AREA DEPENDENCE IN GAUGED GROMOV-WITTEN THEORY

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Abstract. We study the variation of the moduli space of symplectic vortices on a fixed holomorphic curve with respect to the area form. For compact, convex varieties we define symplectic vortex invariants and prove a wall-crossing formula for them. As an application, we prove a vortex version of the *abelianization conjecture* of Bertram, Ciocan-Fontanine, and Kim [4], which related Gromov-Witten invariants of geometric invariant theory quotients by a group and its maximal torus, for vortices on non-trivial bundles.

Contents

Preliminary version.

1. Introduction

For any smooth projective variety X and class $d \in H_2(X,\mathbb{Z})$ let $\overline{M}_{g,n}(X,d)$ denote the moduli space of n-pointed, degree d, genus 0 stable maps to X, equipped with *evaluation maps*

$$\mathrm{ev} = (\mathrm{ev}_1,\dots,\mathrm{ev}_n) : \overline{M}_{g,n}(X,d) \to X^n$$

and a *forgetful map*

$$f : \overline{M}_{g,n}(X,d) \to \overline{M}_{g,n}$$

For any cohomology classes $\alpha \in H(X,\mathbb{Q})^n$ integration over X using the virtual fundamental class $[\overline{M}_{g,n}(X,d)] \in H(\overline{M}_{g,n}(X,d))$ defines a *Gromov-Witten invariant*

$$\langle \alpha;\beta\rangle_{X,d}^{g,n} = \int_{[\overline{M}_{g,n}(X,d)]} \mathrm{ev}^*\alpha \wedge f^*\beta \in \mathbb{Q}.$$

An equivariant version of Gromov-Witten invariants was introduced by Givental [17], by equivariant integration over the same moduli spaces. The resulting invariants

$$\langle \cdot ; \cdot \rangle_{X,d}^{g,n} : H_G(X,\mathbb{Q})^n \otimes H(\overline{M}_{0,n},\mathbb{Q}) \to H(BG,\mathbb{Q})$$

are defined for smooth projective $G_{\mathbb{C}}$-varieties X, see [19] for references. An extension of equivariant Gromov-Witten invariants to arbitrary compact symplectic G-manifolds has still not appeared.

The version of equivariant Gromov-Witten invariants we have in mind still have not been defined algebraically; they should be defined by integration over the moduli space of stable maps to the stack-theoretic quotient of X by $G_{\mathbb{C}}$. From the symplectic point of view, these correspond to pairs (A,u) consisting of a connection A on a principal G-bundle P together with a holomorphic section u of the associated fiber bundle $P(X) := (P \times X)/G$. The space of such pairs has a natural Hamiltonian action of the group of gauge transformations $\mathcal{G}(P)$ with moment map depending on a choice of two-form $\mathrm{Vol}_\Sigma \in \Omega^2(\Sigma)$ on Σ and an invariant inner product $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$. Let $\Phi : X \to \mathfrak{g}^*$ be a moment map, and $P(\Phi)$ the induced map from $P(X)$ to $P(\mathfrak{g})$. The symplectic quotient of the space of pairs above is the *moduli space of symplectic vortices*

$$M(P,X)_\epsilon = \{(A,u), \quad F_A + \epsilon^{-1} u^* P(\Phi)\,\mathrm{Vol}_\Sigma = 0\}/\mathcal{G}(P) \tag{1}$$

and has been studied by Cieliebak, Gaio, Mundet, and Salamon [8], [16], [7]. We denote by $M(\Sigma,X)_\epsilon$ the union of $M(P,X)_\epsilon$ over topological types of principal G-bundles P, and by $M(\Sigma,X,d)_\epsilon$ the subset of (equivariant) degree $d \in H_2^G(X)$. Each $M(\Sigma,X,d)$ admits a natural Gromov compactification $\overline{M}(\Sigma,X,d)_\epsilon$ by allowing u to develop bubbles in the fibers of $P(X)$. Let $\overline{M}_n(\Sigma,X)_\epsilon$ denote the moduli space of vortices with marked points varying on the curve. (A moduli space of vortices with varying complex structure on Σ has been introduced by Mundet-Tian [24]; however, the compactification in this case involves Morse trajectories and will not be considered here.) In good cases, there is an *evaluation map*

$$\mathrm{ev} : \overline{M}_n(\Sigma,X)_\epsilon \to X_G^n$$

and a *forgetful map*

$$f : \overline{M}_n(\Sigma,X)_\epsilon \to \overline{M}_n(\Sigma)$$

to the moduli space of stable n-marked curves with parametrized principal component Σ. We show below that for a convex almost complex structure (that is, one for which transversality for sphere bubbles always holds), if the action of the gauge group is locally free then $\overline{M}_n(\Sigma,X)$ has the structure of a compact topological orbifold. Integrating pull-back classes over the moduli space leads to *vortex invariants*

$$\langle \cdot ; \cdot \rangle_{d,\epsilon} : H_G(X,\mathbb{Q})^n \otimes H(\overline{M}_n(\Sigma),\mathbb{Q}) \to \mathbb{Q}$$

where $d \in H_2^G(X)$ is the degree.

The vortex invariants have two interesting limits, the zero and infinite volume limits $\epsilon \to 0, \infty$. The first limit, in which the moment map is forced to vanish, has been studied by Gaio and Salamon [16] and is related to the Gromov-Witten invariants of a symplectic quotient $X/\!/G$. In the second limit, the curvature is forced to vanish and hence, in genus zero, the bundle must be trivial. This limit is related (in genus zero) to the G-equivariant Gromov-Witten invariants of X. More precisely, our main proposal is that the first limit is a composition (in a homotopical sense) of a morphism of cohomological field theories with the trace associated to the correlators on the quotient, while the second is given by the invariant part of the usual equivariant Gromov-Witten invariants. Both of these limits will be further studied elsewhere.

The purpose of this paper is to study what happens in between, that is, the variation of the moduli space of symplectic vortices with respect to the vortex parameter ϵ, for finite, non-zero values of ϵ. This can be treated as a variation of stability condition, in the geometric invariant theory language. In this general setting, a wall-crossing formulae of Kalkman [27] expresses the difference between the integrals as the form is varied as a sum of residues of fixed point contributions. In two-dimensional gauge theory, the method is quite similar to that used by Thaddeus [41] in his proof of the Verlinde formula. In this case, the polarizing line bundle is provided by the Chern-Simons line bundle over the moduli space of connections, and the relevant moduli space is that of *polarized vortices*, that is, solutions to the vortex equation together with a point in the fiber of the Chern-Simons line bundle. Of course, we would like the moduli space of polarized vortices to have at worst orbifold singularities; in order for this to happen it suffices that the variety X satisfy a genericity condition 5.5.3 which implies that any polystable vortex is stable. Our main result expresses the difference between vortex invariants for two different vortex parameter as a sum of residues over *twisted extended vortex invariants*, arising from reducible vortices with vortex parameter between ϵ_1, ϵ_2. These can be treated as vortices for a smaller structure group G_ζ given as the centralizer of some non-zero vector $\zeta \in \mathfrak{g}$, with target the fixed point set X^ζ. (However, the sphere bubbles lie in X, not in X^ζ.) We remark that wall-crossing formulas for different variations of the vortex equations are proved in [9] and by J. Wehrheim, in progress.

Although the results are only for the convex case, we hope that the same method works for arbitrary compact Hamiltonian actions, using virtual fundamental currents defined using equivariant differential forms on Kuranishi orbifolds. Results of this form have been sketched by Fukaya and collaborators, and some results are proved in Chen-Tian [6]. However, carrying out this program requires substantially more work and would make the paper even longer; we hope to return to this issues in the future.

Using the wall-crossing formula, we prove a vortex version of the abelianization conjecture of [4] which related invariants for the action of G and a maximal torus T with Weyl group $W = N(T)/T$:

$$\langle \alpha; \beta \rangle_{G,\epsilon,d_G} = (\#W)^{-1} \sum_{d_T \mapsto d_G} \langle \rho_T^G \alpha; \beta \rangle_{T,\epsilon,d_T,\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}}$$

where the right-hand side is a sum of twisted vortex invariants involving the Euler class of the index class of the covariant derivative acting on $\mathfrak{g}_\mathbb{C}/\mathfrak{t}_\mathbb{C}$. This formula generalizes one of Martin [30] to the "quantum" (i.e. Gromov-Witten) setting. The idea is to show first that the abelianization holds for sufficiently large vortex parameter ϵ (either because the vortex invariants vanish, as in the case considered in this paper, or more generally by Martin's original argument, as we discuss elsewhere.) We hope to explain elsewhere [43] how to use this formula to prove a version of the original conjecture of [4] (including higher quantum Kirwan corrections.)

The applications we have in mind (for example, the Hori-Vafa conjecture [23, Appendix] which motivated [4]) are to smooth projective varieties. A natural question is whether the moduli spaces of this paper can be constructed algebraically; this would make the construction of virtual fundamental classes substantially easier. However, finding the right stability condition is often easier from the symplectic point of view, and this case is no exception.

We thank A. Ott and F. Ziltener for a number of helpful discussions.

2. Stable curves and cohomological field theories

2.1. **Holomorphic families of stable curves.**

Definition 2.1.1. A *nodal curve* is a collection $\underline{\Sigma} = (\Sigma_1, \ldots, \Sigma_k)$ of smooth, compact Riemann surfaces, together with a collection

$$\underline{w} = \{\{w_1^-, w_1^+\}, \ldots, \{w_m^-, w_m^+\}\} \subset \Sigma$$

of distinct *nodal points* in $\underline{\Sigma}$. For $j = 1, \ldots, m$, we denote by $\Sigma_{i^\pm(j)}$ the components such that $w_j^\pm \in \Sigma_{i^\pm(j)}$. A point $z \in \underline{\Sigma}$ is *smooth* if it is not equal to any of the nodal points. A *marked nodal curve* is a nodal curve together with a collection $\underline{z} = (z_1, \ldots, z_n)$ of distinct, smooth points. A marked nodal curve is *stable* if each component has finite automorphism group, that is, contains at least three special points if genus zero, or one special point if genus one, and *semistable* if each genus zero component has at least two special points.

