1. Introduction

Broadly speaking, I am interested in discrete geometry, and more specifically, discrete complex analysis. There are several discrete theories which have demonstrated analytic characteristics: circle-packing [CP], Mercat’s discrete holomorphicity [M1, M2], and discrete curvature flows [DU] to name a few. My current project has been investigating square and rectangular tilings of plane domains and singular flat surfaces. In previous work, one perspective on such tilings has been to view them as discrete vertex extremal metrics, as pioneered by Schramm [S] and Cannon, Floyd, and Parry [CFP].

Another perspective, discovered and utilized by Dehn [De] and Brooks, Smith, Stone, and Tutte [BSST], has been to view them as circuit solutions to planar electrical networks. In particular, such solutions correspond to rectangular tilings of rectangles: each resistive edge corresponds to a rectangle and the dimensions of the rectangle reflect the current and voltage drop over the edge. This may be viewed as a discrete version of the intuition of Klein, as cited by Poincaré [P]. He obtains a complex structure on a surface by replacing it with a metallic sheet and connecting two points to a battery. The current flow lines and equipotential lines at any point on the surface give locally orthogonal coordinates, and thus a complex structure.

In work with my advisor [CL], we have extended this correspondence to the case of closed surfaces. Following the ideas of Calegari [Cal], we replace the planar network with the edges of a CW decomposition, and a local circuit solution is represented by a discrete harmonic 1-cochain in the cellular homology. Our correspondence results in a family of flat cone metrics that is rectangle-tiled. Many questions about this new object have yet to be explored. These ideas are explained in greater detail in the first section.

A secondary project, which is joint work with my advisor and M. Savas[CLS], has focused on generalizing Ptolemy’s and Casey’s theorems. Ptolemy’s theorem is a classic Euclidean geometry theorem from antiquity, which relates the pairwise distances between four cocyclic points with the simple relation $ac + bd = ef$. Casey’s theorem generalizes this to four circles internal to and tangent to a larger outer circle. Previous work has established Ptolemy’s fundamental relation for $S^2$, $\mathbb{H}^2$, and $\mathbb{R}^{1,1}$; and Casey’s theorem has been established in the first two settings.

We have developed a simple proof technique that reproves these results and allows us to prove a Casey’s theorem in $\mathbb{R}^{1,1}$ and a Ptolemy’s theorem in $dS_2$ and $AdS_2$, the de Sitter spaces. Work is ongoing in proving a Casey’s theorem in $dS_2$ and $AdS_2$. The work has a loose connection with discrete Ricci flow [DU], as Ptolemy’s theorem is vital in analysing the flow when edge switches are made. A brief description of the results and future directions are in the second section below.
2. Square and Rectangle Tilings

2.1. Current Research. The main portion of my thesis research generalizes a classical result of Brooks, Smith, Stone, and Tutte [BSST] relating planar electrical networks and rectangle tilings. An example of this correspondence is shown below:

A planar electrical network is specified with the following data: a planar, connected, directed graph $\Gamma$, one edge $b$ specified as the battery edge, a voltage $V$ applied over $b$, and a resistance function $r : E \to \mathbb{R}^+$ describing the resistance of each edge. With such a network, there is a unique circuit solution to Kirchhoff’s laws $i : E \setminus \{b\} \to \mathbb{R}$ describing the current through each edge.

A rectangle tiling is a covering by rectangles, with disjoint interiors. Horizontal and vertical line segments within such a tiling refer to line segments composed of the sides of the rectangular tiles.

Definition. We say that a rectangle tiling is associated to a circuit solution of a planar electrical network if two sets of conditions are met. First, the following correspondences must hold:

1. Each resistor edge $e$ corresponds to a rectangle $t_e$ with width $|i(e)|$ and height $r(e)|i(e)|$.
2. Each vertex $v$ corresponds to a horizontal segment $h_v$ with length equal to the total current through $v$.
3. Each face $f$ whose boundary consists of resistor edges, corresponds to a vertical segment $u_f$ with length equal to the total voltage drops in one direction as you traverse the boundary edges of $f$.

Secondly, the tiling must reflect the topology of $\Gamma$ and the circuit solution in the following ways:

4. If an edge $e$ carries current flowing out of (into) vertex $v$, then the bottom (top) side of $t_e$ is contained in $h_v$.
5. If an edge $e$ is carrying current clockwise (counterclockwise) about face $f$, then the right (left) side of $t_e$ is contained in $u_f$. 
With this definition, they proved in [BSST] the following theorem:

**Theorem.** Given a network \((\Gamma, b, V, r)\) with a unique circuit solution \(i : E \to \mathbb{R}\), there is a unique associated rectangle tiling of a rectangle of height \(V\) and width \(I\), the total current through the network.

