. We can then write $\sum_w 2^{m-\|w\|} (-1)^{\|w\|+1} t^{|w|+1}$ as

$$\sum_{a_1, \dots, a_m \in \{0, 1\}} (-1)^{\sum a_j + 1} 2^{m - \sum a_j} t^{\sum (a_j j) + 1} =$$

$$-t \sum_{a_1, \dots, a_m \in \{0, 1\}} \prod_{j=1}^m (-1)^{a_j} 2^{1-a_j} t^{a_j j}$$

$$= -t \prod_{j=1}^m \sum_{a_j=0}^1 (-1)^{a_j} 2^{1-a_j} t^{a_j j}$$

$$= -t \prod_{j=1}^m (2 - t^j)$$

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Q_n - example

- ▶ Example for $n = 4$ and 1st three degrees:

	χ_{triv}	χ_{sgn}	χ_3	χ_{reg}	$\chi_{sgn \times reg}$
$m_{\phi,1}$	4	0	1	3	0
$m_{\phi,2}$	26	1	17	36	13
$m_{\phi,3}$	219	54	239	434	273

- ▶ Can write multiplicities in terms of Frobenius formula

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- ▶ Can write multiplicities in terms of Frobenius formula

$$1+t \prod_{k=1}^m (2 - (-t)^{i_k})$$

- ▶ $Tr_{\sigma}(Q_n^!, t) = \frac{1+t \prod_{k=1}^m (2 - (-t)^{i_k})}{1+t}$

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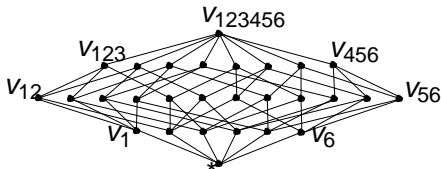
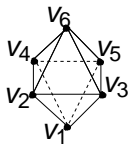
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Octahedron

One example is the algebra associated to the Hasse graph of the octahedron.



- ▶ Automorphism group is k^* times the symmetry group of the octahedron.
- ▶ 10 conjugacy classes, 6 distinct graded trace generating functions
- ▶ Same results for the algebra associated with the Hasse graph of the cube

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Complete Layered Graph

Say there are r_j vertices in level j , $j > 0$ (1 minimal vertex in level 0). The automorphism group of the algebra contains $k^* \times S_{r_1} \times \cdots \times S_{r_n}$.

Theorem

Let m_j be the number of vertices in level j fixed by $\sigma_{r_j} \in S_{r_j}$. Let $\sigma = \sigma_{r_1} \times \cdots \times \sigma_{r_n} \in S_{r_1} \times \cdots \times S_{r_n}$. Then

$$\text{Tr}_\sigma(A(\Gamma_{[r_1, \dots, r_n]}), t) = \frac{1-t}{1-t \sum_{k=0}^n \sum_{j=k}^n (-1)^k m_j (m_{j-1}-1)(m_{j-2}-1) \cdots (m_{j-k+1}-1) m_{j-k} t^k}$$

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Hasse Graph of the Lattice of Subspaces of a Finite-dimensional Vector Space over a Finite Field

- ▶ Denote algebra by $A(L(n, q))$
- ▶ Automorphism group contains $k^* \times PGL_n(F_q)$
- ▶ Write matrix in $PGL_n(F_q)$ in canonical form (we will consider those that can be put into Jordan canonical form)
- ▶ Only need to consider matrices with one eigenvalue

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$A(L(n, q))$ - reduction to one eigenvalue

Proposition

Let $T \in GL_n(F_q)$ be a matrix in Jordan canonical form with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Let $T_{\lambda_j} \in GL_{n_j}(F_q)$ be the submatrix of T containing only those blocks with eigenvalue λ_j . Finally say $\frac{1-t}{1-tf_{\lambda_j}}$ is the graded trace generating function of T_{λ_j} acting on $A(L(n_j, q))$. Then

$$\text{Tr}_T(A(L(n, q)), t) = \frac{1-t}{\prod_{j=1}^k (1-tf_{\lambda_j})}.$$

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$A(L(n, q))$ - proof

Proof.

- ▶ $L(n, q)^T = L(n, q)^{T_{\lambda_1}} \times \cdots \times L(n, q)^{T_{\lambda_k}}$.
- ▶ $f_{\lambda_j} = \sum_{v_1 > \cdots > v_l \geq * } (-1)^{l+1} t^{|v_1| - |v_l|}$
- ▶ (v, w) -entry of Möbius matrix $\hat{\mu}$ (comes from proof of Method 2) is $\sum_{v=v_1 > \cdots > v_l=w \geq * } (-1)^{l+1} t^{|v_1| - |v_l|}$.
- ▶ $\vec{1}^T \hat{\mu} \vec{1} = \sum_{v_1 > \cdots > v_l \geq * } (-1)^{l+1} t^{|v_1| - |v_l|} = f$
- ▶ $\hat{\mu}_{G \times H} = \hat{\mu}_G \otimes \hat{\mu}_H$