Each nodal, marked curve $\underline{\Sigma}$ has a *combinatorial type* $\Gamma(\underline{\Sigma})$, a graph whose vertices are the components and edges are the nodes and markings of $\underline{\Sigma}$. The map $\underline{\Sigma} \mapsto \Gamma(\underline{\Sigma})$ extends to a functor from the category of marked nodal surfaces to the category of graphs. In particular, there is a canonical homomorphism $\mathrm{Aut}(\underline{\Sigma}) \to \mathrm{Aut}(\Gamma(\underline{\Sigma}))$, whose kernel is the product of the automorphism groups of the components of $\underline{\Sigma}$.

Let $\overline{M}_{g,n,\Gamma}$ denote the *coarse moduli space* of isomorphism classes of stable curves of combinatorial type Γ, and $\overline{M}_{g,n}$ the union over combinatorial types

$$\overline{M}_{g,n} = \bigcup_\Gamma \overline{M}_{g,n,\Gamma}.$$

We introduce the following topology on $\overline{M}_{g,n}$, motivated by the universal deformation of a stable curve constructed in (3) below. (A more standard approach would be to show the existence of universal deformations, and then take the topology induced by them. However, for gauged pseudoholomorphic maps we will not have universal deformations and we must define the topology separately. Since we want to treat all moduli problems in this paper in the same way, we treat first the topology on the moduli space of curves as well.)

Definition 2.1.2. A sequence $\underline{\Sigma}_\alpha$ of stable curves *converges* to a stable curve $\underline{\Sigma}$ iff there exists

(a) a sequence of contractions of graphs $\tau_\alpha : \Gamma(\underline{\Sigma}_\alpha) \to \Gamma(\underline{\Sigma})$.
(b) for every node $\{w_j^\pm\}$ collapsed under τ_α, a pair of neighborhoods $W_j^\pm \subset \Sigma_{i^\pm(j)}$ of the nodal points $w_j^\pm$
(c) for every component Σ_i of $\underline{\Sigma}$, a sequence of maps

$$\phi_{i,\alpha} : \Sigma_i - \cup_{w_j^\pm \in \Sigma_i, \tau_\alpha(w_j^\pm) = \emptyset} W_j^\pm \to \underline{\Sigma}_{\alpha,\tau_\alpha(i)}$$

such that

(a) for each α, the maps $\phi_{i,\alpha}$ cover $\underline{\Sigma}_\alpha$;
(b) $\phi_{i,\alpha}^* J_{\Sigma_{\alpha,\tau_\alpha(i)}}$ converges to J_{Σ_i} uniformly in all derivatives on compact sets;
(c) for any nodal point $w_i^\pm$ of $\underline{\Sigma}$ joining components Σ_j, Σ_k, $\phi_{\alpha,j}^{-1} \circ \phi_{\alpha,k}$ converges to w_k uniformly on compact sets in a neighborhood of w_j.
(d) if z_i is contained in Σ_j, then $z_i = \lim_{\alpha\to\infty} \phi_{j,\alpha}^{-1}(z_{i,\alpha})$.

A subset C of $\overline{M}_{g,n}$ is *closed* if any convergent sequence in C has limit point in C, and *open* if its complement is closed.

A stratum $M_{g,n,\Gamma'}$ contains $M_{g,n,\Gamma}$ in its closure if there exists a contraction of graphs $\Gamma \to \Gamma'$. In this case we say that Γ' *degenerates* to Γ.

The coarse moduli space $\overline{M}_{g,n}$ has the structure of a normal projective variety, in particular, a compact Hausdorff space. The original reference by Deligne-Mumford [10] constructs a (Deligne-Mumford) stack with underlying topological space $\overline{M}_{g,n}$. A mostly self-contained construction of the orbifold structure on the moduli space of stable curves which avoids algebraic geometry is given in Robbin-Salamon [37].

The stack constructed by Deligne and Mumford represents the functor that associates to any scheme S the set of families of stable nodal curves over S. Recall that a *family of nodal curves* is a scheme $\underline{\Sigma}_\bullet$ equipped with a proper flat morphism $\pi : \underline{\Sigma}_\bullet \to S$, such that each fiber $\underline{\Sigma}_s$ is a nodal curve.

For our purposes (since the main technique of the paper is non-linear analysis) it is perhaps more natural to avoid algebraic geometry, by restricting to families whose total space $\underline{\Sigma}_\bullet$ is a *nodal complex manifolds*, that is, smooth complex manifold equipped with a finite set of pairs of divisors

$$\{\{D_1^+, D_1^-\}, \ldots \{D_m^+, D_m^-\}\}$$

and identifications $\varphi_j : D_j^+ \to D_j^-$ for $j = 1, \ldots, m$. One can then require that $\pi : \underline{\Sigma}_\bullet \to S$ has only Morse singularities of codimension one as in [37], disjoint from the nodes, and take this as a definition of family. (The definition of [37] does not allow nodes in the total space, but then a single nodal curve does not constitute a family.)

A *family of marked nodal curves* is a family of nodal curves equipped with a family of sections $z_{\bullet,j} : S \to \underline{\Sigma}_\bullet$ disjoint from the singular points. A *deformation* of a marked nodal curve $(\underline{\Sigma}, \underline{z})$ is a germ of a family of marked nodal curves $\underline{\Sigma}_\bullet \to S$ together with an isomorphism of nodal surfaces $\varphi : \underline{\Sigma}_0 \to \underline{\Sigma}$. We often omit the identification of the central fiber from the notation. A deformation $(\underline{\Sigma}_\bullet, \varphi_0)$ of $\underline{\Sigma}$ is *versal* iff any other deformation $(\underline{\Sigma}'_\bullet \to S, \varphi'_0)$ is induced from a map $\varphi : S' \to S$ in the sense that there exists an isomorphism ψ of $\underline{\Sigma}'$ with the fiber product $\underline{\Sigma} \times_S S'$ in a neighborhood of the central fiber $\underline{\Sigma}_0$. A versal deformation is *universal* if the map ψ is the unique such map inducing the identity on $\underline{\Sigma}_0$. We say a deformation has *fixed type* if the combinatorial type of the fiber is constant. A *universal deformation of fixed type* is then a deformation of fixed type, which is universal in the above sense for deformations of fixed type.

Any compact analytic space has a versal deformation, by results of Doaudy [13], Grauert [20], Palamadov [35], with a complex analytic space as base. If $\underline{\Sigma}$ is a smooth marked curve, then there exists a universal deformation $\underline{\Sigma}_\bullet \to S$ if and only if $\underline{\Sigma}$ is stable, and the parameter space S is smooth. The tangent space $T_0 S$ to S is the space of *infinitesimal deformations* of $\underline{\Sigma}$.

Universal deformations for a smooth stable marked curve $\underline{\Sigma}$ can be constructed by a variety of techniques. First, note that the space of infinitesimal automorphisms of $\underline{\Sigma}$ is the space of holomorphic vector fields,

$$\mathrm{Vect}(\underline{\Sigma}) = H^{0,0}(\underline{\Sigma}, T\underline{\Sigma}).$$

Evaluation defines a map

$$H^{0,0}(\underline{\Sigma}, T\underline{\Sigma}) \to \bigoplus_{i=1}^{n} T_{z_i}\underline{\Sigma}.$$

By the stability assumption, this map is an injection. The space of infinitesimal deformations of the marked curve $\underline{\Sigma} = (\underline{\Sigma}, \underline{z})$ is given by

$$T_0 S \cong H^{0,1}(\underline{\Sigma}, T\underline{\Sigma}) \oplus \left(\bigoplus_{i=1}^{n} T_{z_i}\underline{\Sigma} \right) / H^{0,0}(\underline{\Sigma}, T\underline{\Sigma})$$

The universal deformations are also universal in the category of smooth deformations of $\underline{\Sigma}$, by the construction of local slices for the action of diffeomorphisms in [14], [37, Chapter 9].

Universal deformations of a nodal marked curve of constant type can be constructed as follows. Let $\underline{\Sigma}^r$ denote the smooth curve obtained by replacing each node with a pair of markings, that is,

$$(2)\quad (\underline{\Sigma}, \{\{w_1^+, w_1^-\}, \ldots, \{w_m^+, w_m^-\}\}, \{z_1, \ldots, z_n\})^r = (\underline{\Sigma}, \emptyset, \{z_1, \ldots, z_n, w_1^+, w_1^-, \ldots, w_m^+, w_m^-\}).$$

Let $\underline{\Sigma}_\bullet^r \to S$ denote the universal deformation of $\underline{\Sigma}_0^r$. Identifying the nodal points produces a family $\underline{\Sigma}_\bullet \to S$ of nodal curves with the same combinatorial type as $\underline{\Sigma}$ which is the universal deformation of $\underline{\Sigma}$ preserving combinatorial type, which is universal if $\underline{\Sigma}$ is stable.

The following gluing construction produces a universal deformation of a marked nodal curve $\underline{\Sigma}$. We begin with the

Definition 2.1.3. A *local coordinate near a smooth point* $z \in \underline{\Sigma}$ is a neighborhood U of z and a holomorphic isomorphism w_j of U with a neighborhood of 0 in the tangent line $T_z\underline{\Sigma}$, inducing the identity on $T_z\underline{\Sigma}$. A *gluing parameter* for the j-th node is an element $\delta_j \in T^*_{w_j^+}\underline{\Sigma} \otimes T^*_{w_j^-}\underline{\Sigma}$.

Remark 2.1.4. The space of local coordinates near z is convex, since if w_0, w_1 are local coordinates then any combination $tw_0 + (1-t)w_1$ is still holomorphic and has the same differential at z, and so by the inverse function theorem is a holomorphic isomorphism in a neighborhood of z.

Given local coordinates for the nodes of $\underline{\Sigma}$ and $\underline{\delta} = (\delta_1, \ldots, \delta_m)$, define a (possibly nodal) curve $\underline{\Sigma}(\underline{\delta})$ by gluing together small disks around the node z_j by $w_j \mapsto \delta_j^{-1}/w_j$, for every gluing parameter δ_j that is non-zero. That is,

$$(3)\qquad \underline{\Sigma}(\underline{\delta}) = \bigcup_{i=1}^m \Sigma_i - \{w_1^\pm, \ldots, w_m^\pm\}/(w_j \sim \delta_j^{-1}/w_j, j = 1, \ldots, m)$$

for pairs of points in the two components such that both coordinates are defined.