Our result generalizes this theorem to closed surfaces. An example is shown below for genus 2:

\[
\sigma_h = e_1 + e_2 + e_3 + e_4
\]

**Definition.** A rectangular tiling of a flat cone metric \(d\) with holomorphic 1-form \(\alpha\), which specifies horizontal and vertical directions, will be associated to \(\sigma_h\) if it satisfies the same conditions as those mentioned for circuit solutions to planar electrical networks.
Note that now the horizontal and vertical segments may become branched at the cone points. We proved:

**Theorem.** Given such a \((\Sigma, [\sigma], \Gamma, r)\) with a unique harmonic representative \(\sigma_h\) with nonzero coefficients \(c_i\), there is a family of flat cone metrics \(d\) with cone angles in \(2\pi \mathbb{N}\) and holomorphic 1-forms \(\alpha\) that are rectangle-tiled. In each \((d, \alpha)\), these tilings are associated with \(\sigma_h\). In the \(g = 1\) case, we have uniqueness of the \((d, \alpha)\) and the tiling.

The basic idea behind the result is that the local data of \(\sigma_h\) and \(\Gamma\) are used to identify the sides of rectangles that correspond to each edge. The key difficulty, and the source of the non-uniqueness, is the fact that the discrete index of \(\sigma_h\) at certain vertices or faces may be negative. In this case, the method of identification for rectangles representing nearby edges is not uniquely specified, and a branched horizontal or vertical segment results. A simple argument with the discrete Gauss-Bonnet theorem shows that we have index zero at all vertices and faces in the \(g = 1\) case, giving us uniqueness there. For further details, we refer the reader to [CL].

2.2. **Future Research.** As this is very much ongoing work, there are many potential future directions to pursue.

2.2.1. **Simple Extensions.** There are two extensions that we believe strongly in, and should be well within grasp:

1. Modifying the harmonicity constraints at particular vertices and faces should result in flat cone metrics and holomorphic 1-forms on surfaces with boundary. In particular, a boundary component should arise for each vertex or face where closedness, or co-closedness fails. Application of this result to the \(g = 0\) case should result in a new proof of the classical [BSST] theorem.
2. The reverse direction of the correspondence should be established. Given a rectangle tiling of a flat cone metric \(d\) on a surface of genus \(g\), with cone angles in \(2\pi \mathbb{N}\), one should be able to find a CW decomposition of the surface, with a resistance function \(r : E \to \mathbb{R}^+\), and a harmonic 1-cochain \(\sigma_h\) that the tiling is associated to.

2.2.2. **Analogies to Smooth Theory.** Another direction to pursue is the investigation of analogies to the smooth theory of Riemann surfaces. In particular, we feel that the harmonic cochain \(\sigma_h\) should be thought of as the real part of a discrete holomorphic 1-form. In the above result, an investigation of the discrete index shows that the vertices and faces of negative index are to be thought of as being the \(2g - 2\) zeroes (counted with multiplicity) of a holomorphic 1-form on a Riemann surface of genus \(g\). Further analogies should be pursued.
2.2.3. *Image of Our Construction.* Calegari [Cal] states interest in considering the square tilings as giving rise to quadratic differentials and thus, a map from $H_1^1(\Sigma_g)$ to the cotangent space of Teichmüller space. Our investigations at higher genus suggest that the square tilings should really be thought of as giving rise to holomorphic differentials. This is due to the fact that any resulting tiling will have cone angles of $2\pi N$, whereas the flat cone metric associated with a quadratic differential can have more general cone angles $\pi N$. We would still be interested in the analogous question, however, considering which complex structures arise (from the flat cone metrics) and which holomorphic 1-forms result from our construction.

2.2.4. *Discrete Quadratic Differential.* We would also be interested in determining a notion of a discrete quadratic differential, as a square or rectangular tiling of a surface with cone angles of $\pi N$. This is in fact one of the questions that motivated us to first consider such tilings.

2.2.5. *Approximation of Holomorphic Maps.* Yet another direction to consider is the use of these tilings to approximate holomorphic maps. Our data is merely combinatorial at this point, but one might ask to incorporate geometric data via an appropriate choice of the resistances $r : E \to \mathbb{R}^+$. Professor Hersonsky is nearing the completion of such a program for multiply-connected domains in the plane [H1, H2, H3, H4]. With his ideas on choosing resistances, we might hope to achieve similar results for closed surfaces, or surfaces with boundary.

2.2.6. *Related Fields and Works.* Lastly, we mention three related works/subfields that we would like to explore for potential applications of our construction. The first is the study of translation surfaces, which are flat cone metrics resulting from identification of plane polygons along parallel sides. By construction, our flat cone metrics are translation surfaces, resulting from identifications of squares or rectangles along parallel sides. We plan to investigate the extensive literature beginning with the following references: [Ma, MT, Z].