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$A(L(n, q))$ - proof continued

Proof.

$$\begin{aligned}
 f_{\lambda_1 \times \dots \times \lambda_k} &= \\
 \vec{1}_{L(n, q)^{T_{\lambda_1} \times \dots \times L(n, q)^{T_{\lambda_k}}} \hat{\mu}_{L(n, q)^{T_{\lambda_1} \times \dots \times L(n, q)^{T_{\lambda_k}}} \vec{1}_{L(n, q)^{T_{\lambda_1} \times \dots \times L(n, q)^{T_{\lambda_k}}} & \\
 = (\vec{1}^T \otimes \dots \otimes \vec{1}^T) (\hat{\mu}_{L(n, q)^{T_{\lambda_1}}} \otimes \dots \otimes \hat{\mu}_{L(n, q)^{T_{\lambda_k}}}) (\vec{1} \otimes \dots \otimes \vec{1}) & \\
 = (\vec{1}_{L(n, q)^{T_{\lambda_1}}} \hat{\mu}_{L(n, q)^{T_{\lambda_1}}} \vec{1}_{L(n, q)^{T_{\lambda_1}}}) \cdots (\vec{1}_{L(n, q)^{T_{\lambda_k}}} \hat{\mu}_{L(n, q)^{T_{\lambda_k}}} \vec{1}_{L(n, q)^{T_{\lambda_k}}}) & \\
 = f_{\lambda_1} \cdots f_{\lambda_k}. &
 \end{aligned}$$



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$A(L(n, q))$ - counting subspaces

- ▶ T has one eigenvalue λ , k blocks
- ▶ $V = V_{[a]} = V_{a_1} \oplus \cdots \oplus V_{a_k}$ indecomposable
- ▶ $W = W_{[i]} = W_{i_1} \oplus \cdots \oplus W_{i_{r'}} \oplus W_{i_{r'+1}} \oplus \cdots \oplus W_{i_r}$ - invariant subspace of V
- ▶ $i_1 \geq i_2 \geq \cdots \geq i_{r'} > 1, i_{r'+1} = \cdots = i_r = 1, r \leq k.$
- ▶ $V_{[a-1]} = V_{a_1-1} \oplus \cdots \oplus V_{a_k-1}$
- ▶ $N_{[a,i]}$ - number of invariant subspaces $W_{[i]}$ of $V_{[a]}$

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$A(L(n, q))$ - counting subspaces

Proposition

The number of invariant $W_{[i]} \subseteq V_{[a]}$ is

$$N_{[a,i]} = (q^{k-r})^{r'} \binom{k-r'}{r-r'}_q N_{[a-1,i-1]}.$$

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$A(L(n, q))$ - counting subspaces

Proposition

The number of invariant $W_{[i]} \subseteq V_{[a]}$ is

$$N_{[a,i]} = (q^{k-r})^{r'} \binom{k-r'}{r-r'}_q N_{[a-1,i-1]}.$$

This proposition and the previous are enough to allow us to calculate the graded trace for any matrix in Jordan canonical form acting on $A(L(n, q))$

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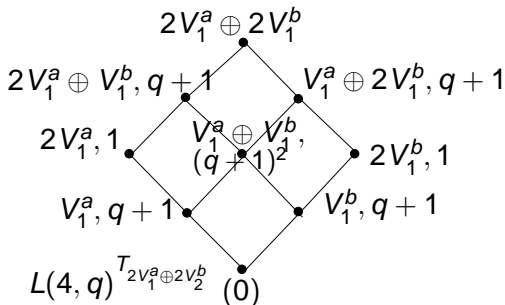
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An Example

$$\begin{array}{l}
 V_1^a \oplus V_1^a \oplus V_1^b \oplus V_1^b \\
 W_1^a \oplus W_1^a \oplus W_1^b, W_1^a \oplus W_1^b \oplus W_1^b \\
 W_1^a \oplus W_1^a, W_1^a \oplus W_1^b, W_1^b \oplus W_1^b \\
 W_1^a, W_1^b
 \end{array}
 \begin{array}{l}
 1 \\
 q+1, q+1 \\
 1, (q+1)^2, 1 \\
 q+1, q+1
 \end{array}$$



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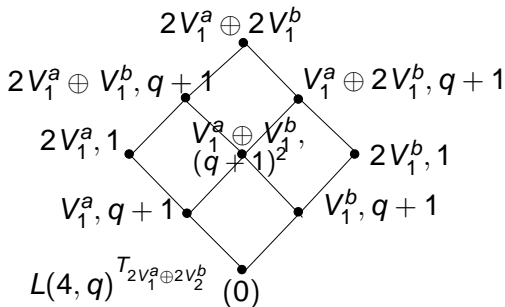
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$$\text{Tr}_T = \frac{1 - t}{1 - t((q + 3) - 2(q + 1)t + qt^2)^2}$$

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