The gluing construction works in families as follows. Let $\underline{\Sigma}_\bullet \to S$ be a family of nodal curves of the same combinatorial type, with nodal points $(w_{\bullet,j}^\pm)_{j=1}^m$. Let $I^{j,\pm}$ resp. $\underline{I}$ denote the vector bundle over S whose fiber at $s \in S$ is the tangent line at the j-node resp. tensor product of cotangent lines at the nodes,

$$I_s^{j,\pm} = T_{w_{j,s}^\pm}\underline{\Sigma}_s$$

resp.

$$(4)\qquad \underline{I}_s = \bigoplus_{j=1}^m T^*_{w_{j,s}^-}\underline{\Sigma}_s \otimes T^*_{w_{j,s}^+}\underline{\Sigma}_s.$$

A *holomorphic system of local coordinates for the j-th node of* $\underline{\Sigma}$ is a holomorphic map κ_j from a neighborhood U_j of the zero section in $I^{j,\pm}$ to $\underline{\Sigma}$ which is an isomorphism onto its image and induces the identity at zero. Given a holomorphic system $\underline{\kappa} = (\kappa_1^+, \kappa_1^-, \ldots, \kappa_m^+, \kappa_m^-)$ of local coordinates near the nodes, the gluing construction (3) produces a family of curves $\#^\kappa\underline{\Sigma} \to U$, where $\#S$ is a neighborhood of the zero section in I. Applying the gluing construction (3) gives a family

$$\#^{\kappa}\underline{\Sigma}_{\bullet} \to \#S$$

over an open neighborhood U of the zero section in the bundle I of gluing parameters. By [40, Proposition 2.4] $\#\underline{\Sigma}^{\kappa}_{\bullet}$ is a universal deformation of any of its fibers, and in particular is independent up to isomorphism of deformations of the choice of local coordinates κ. For later reference, we record the following as a theorem, since it is exactly this point at which the symplectic construction of universal deformations of stable maps becomes problematic:

Theorem 2.1.5. *The action of automorphisms* $\mathrm{Aut}(\underline{\Sigma})$ *of* $\underline{\Sigma}$ *extends to an action of* $\mathrm{Aut}(\underline{\Sigma})$ *on the universal deformation* $\underline{\Sigma}_{\bullet} \to S$*, possibly after shrinking* S*. Furthermore, any two fibers* $\underline{\Sigma}_{\bullet}$ *are isomorphic, if and only if they are related by an automorphism of* $\underline{\Sigma}$*, again after shrinking* S [40, Lemma 2.7]*.*

The first claim follows from universality. The proof of the second claim depends strongly on analyticity. In particular, the subset of $S \times S$ defined as the set of points (s_1, s_2) such that the fibers over s_1, s_2 are isomorphic is analytic, and so after shrinking S any component contains 0, see [40, Section 2]. The maps

$$S \to \overline{M}_{g,n}, s \mapsto [\underline{\Sigma}_s]$$

are continuous, by Definition 2.1.2, and homeomorphisms onto their image, by the previous discussion. It follows as in [32, 5.6.5] that the topology in Definition 2.1.2 is the same as that induced by S, so that (if one prefers to think of $\overline{M}_{g,n}$ as an orbifold) the maps above provide $\overline{M}_{g,n}$ with the orbifold charts. See also [37, Theorem 16.6] which however uses a different definition of convergence.

2.2. **Partially smooth families of stable curves.** We extend the definition of families and deformations to smooth and partially smooth settings as follows.

Definition 2.2.1. A *smooth* family of curves of fixed type is a fiber bundle $\underline{\Sigma}_{\bullet} \to S$ and a smoothly varying complex structure on the nodal fibers $\underline{\Sigma}_s$. The category of families fibers naturally over the category of smooth base manifolds and smooth maps. A *partially smooth* family of curves is a family $\underline{\Sigma}_{\bullet} \to S$ over a stratified space $S = \cup_{\Gamma} S_{\Gamma}$ such that each stratum $\underline{\Sigma}_{\Gamma,\bullet}$ is a smooth family of fixed type over S_{Γ}. The category of families fibers naturally over the category of stratified base manifolds and stratified maps, that is, continuous maps whose restriction to each fiber is smooth. Deformations of partially smooth families are defined as before. A *universal partially smooth deformation* of $\underline{\Sigma}$ is a partially smooth family $\underline{\Sigma}_{\bullet}$ together with an identification $\varphi : \underline{\Sigma}_0 \to \underline{\Sigma}$, such that any other partially smooth family $\underline{\Sigma}'_{\bullet} \to S'$ is obtained by pull-back by maps $\psi : S' \to S$, $\phi : \underline{\Sigma} \times_S S' \to \underline{\Sigma}'$, and any two identifications ϕ, ϕ' inducing the identity on $\underline{\Sigma}$ are equal. A *smooth system of local coordinates for the* j*-th node of* $\underline{\Sigma}$ is a smooth map κ_j from a neighborhood $U_{j,\pm}$ of the zero section in $I^{j,\pm}$ to $\underline{\Sigma}$ which is an isomorphism onto its image and induces the identity at zero.

Applying the gluing construction (3) gives a smooth family $\#^\kappa \underline{\Sigma}_\bullet \to \#^\kappa S$ over an open neighborhood $\#^\kappa S$ of the zero section in the bundle I of gluing parameters. We then have analogs of the properties discussed before in the holomorphic context:

Proposition 2.2.2. (a) *The family $\#^\kappa \underline{\Sigma}_\bullet \to \#^\kappa S$ is a partially smooth universal deformation of any of its fibers, and in particular independent up to isomorphism of deformations of the choice of local coordinates κ.*
(b) *The action of automorphisms* $\mathrm{Aut}(\underline{\Sigma})$ *of $\underline{\Sigma}$ extends to an action of* $\mathrm{Aut}(\underline{\Sigma})$ *on $\#^\kappa \underline{\Sigma}_\bullet \to \#^\kappa S$, possibly after shrinking $\#^\kappa S$.*
(c) *Any two fibers of $\#^\kappa \underline{\Sigma}_\bullet$ are isomorphic, if and only if they are related by an automorphism of $\underline{\Sigma}$, again after shrinking $\#^\kappa S$.*

We sketch a proof of the last claim. Let $\underline{\Sigma}^\kappa_\bullet \to S^\kappa$ be a universal deformation constructed using a smooth family of local coordinates, and $\underline{\Sigma}_\bullet \to S$ a universal deformation using holomorphic family of local coordinates. The map $S^\kappa \to S$ is an equivariant diffeomorphism on the open stratum, since it is homotopic to the identity (by interpolating the two systems of local coordinates) and the map of tangent spaces is an isomorphism (by a non-trivial index computation which we do not give here). A similar computation holds for the other strata, which shows that $S^\kappa \to S$ is an equivariant homeomorphism everywhere, and so the claim for the smooth family follows from the corresponding holomorphic statement in Theorem 2.1.5. We hope that these matters will be explained in more detail in the work of Hofer et al.

These orbifold charts are compatible at least if the local coordinates near the nodes are constructed inductively as follows. Let Γ' be a combinatorial type degenerating to Γ. Local coordinates for the nodes of $\overline{M}_{g,n,\Gamma}$ induce local coordinates for $\overline{M}_{g,n,\Gamma'}$, in a neighborhood of $M_{g,n,\Gamma}$, via the gluing construction (3).

Definition 2.2.3. A *compatible system of local coordinates* for $\overline{M}_{g,n}$ is a system of local coordinates for the nodes of each stratum $\overline{M}_{g,n,\Gamma}$, so that the local coordinates on any stratum $\overline{M}_{g,n,\Gamma'}$ are induced from those on $\overline{M}_{g,n,\Gamma}$, in a neighborhood of $\overline{M}_{g,n,\Gamma}$.

Compatible systems of local coordinates can be constructed by induction on the dimension of $\overline{M}_{g,n,\Gamma}$, using convexity on the space of local coordinates in Remark 2.1.4.

One can also modify the gluing map in the gluing construction above. In the language of Hofer, Wysocki and Zehnder,

Definition 2.2.4. A *gluing profile* is a diffeomorphism $\varphi : (0,1] \to [0,\infty)$.

(a) $\varphi(\delta) = -1 + 1/\delta$ will be called the *(inverted) linear gluing profile*,
(b) $\varphi(\delta) = e^{1/\delta} - e$ will be called the *exponential gluing profile*, and
(c) $\varphi(\delta) = -\ln(\delta)$. the *logarithmic gluing profile*.

The set of gluing profiles naturally forms a partially ordered set, as follows . We write $\varphi_1 \geq \varphi_0$ and say φ_1 is *softer than* φ_0 if $\varphi_1^{-1}\varphi_0 : [0,1] \to [0,1]$ is smooth. We write $\varphi_1 > \varphi_0$ and say that φ_1 is *strictly softer* than φ_0 if the derivatives of $\varphi_1^{-1}\varphi_0 : [0,1] \to [0,1]$ vanish at 0. Thus, the exponential gluing profile, linear gluing profile, and logarithmic gluing profile form a decreasingly soft sequence in this partial order.

Fix a gluing profile φ, and consider once again the gluing construction. Given a nodal curve $\underline{\Sigma}$ of local coordinates near the nodes, and a collection of gluing parameters $\delta = (\delta_1, \ldots, \delta_m)$, define a curve $\underline{\Sigma}(\underline{\delta}, \underline{\varphi})$ by gluing together small disks:

$$\underline{\Sigma}(\underline{\delta}, \varphi) := \bigcup_{i=1}^{m} \Sigma_i - \{w_1^{\pm}, \ldots, w_m^{\pm}\}/(w_j \sim \exp(\varphi(|\delta_j|) + i \arg(\delta_j))/w_j). \tag{5}$$

More generally, given a family $\underline{\Sigma}_\bullet \to S$ of curves of a fixed combinatorial type and a system of local coordinates near the nodes $\underline{\kappa}$, the construction (5) produces a family of curves

$$\#^{\underline{\kappa},\underline{\varphi}}\underline{\Sigma}_\bullet \to \#^{\underline{\kappa},\underline{\varphi}}S.$$

The map

$$\#^{\underline{\kappa},\underline{\varphi}}S/\operatorname{Aut}(\underline{\Sigma}_0) \to \overline{M}_{g,n}, \quad s \mapsto [\#\underline{\Sigma}_s]$$

is a homeomorphism but not a diffeomorphism onto its image, for the standard smooth structure on $\overline{M}_{g,n}$.