Secondly, there is an interesting work by Kenyon [K] in which he develops an invariant, which he calls the $J$-invariant, associated to a rectilinear polygon, which gives information about the square tileability of the polygon. He uses it to determine the square-tileable Euclidean tori and determines necessary conditions for square-tileability of surfaces. We plan to read this article carefully and see if our construction has anything more to say about such surfaces.

Finally, there is a connection to statistical physics, which is the first application of Mercat’s work [M1, M2]. There is also related recent work by Chelkak and Smirnov [CS]. We plan to read these references.
3. Ptolemy’s and Casey’s Theorem Extensions

3.1. Current Research. Two simple diagrams along with the relation \( ac + bd = ef \) quickly illustrate the content of Ptolemy’s and Casey’s theorems [Cas]:

![Diagram 1]

![Diagram 2]

**Figure 1.** From Wolfram Mathworld and Wikipedia, respectively

An equivalent statement of Ptolemy’s theorem is the trigonometric identity, which holds when \( A + B + C + D = \pi \):

\[
\sin(A + B) \sin(B + C) = \sin(A) \sin(C) + \sin(B) \sin(D)
\]

This same identity may be used to prove Casey’s theorem, by expressing the length of a tangent segment between two circles \( C_i \) and \( C_j \) (tangent to a larger circle \( C \)) in terms of the radii of the circles \( r_i, r_j \), and \( R \) and angle between the points where \( C_i \) and \( C_j \) contact \( C \). It is not hard to show that Casey’s theorem is equivalent to (1) multiplied by a function:

\[
F(r_1, r_2, r_3, r_4, R) = 2\sqrt{(R - r_1)(R - r_2)(R - r_3)(R - r_4)}
\]

We call such a function a *universal function*.

3.1.1. Spherical and Hyperbolic Extensions. We reprove suitable analogues in \( S^2 \) and \( \mathbb{H}^2 \) using (1) and analogous universal functions [CLS]. In these analogues, one replaces lengths \( l \) by \( \sin(l/2) \) and \( \sinh(l/2) \), respectively. Previously, the spherical case was discovered and proven by Frobenius [F] and Darboux [Da], while the hyperbolic case was first done by Kubota [Ku]. Valentine also established Ptolemy’s theorem in both settings in a different fashion [V1, V2]. Our proofs provide a unified approach to all of these results.

3.1.2. Minkowski Plane Extensions. We also prove analogues in \( \mathbb{R}^{1,1} \), the Minkowski plane [CLS]. In this case, a *circle* of radius \( r \) about a point \( p \) is the set of points \( \{q | \|p - q\| = r\} \). Note the resulting set is a hyperbola in the plane, and we refer to the two connected components as the left and right *arms* of the circle. Without loss of generality, we may focus on the unit circle about the origin which is the set
of points \( \{\pm(\cosh \theta, \sinh \theta) \mid \theta \in \mathbb{R}\} \). Let us refer to the value of \( \theta \) as the argument of a point on the unit circle.

With this, we define the distances that allow us to state the Minkowski analogue. The distance between two points with arguments \( \theta_1 \) and \( \theta_2 \) on the same arm is defined to be \( 2i \sinh(|\theta_1 - \theta_2|) \). For points on opposite arms, it is defined to be \( 2 \cosh(\theta_1 - \theta_2) \). The following theorem is proven.

**Theorem.** For four cocyclic points in \( \mathbb{R}^{1,1} \), with distances defined above, the relation \( ac + bd = ef \) still holds.

This Ptolemy theorem can be thought of as the analogue of the Ptolemy trigonometric identity (1) with hyperbolic trigonometric functions. Previously, Smith [Sm] first established an analogue of Ptolemy’s theorem in \( \mathbb{R}^{1,1} \).

A Casey’s theorem analogue follows with a little more work from the above theorem. We require that circles be tangent at points on the same arms, and we consider the lengths of tangent segments between them that join the same arm. Through a simple geometric lemma, we find that such tangent segments only exist when two circles are tangent to the same arm of the larger circle. Then such segments have negative magnitude with respect to the inner product, and we take their length to be the square root of the absolute value of the magnitude. An example of a tangent segment between two circles tangent to a larger circle is shown below:

As such, we are left with considering Casey’s theorem only when all four circles are tangent to the same arm of the larger circle. We prove with a universal function approach:

**Theorem.** For four circles tangent to a larger one in \( \mathbb{R}^{1,1} \), with lengths of tangent segments as defined above, the relation \( ac + bd = ef \) still holds.