The universal deformations constructed from a gluing profiles and smooth families of local coordinates provide orbifold charts for $\overline{M}_{g,n}$. For any gluing profile φ and any collection $\underline{\kappa}$ of local coordinates near the nodes, the gluing map

$$\#^{\varphi,\kappa} : \#^{\kappa,\varphi}S/\operatorname{Aut}(\underline{\Sigma}_0) \to \overline{M}_{g,n}$$

is a homeomorphism onto its image, for sufficiently small parameter space $\#S$. Indeed, in the logarithmic gluing profile $\varphi_{\log}$, this holds by Proposition 2.2.2(c), and for arbitrary φ, the map $\#^{\varphi,\kappa}$ factors through $\#^{\varphi_{\log},\kappa}$, which proves the claim. If the local coordinates are constructed inductively as in Definition 2.2.3, then the gluing maps provide $\overline{M}_{g,n}$ with a compatible set of orbifold charts. We denote by $\overline{M}_{g,n}^{\kappa,\varphi}$ the smooth structure on $\overline{M}_{g,n}$ defined by the local coordinates κ near the nodes and the gluing profile φ.

The forgetful maps with respect to these non-standard smooth structures have regularity properties that are worse than those with respect to the standard smooth structure. For $2g + n > 3$ we have forgetful morphisms $f_j : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$ by forgetting the j-th marking and collapsing unstable components. There are two possibilities: a genus zero component with one marking and two nodes is replaced by a point; a genus zero component with two markings and one node is replaced by a single marking. For any gluing profile, the maps f_j are smooth away from the locus where collapsing occurs. Near the locus of curves containing components with two nodes and one marking, the map on gluing parameters for the collapsed component is

$$\gamma : \delta_j, \delta_{j-1} \mapsto \varphi^{-1}(\varphi(\delta_j) + \varphi(\delta_{j-1})).$$

For example, in the logarithmic gluing profile we have $\gamma(\delta_{j-1}, \delta_j) = \delta_{j-1}\delta_j$, which is smooth, while for the linear gluing profile collapsing a component gives the map

$$(\delta_j, \delta_{j-1}) \mapsto \delta_j \delta_{j-1}/(\delta_j + \delta_{j-1})$$

in the local gluing parameters, which is not smooth. At the locus of curves with one node and two markings, the map $\overline{M}_{g,n} \to \overline{M}_{g,n-1}$ simply forgets the additional component and creates a new marking at the former node. For nearby curves, assuming that the local coordinate is induced from an isomorphism with the projective line, the forgetful morphism creates a new marking at $\exp(\varphi(\delta))^{-1}w$, in the local coordinate on the remaining component. Thus the forgetful morphism is smooth near the locus of one node, two marking components if the local coordinates are standard and $\delta \mapsto \exp(\varphi(\delta))^{-1}$ is smooth, that is, φ is at least as hard as the logarithmic gluing profile.

Finally we discuss regularity of the identity map on $\overline{M}_{g,n}$, equipped with two different smooth structures. Given two gluing profiles φ_0, φ_1 and a fixed compatible system of local coordinates, the identity map $\overline{M}_{g,n}^{\varphi_0} \to \overline{M}_{g,n}^{\varphi_1}$ is smooth if φ_0 is softer than φ_1. In particular, if φ_0 is the exponential gluing profile and φ_1 is the logarithmic gluing profile defining the standard smooth structure, then the identity map is smooth for a fixed system of local coordinates.

2.3. **Cohomological field theory.** The boundary of $\overline{M}_{g,n}$ (that is, the complement of the open locus of curves $M_{g,n}$ with smooth domain) consists of the following divisors:

(a) if $g > 0$, a divisor
$$\iota_{g-1,n+2} : D_{g-1,n} \to \overline{M}_{g,n}$$
equipped with an isomorphism
$$\varphi_{g-1,n+2} : D_{g-1,n} \to \overline{M}_{g-1,n+2}.$$
The inclusion is obtained by identifying the last two marked points.

(b) for each splitting $g = g_1 + g_2, \{1, \ldots, n\} = I_1 \cup I_2$ with $2g_j + |I_j| \geq 3, j = 1, 2$, a divisor
$$\iota_{g_1+g_2,I_1\cup I_2} : D_{g_1+g_2,I_1\cup I_2} \to \overline{M}_{g,n}$$
corresponding to the formation of a separating node, splitting the surface into pieces of genus g_1, g_2 with markings I_1, I_2, equipped with an isomorphism
$$\varphi_{g_1+g_2,I_1\cup I_2} : D_{g_1+g_2,I_1\cup I_2} \to \overline{M}_{g_1,|I_1|} \times \overline{M}_{g_2,|I_2|}$$
(except in the cases $I_1 = I_2 = \emptyset$ and $g_1 = g_2$ in which case there is an additional automorphism.)

The pull-back $\iota^*_{g_1+g_2,I_1\cup I_2}\beta$ of any class $\beta \in H^*(\overline{M}_{g,n}, \mathbb{Q})$ has a Kunneth decomposition

$$\iota^*_{g_1+g_2,I_1\cup I_2}\beta = \sum_{j\in J} \beta_{1,j} \otimes \beta_{2,j} \tag{6}$$

for some index set J and classes $\beta_{k,j} \in H^\bullet(\overline{M}_{g_1,|I_1|}, \mathbb{Q})$. For each of the above divisors, let

$$\gamma_\cdot \in H^2(\overline{M}_{g,n})$$

denote the dual class to the divisor $D_\cdot$.

Let R be a ring.

Definition 2.3.1. (Kontsevich-Manin) An (even) cohomological field theory (CohFT) with values in R is an R-module V equipped with a symmetric non-degenerate R-valued bilinear form $\langle \cdot, \cdot \rangle_V$ and collection of *correlators*

$$V^n \otimes H^\bullet(\overline{M}_{g,n}, \mathbb{Q}) \to R, \quad (\alpha, \beta) \mapsto \langle \alpha; \beta \rangle_{g,n}$$

satisfying the following two splitting axioms:

$$\begin{aligned} \langle \alpha; \beta \wedge \gamma_{g-1,n+2} \rangle_{g,n} &= \sum_k \langle \alpha, \delta_k, \delta^k; \iota^*_{g-1,n} \beta \rangle_{g-1,n+2} \\ \langle \alpha, ; \beta \wedge \gamma_{g_1+g_2, I_1 \cup I_2} \rangle &= \sum_k \langle \alpha_{I_1}, \delta_k; \cdot \rangle_{g_1, |I_1|+1} \langle \alpha_{I_2}, \delta^k; \cdot \rangle_{g_2,|I_2|+1} (\iota^*_{g_1+g_2, I_1 \cup I_2} \beta) \end{aligned}$$

where the dots indicate insertion of the Kunneth components of $\iota^*_{g_1+g_2,I_1\cup I_2}\beta$, $\delta_k \in V$ is a basis and $\delta^k \in V$ a dual basis. (There is an additional factor of 2 in the exceptional case $g_1 = g_2$, $I_1 = I_2 = \emptyset$ arising from the additional automorphism.)

That is, if β is as in (6) then

$$\langle \alpha; \beta \wedge \gamma_{g_1+g_2,I_1\cup I_2} \rangle = \sum_{j \in J} \langle \alpha_{I_1}, \delta_k; \beta_{1,j} \rangle_{g_1,|I_1|+1} \langle \alpha_{I_2}, \delta^k; \beta_{2,j} \rangle_{g_2,|I_2|+1}.$$

More generally, if V is $\mathbb{Z}_2$-graded then the above formula with appropriate signs leads to a general definition of CohFT. However, we will consider only the even case.

Using the pairing any CohFT gives rise to *composition maps*

$$\mu^{g,n} : V^n \otimes H^\bullet(\overline{M}_{g,n+1}) \to V \tag{7}$$

defined by the equation

$$\langle \mu^{g,n}(\alpha_1, \ldots, \alpha_n; \beta), \alpha_0 \rangle_V = \langle \alpha_0, \ldots, \alpha_n; \beta \rangle_{g,n+1}.$$

The various relations on the divisors in $\overline{M}_{g,n}$ give rise to relations on the maps $\mu^{g,n}$. In particular the relation

$$[D_{0,\{0,3\}\cup\{1,2\}}] = [D_{0,\{0,1\}\cup\{2,3\}}]$$

in $H^2(\overline{M}_{0,4})$ implies that

$$\mu^{0,2} : V^2 \to V$$

is associative.

3. Equivariant symplectic geometry

3.1. **Equivariant cohomology and localization.** Let G be a compact, connected group and $EG \to BG$ the universal bundle. For any G-space X, we denote by $X_G = (X \times EG)/G$ the *homotopy quotient* of X, and by $H_G(X) := H(X_G)$ the *equivariant cohomology* of X. The following properties of equivariant cohomology are well-known:

(a) If X has a free G-action making the projection $X \to X/G$ into a principal G-bundle then the projection $X_G \to X/G$ has contractible fiber EG and pull-back $X_G \to X/G$ induces an isomorphism $H_G(X) \cong H(X/G)$.
(b) If G' is a normal subgroup and G' acts freely making X into a principal G'-bundle then the projection $X_G \to (X/G')_{G/G'}$ has contractible fiber and pull-back induces an isomorphism $H_G(X) \cong H_{G/G'}(X/G')$.
(c) If the action of G' is only locally free, then the projection $X_G \to (X/G')_{G/G'}$ has fibers with torsion cohomology and pull-back induces an isomorphism in rational cohomology $H_G(X,\mathbb{Q}) \cong H_{G/G'}(X/G',\mathbb{Q})$.

Any homomorphism $G' \to G$ induces an G'-action on X and a homomorphism $H_G(X) \to H_{G'}(X)$. In particular, suppose that $T \subset G$ is a maximal torus with Weyl group $W = N(T)/T$. The group W naturally acts on ET, hence X_T, by homeomorphisms, and induces an action of W on the equivariant cohomology $H_T(X)$. We denote by $H_X(T)^W$ the invariant part. The inclusion induces a map $H_G(X) \to H_T(X)$; with image contained in $H_X(T)^W$. Over the rationals we obtain an isomorphism

$$H_G(X,\mathbb{Q}) \to H_T(X,\mathbb{Q})^W; \tag{8}$$

see [22]. The G-space X is *equivariantly formal* if the cohomology spectral sequence for the projection $(X \times EG)/G \to BG$ collapses, or equivalently, inclusion of the fiber induces an injection $H(X) \to H_G(X)$. If X is compact and smooth, there is an *equivariant fundamental class* $[X]_G \in H^G_{\dim(X)}(X)$. Product with $[X]_G$ defines a map

$$\int_X : H_G^\bullet(X,\mathbb{Z}) \to H_G^{\bullet-\dim(X)}(\mathrm{pt},\mathbb{Z}).$$

If X is equivariantly formal, this induces an isomorphism over the rationals (fiberwise Poincaré duality)

$$H_G^\bullet(X,\mathbb{Q}) \to \mathrm{Hom}_{H_G^\bullet(\mathrm{pt},\mathbb{Q})}(H_G^\bullet(X,\mathbb{Q}), H_G^{\bullet-\dim(X)}(\mathrm{pt},\mathbb{Q})) \tag{9}$$

see for example [18], [5].

If X is a smooth G-manifold, then the equivariant cohomology with real coefficients can be realized via equivariant differential forms in the Cartan model. The action of G on X gives rise to a homomorphism

$$\mathfrak{g} \to \mathrm{Vect}(X), \qquad \xi \mapsto \xi_X.$$

We denote by

$$\Omega_G(X) = \mathrm{Hom}(\mathfrak{g}, \Omega(X))^G$$

the space of equivariant forms on X with smooth coefficients, and $H_G^{dR}(X,\mathbb{R})$ the cohomology of the equivariant de Rham operator

$$\mathrm{d}_G \in \mathrm{End}(\Omega_G(X))[1], \quad (\mathrm{d}_G\alpha)(\zeta) := (\mathrm{d}\alpha)(\zeta) + (\iota(\zeta_X)\alpha)(\zeta).$$

Then $H_G^{dR}(X,\mathbb{R})$ is naturally isomorphic to $H_G(X,\mathbb{R})$, see [21].

Suppose that X is smooth and compact and G is torus. *Equivariant localization* says that integration of an equivariantly closed form over X factors through the fixed point set of any one-parameter subgroup. Let $\beta \in \mathfrak{g}$, and X^β the fixed point set of β. Let N^β be the normal bundle of the embedding $X^\beta \to X$. The intersection pairings on X are related to those on X^β by

$$\int_X \alpha = \int_{X^\beta} \iota_\beta^* \alpha \wedge \mathrm{Eul}(N_\beta)^{-1}. \tag{10}$$

3.2. **Hamiltonian group actions.** Let G be a compact, connected Lie group with complexification $G_{\mathbb{C}}$ and Lie algebra $\mathfrak{g}$. Let (X,ω) denote a compact symplectic manifold with an action of G preserving ω. The action is *Hamiltonian* if there exists an equivariant *moment map* $\Phi : X \to \mathfrak{g}^*$ satisfying

$$\iota(\xi_X)\omega = -\mathrm{d}(\Phi,\xi), \qquad \forall \xi \in \mathfrak{g},$$

or equivalently if the symplectic form ω has an equivariantly closed extension

$$\omega_G(\zeta) := \omega + (\Phi,\zeta) \in \Omega_G^2(X).$$

A *polarization* of X is a G-equivariant line bundle $\pi : L \to X$ equipped with a connection $\alpha \in \Omega^1(L_1)^G$ with $\mathrm{d}_G\alpha = -\pi^*\omega_G$. In particular, the curvature of α is ω. The *symplectic quotient* of X by G is

$$X/\!\!/G := \Phi^{-1}(0)/G.$$

If G acts freely on $\Phi^{-1}(0)$, then $X/\!\!/G$ naturally has the structure of a symplectic manifold with symplectic form $\omega/\!\!/G \in \Omega^2(X/\!\!/G)$ defined by

$$p^*(\omega/\!\!/G) = \iota^*\omega$$

where

$$\begin{array}{ccc} \Phi^{-1}(0) & \xrightarrow{\iota} & X \\ \Big\downarrow{\scriptstyle p} & & \\ X/\!\!/G & & \end{array}$$

are the inclusion and projection respectively. If $L \to X$ is a polarization, then the line bundle

$$L/\!\!/G := (L|\Phi^{-1}(0))/G$$

with connection defined by

$$(\alpha/\!\!/G) \in \Omega^1((L/\!\!/G)_1), \qquad \hat{p}^*(\alpha/\!\!/G) = \hat{\iota}^*\alpha$$

is a polarization of $X/\!\!/G$; here $\hat{p}, \hat{\iota}$ are the lift of p, ι to $L|\Phi^{-1}(0)$.

3.3. **Wall-crossing formulae.** Suppose that $G = U(1)$. We denote by κ_ϵ the Kirwan map

$$\kappa_\epsilon : H_{U(1)}(X) \to H(X_\epsilon).$$

For the following, see Kalkman [27] or Lerman [29].

Theorem 3.3.1. *Let X be a Hamiltonian $U(1)$-action, and $\alpha \in H_{U(1)}(X)$. The integrals of the forms $\kappa_\epsilon(\alpha)$ over the quotients X_ϵ satisfy the wall-crossing formula*

$$\int_{X_{\epsilon_2}} \kappa_{\epsilon_2}(\alpha) - \int_{X_{\epsilon_1}} \kappa_{\epsilon_1}(\alpha) = \operatorname{Res}_\xi \left(\sum_{F \subset (\phi /\!\!/ G)^{-1}(\epsilon_1,\epsilon_2)^{U(1)}} \int_F \iota_F^* \alpha \wedge \operatorname{Eul}(N_F)^{-1} \right) \tag{11}$$

where ξ is the equivariant parameter, and Res_ξ *denotes the residue at* 0.

A proof suggested by Lerman proceeds as follows. For any interval $[\epsilon_1, \epsilon_2]$ let $\mathbb{P}^1_{[\epsilon_1,\epsilon_2]}$ be the two-sphere equipped with the standard $U(1)$-action by rotation and equivariant symplectic form with moment image $[\epsilon_1, \epsilon_2]$. Define the *symplectic cut* of X at $[\epsilon_1, \epsilon_2]$ by

$$X_{[\epsilon_1,\epsilon_2]} := (X \times \mathbb{P}^1_{[\epsilon_1,\epsilon_2]}) /\!\!/ U(1) \tag{12}$$

$$\cong \phi^{-1}(\epsilon_1)/U(1) \cup \phi^{-1}(\epsilon_1,\epsilon_2) \cup \phi^{-1}(\epsilon_2)/U(1). \tag{13}$$

$X_{[\epsilon_1,\epsilon_2]}$ has the structure of a Hamiltonian $U(1)$-manifold, with fixed point set

$$X^{U(1)}_{[\epsilon_1,\epsilon_2]} = X_{\epsilon_1} \cup \phi^{-1}((\epsilon_1,\epsilon_2))^{U(1)} \cup X_{\epsilon_2}.$$

We denote by

$$\kappa_{[\epsilon_1,\epsilon_2]} : H_{U(1)}(X) \to H_{U(1)}(X_{[\epsilon_1,\epsilon_2]})$$

the canonical map given by the Cartan construction. Taking the residue of the integral of $\kappa_{[\epsilon_1,\epsilon_2]}(\alpha)$ over $X_{[\epsilon_1,\epsilon_2]}$ gives

$$\operatorname{Res}_\xi \int_{X_{[\epsilon_1,\epsilon_2]}} \kappa_{[\epsilon_1,\epsilon_2]} \alpha = 0$$

since the integral is polynomial. On the other hand, localization (10) expresses the residue as

$$\int_{X_{\epsilon_2}} \kappa_{\epsilon_2}(\alpha) \operatorname{Eul}(N_{\xi_2})^{-1} + \int_{X_{\epsilon_1}} \kappa_{\epsilon_2}(\alpha) \operatorname{Eul}(N_{\xi_1})^{-1} + \operatorname{Res}_\xi \left(\sum_{F \subset \phi^{-1}(\epsilon_1,\epsilon_2)^{U(1)}} \int_F \iota_F^* \alpha \wedge \operatorname{Eul}(N_F)^{-1} \right) \tag{14}$$

where N_{ξ_j} is the normal bundle of X_{ξ_j} in $X_{[\xi_1,\xi_2]}$ for $j = 1, 2$. The residues of the first two terms come exclusively from the first term in the expansion of the inverted Euler classes, since α is top degree, and the wall-crossing formula follows.