3.1.3. *de Sitter and Anti de Sitter Extensions.* We now describe a Ptolemy analogue on \( dS^2 \) and \( AdS^2 \), the Lorentzian analogues of \( S^2 \) and \( \mathbb{H}^2 \), respectively. Let us recall that \( dS^2 \) and \( AdS^2 \) are embedded in \( \mathbb{R}^{2,1} \) and \( \mathbb{R}^{2,1} \), respectively, as the set of points
\{(x, y, z) \mid x^2 + y^2 - z^2 = 1\}. They inherit the ambient pseudometric, so as Lorentzian spaces, they differ only by a sign switch on the pseudometric. As such, an analogue may be found on just one of these spaces, with the corresponding analogue on the other easily obtained by some sign changes. For this discussion, we will be in \(dS_2\). For clarity, the ambient pseudometric on \(\mathbb{R}^{2,1}\) is \(ds^2 = dx^2 + dy^2 - dz^2\).

Let us recall that geodesics in \(dS_2\) are intersections of codimension one subspaces of \(\mathbb{R}^{2,1}\) with \(dS_2\). Circles in \(dS_2\) are intersections of hyperplanes in \(\mathbb{R}^{2,1}\) (so they need not include the origin) with \(dS_2\). Geodesics and circles may be split up into three types categorized by the magnitude of the normal vector to the intersecting hyperplane. A circle/geodesic is spacelike, lightlike, or timelike if the normal vector has negative, zero, or positive magnitude, respectively.

The topology of the geodesics and circles change with the type, and for circles they even change within the timelike and lightlike types. This leads to many cases for descriptions of Ptolemy and Casey analogues. For simplicity and brevity, we include just one such case, which gives a flavor of the results.

We consider timelike circles with positive radius. Up to isometries of \(\mathbb{R}^{2,1}\) (which restrict to isometries of \(dS_2\)), these are intersections of \(dS_2\) with hyperplanes \(\{x = r\}\) where \(0 \leq r < 1\). Such circles are topologically identical to our circles in \(\mathbb{R}^{1,1}\), and we again refer to the two connected components as arms.

It is not difficult to show that any two points on the same arm are timelike-separated, meaning that the geodesic connecting the two is timelike. For two such points \(p, q\), we define the distance between them, \(d\), to satisfy \(\cosh d = \langle p, q \rangle\) where we are using the ambient pseudometric. Similarly, it is not hard to show that any two points on opposite arms are spacelike-separated, meaning that the geodesic connecting the two is spacelike. Here, for two such points \(p, q\), we define the distance between them to satisfy \(\cos d = \langle p, q \rangle\).

Now, just as in the analogues in \(S^2\) and \(H^2\) we replace the distances with simple expressions in terms of them. For points on the same arm, we replace the distance \(d\) with \(i \sinh(d/2)\); and for points on opposite arms, we replace the distance \(d\) with \(\sin(d/2)\). With this, the following result follows directly from the Ptolemy analogue in \(\mathbb{R}^{1,1}\):

**Theorem.** For four cocyclic points on a positive-radius timelike circle in \(dS_2\), with terms as defined in the expressions above, the relation \(ac + bd = ef\) still holds.

For the other cases of the analogue, which are similar, we refer to our paper [CLS].

3.2. **Future Research.**

3.2.1. **Casey’s Theorem in de Sitter and Anti de Sitter spaces.** As can be seen, we have not finished formulating and proving an analogue for Casey’s theorem in these geometries. The many different types of circles and geodesics complicates matters, and we are in the process of working through these cases.

3.2.2. **Angle Ptolemy relations.** In some cases that we have considered above, Ptolemy relations are dual to “angle Ptolemy relations,” via inner products on ambient spaces.
containing the geometry. These relations are amongst the angles of intersection of geodesics tangent to an inner circle. For example, the Ptolemy analogue for spacelike circles in $dS_2$ (not mentioned above), is dual to an angle Ptolemy relation on $\mathbb{H}^2$, via the pseudometric on $\mathbb{R}^{2,1}$. Recall that the subset $\{(x, y, z) \mid x^2 + y^2 - z^2 = -1, z > 0\}$ in $\mathbb{R}^{2,1}$ is a model of $\mathbb{H}^2$. We would like to uncover such relations in as many geometries as possible.

3.2.3. More Cocyclic points. Another direction of further generalization to Ptolemy’s theorem is to consider more than four cocyclic points. In a work of Guo and Sonmez [GS], they demonstrate that the same set of Ptolemy-like relations with $n$ cocyclic points hold in $\mathbb{R}^2$, $S^2$, and $\mathbb{H}^2$. In preliminary work with Matt Russell, we have demonstrated that this same set of Ptolemy-like relations hold in $\mathbb{R}^{1,1}$. We aim to extend this to $dS_2$ and $AdS_2$ as well.

References


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