We apply Kalkman's wall-crossing formula to the following situation. Suppose that $\omega_0 \in \Omega^2(X)$ is a symplectic form on a compact manifold X, $\omega_1 \in \Omega^1(X)$ is a closed two-form, and

$$\omega_\epsilon = \omega_0 + \epsilon\omega_1.$$

The action of G on $X_\epsilon := (X, \omega_\epsilon)$ is Hamiltonian with moment map

$$\Phi_\epsilon = \Phi_0 + \epsilon\Phi_1 : X \to \mathfrak{g}^*.$$

Consider the symplectic quotients

$$X_{0,\epsilon} = X_\epsilon /\!\!/ G = \Phi_\epsilon^{-1}(0)/G.$$

We wish to study the dependence of $X_{0,\epsilon}$ on G. Suppose that (ω_1, Φ_1) is the equivariant curvature of an equivariant Hermitian line bundle-with-connection $\pi : L \to X$ with connection $\alpha_1 \in \Omega^1(L_1)$; here L_1 is the unit circle bundle L. That is,

$$\pi^*\omega_1 = \mathrm{d}\alpha_1, \quad (\Phi_1, \xi) = \alpha_1(\xi_L), \quad \xi \in \mathfrak{g}.$$

Let $\phi : L \to \mathbb{R}$ denote the norm. The restriction of ϕ to the complement of the zero section is smooth. Equip the complement of the zero section $L - 0$ with the closed two-form $\pi^*\omega_0 + \mathrm{d}(\phi, \alpha_1)$. The action of $U(1)$ on the fibers of $L - 0$ is then Hamiltonian with moment map ϕ, the action of G on $L - 0$ is Hamiltonian with moment map given by $\Phi_L := \pi^*\Phi_0 + \phi\pi^*\Phi_1$. There is a symplectomorphism of symplectic quotients

$$(L - 0/\!\!/ G)/\!\!/_\epsilon U(1) \to X_\epsilon, \quad l \mapsto \pi(l).$$

As usual, $L - 0/\!\!/G$ is smooth if the action of G is free on the zero level set. An element $\xi \in \mathfrak{g}$ acts trivially on an element $y \in \Phi_L^{-1}(0)$ if and only if $\pi(y)$ is ξ-fixed and ξ acts trivially on the fiber over $\pi(y)$, that is, $(\Phi_1(\pi(y)), \xi) = 0$. Hence $X/\!\!/G$ is a smooth orbifold if for each element $\xi \in G$, the fixed point locus X_ξ satisfies $(\Phi_1(X_\xi), \xi) \neq 0$. For example, this condition holds whenever X is a flag variety and ω_1 is a sufficiently generic invariant closed 2-form.

Kalkman's wall-crossing formula (16) compares the pairings on X_{0,ϵ_1} and X_{0,ϵ_2}. Let us examine the structure of the fixed point set of $U(1)$ on $L - 0/\!\!/G$. Necessarily, any fixed point $[l] \in (L - 0/\!\!/G)^{U(1)}$ has the property that $zl = \exp(\xi)l$ for some $\xi \in \mathfrak{g}$. The projection defines a surjection

$$(L - 0/\!\!/G)^{U(1)} \to \bigcup_{\xi \in \mathfrak{g} - \{0\}} X^\xi/G. \tag{15}$$

Any fixed point component $\tilde{F} \subset (L - 0/\!\!/G)^{U(1)}$ maps diffeomorphically onto a component of X^ξ for some ξ. The fiber of any component $F \subset X^\xi$ is the quotient W/W_ξ, where $W = N(T)/T$ is the Weyl group and W_ξ is the stabilizer of ξ under the adjoint action. The action of $U(1)$ on the normal bundle of $\tilde{F}$ identifies with the action of the on parameter subgroup generated by ξ on the normal bundle of F. This yields

Corollary 3.3.2. *Let X_ϵ be a family of compact Hamiltonian G-manifolds, and $\epsilon_1, \epsilon_2 \in \mathbb{R}$ such that the G acts locally freely on $\Phi^{-1}(\epsilon_j), j = 1, 2$, and with only one-dimensional stabilizers on the union of level sets $\Phi^{-1}(\epsilon), \epsilon \in (\epsilon_1, \epsilon_2)$. Then*

(16)
$$\int_{X_{0,\epsilon_2}} \kappa_{0,\epsilon_2}(\beta) - \int_{X_{0,\epsilon_1}} \kappa_{0,\epsilon_2}(\beta) = \sum_{\epsilon\in[\epsilon_1,\epsilon_2]} \sum_{\zeta} \frac{|W_\zeta|}{|W|} \operatorname{Res}_{(\xi,\zeta)} \sum_{F\subset X^\zeta} \int_F \iota_F^* \beta \wedge \operatorname{Eul}(N_F)^{-1}$$

where the sum is over fixed point components $F \subset X^\zeta$ with $F \cap \Phi_\epsilon^{-1}(0) \neq \emptyset$ and the sum over ζ represents the sum over one-parameter subgroups of G, up to conjugacy.

Remark 3.3.3. In fact, the argument does not use the Hamiltonian structure on X; the same formula holds for any differentiable $U(1)$-manifold X equipped with a proper map $\phi : X \to \mathbb{R}$ with the property that $U(1)$ acts freely on the regular values of ϕ.

A more complicated but also more conceptual proof of the above formula can be given by comparing the contributions to the two terms on the left-hand side from the non-abelian localization formula [44].

3.4. **Characteristic maps for orbibundles.** The material in this section is well-known, but we could not find a good reference. Recall that

Definition 3.4.1. A *topological groupoid* X consists of

(a) a pair of topological spaces X_0, X_1;
(b) head and tail maps $t, h : X_1 \to X_0$;
(c) a composition map $m : X_1 \times_{t,h} X_1 \to X_1$;
(d) an inverse map $i : X_1 \to X_1$

satisfying associativity and invertibility axioms

$$m \circ (\operatorname{Id} \times m) = m \circ (m \times \operatorname{Id}), \qquad m \circ (i \times \operatorname{Id}) = m \circ (\operatorname{Id} \times i).$$

The *coarse moduli space* $[X]$ of X is the space of isomorphism classes of objects of X, with topology induced by that on X_0. The *classifying space* of X (or more accurately, its geometric realization) is the topological space defined by

$$BX = \bigcup_{n\geq 0} \Delta_n \times (X_1 \times_{X_0} \ldots \times_{X_0} X_1)/\sim$$

where Δ_n is the standard n simplex and the quotient $\sim$ is by the relations generated by the face and degeneracy maps as in [39].

The classifying space construction gives a covariant functor from topological groupoids to topological spaces, that is, any morphism of groupoids $X \to Y$ induces a continuous map $BX \to BY$. The map from X_0 to $[X]$ induces a continuous map

$$\pi : BX \to [X]$$

whose fibers are isomorphism to the classifying space of the automorphism groups,

$$\pi^{-1}(x) = B\,\mathrm{Aut}(x).$$

Proposition 3.4.2. *Let X be a topological groupoid with only finite automorphism groups. Pull-back induces an isomorphism in rational cohomology*

$$H(BX, \mathbb{Q}) \to H([X], \mathbb{Q}). \tag{17}$$

Proof. Since any such classifying space has torsion cohomology,

$$H^k(B\,\mathrm{Aut}(x), \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

□

A topological *orbifold* is a proper étale topological groupoid X, see e.g. [31] for references.

Proposition 3.4.3. *For any orbifold X, the coarse moduli space $[X]$ has the structure of a rational homology manifold:*

$$H_k([X], [X] - \{x\}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = \dim(X) \\ 0 & \textit{otherwise} \end{cases}.$$

In particular, if X is compact then X has a rational fundamental class $[X] \in H_{\dim(X)}(X, \mathbb{Q})$.

Proof. It suffices to consider the case that X_0 is a vector space and X_1 is the space of morphisms given by the action of a finite group G on X_0, in which case $X_0 \to BX$ is the fiber of $BX \to BG$. Since BG has torsion cohomology, the result follows from the spectral sequence for the fibration. □

Let G be a topological group. A *groupoid principal G-bundle* is a topological groupoid P equipped with a morphism of groupoids $P \to X$, such that each space $P_0 \to X_0$ is a topological principal G-bundle and the structure maps of P are morphisms of principal G-bundles. The classifying space construction gives a topological principal G-bundle

$$BP \to BX.$$

Taking a classifying map for the bundle BP induces a map

$$H(BG, \mathbb{Q}) \to H(BX, \mathbb{Q}).$$

Combining this with (17) produces a *rational characteristic class* map

$$H(BG, \mathbb{Q}) \to H([X], \mathbb{Q}). \tag{18}$$

In particular, for moduli spaces such as the moduli space of curves $\overline{M}_{g,n}$, bundles such as the tangent bundle are orbifold vector bundles and so define honest vector bundles on the classifying space $B\overline{M}_{g,n}$, which then define rational characteristic classes on the coarse moduli space $\overline{M}_{g,n}$.

4. Equivariant Gromov-Witten theory

In this section we discuss equivariant Gromov-Witten theory from the symplectic point of view, for convex target in genus zero. The construction is mostly a repetition of arguments of Ruan-Tian [38] and in genus zero, McDuff-Salamon [32]. However, we give details since we have to repeat the construction for gauged stable maps later.

4.1. **Stable maps.** Let (X, ω) be a compact symplectic manifold and $\mathcal{J}(X)$ the space of compatible almost complex structures on X. Let $J \in \mathcal{J}(X)$.

Definition 4.1.1. A *marked nodal J-holomorphic map* to X consists of a nodal curve $\underline{\Sigma}$, a collection $\underline{z} = (z_1, \ldots, z_n)$ of distinct, smooth points on $\underline{\Sigma}$, and a J-holomorphic map $\underline{u} : \underline{\Sigma} \to X$. The pair $(\underline{\Sigma}, \underline{u})$ is *stable* if each component Σ_i for which u_i is constant has at least three special (nodal or marked) points. An *isomorphism* of marked nodal maps from $(\underline{\Sigma}_0, \underline{z}_0, \underline{u}_0)$ to $(\underline{\Sigma}_1, \underline{z}_1, \underline{u}_1)$ is an isomorphism of nodal surfaces $\underline{\varphi} : \underline{\Sigma}_0 \to \underline{\Sigma}_1$ such that $\varphi(z_{0,i}) = z_{1,i}$ for $i = 1, \ldots, n$ and $\underline{u}_1 \circ \underline{\varphi} = \underline{u}_0$. The *degree* of stable map is the sum of the degrees of the components, $\deg(\underline{u}) = \sum_{i=1}^m \deg(u_i)$.

Let $\overline{M}_{g,n,\Gamma}(X, d)$ denote the coarse moduli space of isomorphism classes of stable maps of combinatorial type Γ and degree d, and $\overline{M}_{g,n}(X, d)$ the union over combinatorial types

$$\overline{M}_{g,n}(X, d) = \bigcup_{\Gamma} \overline{M}_{g,n,\Gamma}(X, d).$$

We introduce a topology on $\overline{M}_{g,n}(X, d)$ as follows. We say a graph Γ_1 is a *contraction by an edge* of Γ_0 by $e \in \operatorname{Edge}(\Gamma_0)$ if Γ_1 is obtained from Γ_0 by identifying the head and tail of Γ_0 and removing the edge e. We say that Γ_1 is a *contraction* of Γ_0 if Γ_1 is obtained by a successive contraction of edges. In this case, there are canonical surjections $\operatorname{Vert}(\Gamma_0) \to \operatorname{Vert}(\Gamma_1)$ and a canonical inclusion $\operatorname{Edge}(\Gamma_1) \to \operatorname{Edge}(\Gamma_0)$ whose complement is the set of contracted edges. If e is a contracted edge, we write $\tau(e) = \emptyset$.

Definition 4.1.2. A sequence $\underline{u}_\alpha : \underline{\Sigma}_\alpha \to X$ of stable maps *Gromov converges* to a stable map $\underline{u} : \underline{\Sigma} \to X$ iff there exists

(a) a sequence of contractions $\tau_\alpha : \Gamma(\underline{\Sigma}_\alpha) \to \Gamma(\underline{\Sigma})$
(b) for every node $\{w_j^\pm\}$ collapsed under τ_α, a pair of neighborhoods $W_j^\pm \subset \Sigma_{i^\pm(j)}$ of the nodal points $w_j^\pm$;
(c) for every component Σ_i of $\underline{\Sigma}$, a sequence of maps

$$\phi_{i,\alpha} : \Sigma_i - \bigcup_{w_j^\pm \in \Sigma_i, \tau_\alpha(w_j^\pm) = \emptyset} W_j^\pm \to \underline{\Sigma}_{\alpha,\tau_\alpha(i)}$$

such that

(a) $\phi_{i,\alpha}^* J_{\Sigma_{\alpha,\tau_\alpha(i)}}$ converges to J_{Σ_i} uniformly in all derivatives on compact sets;

(b) $u_{\tau_\alpha(i),\alpha} \circ \phi_{i,\alpha}$ converges uniformly on compact subsets of the complement of $W_i \subset \Sigma_i$ to $u_{i,\infty}$,
(c) for any bubble point $w \in W_i$ on Σ_i, the energy lost

$$m(w) := \lim_{\epsilon \to 0} \lim_{\alpha \to \infty} E(u_\alpha \circ \phi_{i,\alpha}; B_\epsilon(w_j))$$

is equal to the sum of the energies on the components of $\underline{u}$ attached to w_j.
(d) for any nodal point $w_i^\pm$ of $\underline{\Sigma}$ joining components Σ_j, Σ_k, $\phi_{\alpha,j}^{-1} \circ \phi_{\alpha,k}$ converges to w_k uniformly on compact sets in a neighborhood of w_j.
(e) if z_i is contained in Σ_j, then $z_i = \lim_{\alpha \to \infty} \phi_{j,\alpha}^{-1}(z_{i,\alpha})$.

A subset C of $\overline{M}_{g,n}(X,d)$ is *Gromov closed* if any convergent sequence in C has limit point in C, and *Gromov open* if its complement is closed.

We will need the following alternative, less canonical description of Gromov convergence. Suppose that $\underline{\Sigma}_0$ is a nodal marked curve, $\underline{u}_0 : \underline{\Sigma}_0 \to X$ a stable map. Suppose that $\underline{\Sigma}_0$ is stable. Let $t_1, \ldots, t_k$ be coordinates on the stratum $M_{g,n,\Gamma}$ of the moduli space of marked curves containing $\underline{\Sigma}_0$, and $\epsilon_1, \ldots, \epsilon_m$ a collection of gluing parameters. Any nearby curve $\underline{\Sigma}_1$ is obtained by the gluing construction from a curve in $\overline{M}_{g,n,\Gamma}$ using local coordinates $z_1^\pm, \ldots, z_m^\pm$ and a set of gluing parameters $\epsilon_1, \ldots, \epsilon_m$, by identifying neighborhood $z_i^+ = \delta_i / z_i^-$. Let Σ_0' denote the smooth curve obtained by removing small balls around the nodes of $\underline{\Sigma}_0$. The gluing construction identifies Σ_0' with a subset of $\underline{\Sigma}_1$.

Definition 4.1.3. We say that $\underline{\Sigma}_1, \underline{u}_1$ is *ϵ-close* to $\underline{\Sigma}_0, \underline{u}_0$ if

$$\underline{u}_1 | \Sigma_0' = \exp_{\underline{u}_0 \Sigma_0'} \xi_1$$

for some $\xi_1 \in \Omega^0(\Sigma_0', (u_0')^* TX)$ with $\|\xi_1\|_{L^2} < \epsilon$, where

$$E(\underline{u}_1 | \underline{\Sigma_1} - \Sigma_0') < E(\underline{u}_0 | \underline{\Sigma_0} - \Sigma_0') + \epsilon, \quad \sum_{i=1}^m |\delta_i|^2 < \epsilon, \quad \sum_{j=1}^k |t_j|^2 < \epsilon.$$

Let $B_\epsilon(\underline{\Sigma}_0, \underline{u}_0)$ denote the space of stable maps ϵ-close to $\underline{\Sigma}_0, \underline{u}_0)$. Define a *pseudodistance function*

$$\rho_\epsilon : B_\epsilon(\underline{\Sigma}_0, \underline{u}_0) \to \mathbb{R}$$

by

$$\rho_\epsilon(\underline{\Sigma}_1, \underline{u}_1) = \sum |t_j|^2 + \sum |\delta_i|^2 + \|\xi_0'\|^2 + |E(\underline{u}_1 | \underline{\Sigma_1} - \Sigma_0') - E(\underline{u}_0 | \underline{\Sigma_0} - \Sigma_0')|.$$

More generally, suppose that $\underline{\Sigma}_0$ has unstable components. We suppose, for simplicity, that there is a single unstable component Σ_i with a single special (necessarily nodal) point. By assumption, the map u_i is non-trivial on Σ_i. In particular, it's differential $\mathrm{d}u_i$ is non-zero almost everywhere. Choose points additional marked points $z_1', z_2' \in \Sigma_i$ at which $\mathrm{d}u_i$ is non-vanishing, and transverse (locally defined) hypersurfaces $H_1, H_2 \subset X$. Requiring $u_i(z_i') \in H_i, i = 1, 2$ fixes the parametrization of the component Σ_i. The same formula now gives a pseudodistance function ρ_ϵ, depending on the choice of hypersurfaces.

Remark 4.1.4. We prefer the functions ρ_ϵ over the ones defined on [32, p. 134] because they are smooth on the regular locus in any stratum.

Lemma 4.1.5. *For ϵ sufficiently small, $(\underline{\Sigma}_\alpha, \underline{u}_\alpha)$ Gromov converges to $(\underline{\Sigma}_1, \underline{u}_1)$, if and only if the coordinates $t_{j,\alpha}, \epsilon_{i,\alpha}$ for Σ_α converge to the coordinates $t_{1,\alpha}, \epsilon_{1,\alpha}$ for Σ_1 and ξ_α converges to ξ_1.*

Proof. The forward direction is immediate from the definition of Gromov convergence. Suppose that the gluing parameters $\epsilon_{i,\alpha}$ for Σ_α converge to the parameters $\epsilon_{1,\alpha}$ for Σ_1 and ξ_α converges to ξ_1. It suffices to show that $\underline{u}_\alpha \circ \phi_{\alpha_i}$ converge to u_i on any compact subset K of Σ_i. We may assume that ϵ is sufficiently small so that no bubbling occurs on the cylinders connecting the components of Σ_0' in $\underline{\Sigma}_\alpha$, and we have exponential decay on these cylinders. Hence $\underline{u}_\alpha \circ \phi_{\alpha_i} | K$ converges to some map, which is necessarily equal to u_i on $K \cap \Sigma_0'$. By unique continuation, the limit of $\underline{u}_\alpha \circ \phi_{\alpha_i} | K$ equals u_i. □

Proposition 4.1.6. *The Gromov open sets form a topology for which any convergent sequence is Gromov convergent. Furthermore, any convergent sequence has a unique limit.*

Proof. By [32, Lemma 5.6.5] it suffices to show that the function ρ_ϵ satisfies the following properties:

(a) $\rho_\epsilon(\underline{\Sigma}_1, \underline{u}_1) = 0$ only if $(\underline{\Sigma}_1, \underline{u}_1) = (\underline{\Sigma}_0, \underline{u}_0)$.
(b) $(\underline{\Sigma}_\alpha, \underline{u}_\alpha)$ converges to $(\underline{\Sigma}_0, \underline{u}_0)$ if and only if $\rho_\epsilon(\underline{\Sigma}_\alpha, \underline{u}_\alpha)$ converges to 0.
(c) Suppose that $(\underline{\Sigma}_\alpha, \underline{u}_\alpha)$ converges to $(\underline{\Sigma}_1, \underline{u}_1)$. Then $\limsup_\alpha \rho_\epsilon(\underline{\Sigma}_\alpha, \underline{u}_\alpha) \geq \rho_\epsilon(\underline{\Sigma}_1, \underline{u}_1)$.

(a) Suppose $\rho_\epsilon(\underline{\Sigma}_1, \underline{u}_1) = 0$. Then $\underline{\Sigma}_0 = \underline{\Sigma}_1$, and $\underline{u}_0 | \Sigma_0' = \underline{u}_1 | \Sigma_0'$ implies that $\underline{u}_0 = \underline{u}_1$, by unique continuation. (b,c) follow from Lemma 4.1.5. □

Theorem 4.1.7 (Gromov compactness)**.** *$\overline{M}_{g,n}(X,d)$ is a compact, Hausdorff space.*

Strikingly, there seems to be no reference containing a complete proof of this theorem. For references and discussion, see for example [25, Theorem 1.8]. The strategy for constructing a smooth structure on a subset of $\overline{M}_{g,n}(X,d)$ is similar to that for $\overline{M}_{g,n}$. Namely, we wish to construct universal deformations in a smooth sense.

Definition 4.1.8. A *partially smooth* family of nodal J-holomorphic maps $(\underline{\Sigma}_\bullet, \underline{u}_\bullet)$ consists of a partially smooth family of nodal surfaces $\underline{\Sigma}_\bullet \to S$ together with a continuous map $\underline{u}_\bullet : \underline{\Sigma}_\bullet \to X$ such that the restriction $\underline{u}_s$ of $\underline{u}$ to any fiber $\underline{\Sigma}_s$ is smooth, and the restriction of $\underline{u}$ to any stratum $\underline{\Sigma}_\Gamma$ is smooth. A *partially smooth deformation* a stable J-holomorphic map $(\underline{\Sigma}, \underline{u})$ is a germ of a partially smooth family $(\underline{\Sigma}_\bullet, \underline{u}_\bullet)$ together with an isomorphism of nodal maps $\varphi : \underline{\Sigma} \to \underline{\Sigma}_0$ such that $\varphi^* \underline{u}_0 = \underline{u}$. The deformation $(\underline{\Sigma}, \underline{u})$ is *versal* if any other (germ of) family of marked, nodal curves $(\underline{\Sigma}', \underline{\Sigma}_0) \to (S', 0)$ is induced from a map $\psi : S' \to S$ in the sense that there exists an isomorphism $\underline{\Sigma}' = \underline{\Sigma} \times_S S'$ in a neighborhood of the central fiber $\underline{\Sigma}_0$, and $\underline{u}'$ is obtained by composing projection on the first factor with $\underline{u}$. The deformation is *universal* if the map ψ is the unique map inducing the identity on $\underline{\Sigma}_0$.

We construct smooth universal deformations of regular stable maps of a fixed combinatorial type. Let $\underline{u} : \underline{\Sigma} \to X$ be a stable map. Let $\underline{\Sigma}_\bullet \to S$ be the minimal versal deformation of $\underline{\Sigma}$ of fixed type. We may assume after shrinking S that $\underline{\Sigma}_\bullet$ is trivial as a smooth fiber bundle, and that the variation of complex structure occurs outside of a fixed neighborhood of the nodal points; that is, $\underline{\Sigma}_\bullet$ is given by a family $j_\bullet$ of complex structures on $\underline{\Sigma}$. For $p > 2$ define

$$\mathcal{B} = S \times \mathrm{Map}(\underline{\Sigma}, X)^{1,p}$$

and fiber bundle $\mathcal{E} \to \mathcal{B}$ with

$$\mathcal{E}_{s,\underline{u}} = \Omega^{0,1}(\underline{\Sigma}, u^*TX)_{j_s} \times X^{2m}.$$

The subscript indicates the the space of $(0,1)$-forms is with respect to the pair (j_s, J). The Cauchy-Riemann operator together with the evaluations at the nodes defines a section

$$\overline{\partial} : \mathcal{B} \to \mathcal{E}, \quad u \mapsto (\overline{\partial} u, u(w_j^\pm), j = 1, \ldots, m)$$

whose linearization is Fredholm. Let

$$\Delta := \{(x_1, x_1, x_2, x_2, \ldots, x_m, x_m) | x_1, \ldots, x_m \in X\} \subset X^{2m}$$

denote m copies of the diagonal. The vertical tangent space to $\mathcal{E}$ at the zero section is given by

$$T_{0,x}\mathcal{E}_{s,\underline{u}} \cong \Omega^{0,1}(\underline{\Sigma}^r, \tilde{u}^*TX)_{j_s} \oplus \bigoplus_j T^*_{u(z_j)}X =: \Omega^{0,1}(\underline{\Sigma}, u^*TX)_{j_s};$$

that is, we define the space of $(0,1)$-forms on the nodal surface $\underline{\Sigma}$ by appending the tensor product of tangent spaces at the nodes. Let $\tilde{D}_{\underline{\Sigma},\underline{u}}$ denote the *parametrized linear operator*

$$\tilde{D}_{\underline{\Sigma},\underline{u}} : T_{s,\underline{u}}\mathcal{B} \to T_{0,\underline{x}}\mathcal{E}_{U,\underline{u}}, \quad v \mapsto D_v \overline{\partial}. \tag{19}$$

This has an explicit description in terms of the connection ∇ associated to the modified Levi-Civita connection,

$$\tilde{D}_{\underline{\Sigma},\underline{u}}(\zeta, \xi) := \pi^{0,1}_{\underline{\Sigma}}(\nabla \xi - \frac{1}{2} J(u) \mathrm{d}u \zeta - \frac{1}{2} J_u (\nabla_\xi J)_u \overline{\partial} u). \tag{20}$$

Definition 4.1.9. $\underline{\Sigma}, \underline{u}$ is *parametrized regular* with respect to the given deformation of $\underline{\Sigma}$ iff $\tilde{D}_{\underline{\Sigma},\underline{u}}$ is transverse to $0 \times \Delta$, and *parametrized regular* iff it is parametrized regular for some deformation of $\underline{\Sigma}$.

Example 4.1.10. Constant maps are parametrized regular if and only if $\underline{\Sigma}$ has genus zero, since the cokernel of $\tilde{D}_{\underline{\Sigma},\underline{u}}$ in this case is the tensor product of the space of infinitesimal deformations of the curve with the tangent space of the target.

We wish to prove

Theorem 4.1.11. *A parametrized regular nodal pseudoholomorphic map* $(\underline{\Sigma}, \underline{u})$ *admits a smooth universal deformation* $(\underline{\Sigma}_\bullet, \underline{u}_\bullet) \to S$ *of constant type if and only if* $(\underline{\Sigma}, \underline{u})$ *is stable. The family* $(\underline{\Sigma}_\bullet, \underline{u}_\bullet) \to S$ *is a universal deformation of any of its fibers, and two fibers* $\underline{\Sigma}_{s_j}, \underline{u}_{s_j}$ *are isomorphic if and only if they are related by the action of* $\operatorname{Aut}(\underline{\Sigma}, \underline{u})$.

Sketch of proof. Let $(\underline{\Sigma}_0, \underline{u}_0)$ be a stable map and $(\underline{\Sigma}, \underline{\Sigma}_0) \to (S, 0)$ the versal deformation of $\underline{\Sigma}_0$ constructed in (3). Choose a coordinate chart for parameter space S,

$$\exp_{\underline{\Sigma}} : U_s \to S$$

where $U_s \subset T_s S$ is a neighborhood of $0 \in T_s S$. We may write any map $\underline{u}_1$ close to $\underline{u}$ as

$$\underline{u}_1 = \exp_{\underline{u}}(\underline{\xi}_1)$$

for some $\underline{\xi}_1 \in \Omega^0(\underline{\Sigma}, \underline{u}^* TX)$. Let

$$\Phi_{\underline{u}}(\underline{\xi}_1) : \underline{u}^* TX \to \exp_{\underline{u}}(\underline{\xi}_1)^* TX$$

denote parallel transport with respect to the Hermitian connection. Acting on the space of $(0,1)$-forms parallel transport induces an isomorphism

$$\Phi_{\underline{u}}(\underline{\xi}_1)^{-1} : \Omega^{0,1}(\underline{\Sigma}, \exp_{\underline{u}}(\underline{\xi}_1)^* TX; j) \to \Omega^{0,1}(\underline{\Sigma}, (\underline{u}_{\underline{\delta}})^* TX; \underline{j}).$$

One can identify $\Omega^{0,1}(\exp_{\underline{\Sigma}}(\zeta), u^* TX)$ with $\Omega^{0,1}(\underline{\Sigma}, u^* TX)$ by composing the inclusion

$$\Omega^{0,1}(\exp_{\underline{\Sigma}}(\zeta), u^* TX) \to \Omega^1(\exp_{\underline{\Sigma}}(\zeta), u^* TX)_{\mathbb{C}} = \Omega^1(\underline{\Sigma}; u^* TX)_{\mathbb{C}}$$

with the projection

$$\Omega^1(\underline{\Sigma}; \underline{u}^* TX)_{\mathbb{C}} \to \Omega^{0,1}(\underline{\Sigma}, \underline{u}^* TX).$$

We denote by

$$\Phi_{\underline{\Sigma}}(\zeta)^{-1} : \Lambda^{0,1}(\exp_{\underline{\Sigma}}(\zeta), u^* TX) \to \Lambda^{0,1}(\underline{\Sigma}, \underline{u}^* TX)$$

parallel transport with respect to the connection induced by the trivialization. By composing $\Phi_{\underline{u}}(\underline{\xi}_1)^{-1}$ and $\Phi_{\underline{\Sigma}}(\zeta)^{-1}$ we obtain an identification

$$\Phi^{-1}_{\underline{\Sigma}, \underline{u}} : \Omega^{0,1}(\exp_{\underline{\Sigma}}(\zeta), \exp_{\underline{u}}(\underline{\xi}_1)^* TX) \to \Omega^{0,1}(\underline{\Sigma}, \underline{u}^* TX).$$

Define

$$\begin{aligned} \mathcal{F}_{\underline{\Sigma}, \underline{u}} : T_0 S \times \Omega^0(\underline{\Sigma}, \underline{u}^* TX) &\to \Omega^{0,1}(\underline{\Sigma}, \underline{u}^* TX) \\ (\zeta, \underline{\xi}_1) &\mapsto \Phi_{\underline{\Sigma}, \underline{u}}(\zeta, \underline{\xi}_1)^{-1}(\overline{\partial}_{\exp_{\underline{\Sigma}}(\zeta)}(\exp_{\underline{u}}(\underline{\xi}_1))). \end{aligned}$$

The operator $\tilde{D}_{\underline{\Sigma}, \underline{u}}$ is the linearization of $\mathcal{F}_{\underline{\Sigma}, \underline{u}}$. The implicit function theorem implies that if $\underline{\Sigma}, \underline{u}$ is parametrized regular then the zero set of $\mathcal{F}_{\underline{\Sigma}, \underline{u}}$ is modelled locally on a neighborhood of 0 in $\ker(\tilde{D}_{\underline{\Sigma}, \underline{u}})$. Regularity is [32, Section B.4]. Thus we obtain a smooth family of stable maps in a neighborhood of 0 in $\ker(\tilde{D}_{\underline{\Sigma}, \underline{u}})$. The action of $\operatorname{Aut}(\underline{\Sigma}, \underline{u})$ on the space of stable maps induces an inclusion $\operatorname{aut}(\underline{\Sigma}, \underline{u}) \to \ker(\tilde{D}_{\underline{\Sigma}, \underline{u}})$. Let $\tilde{S}$ be a neighborhood of 0 in a complement of $\operatorname{aut}(\underline{\Sigma}, \underline{u})$, and $(\underline{\Sigma}_\bullet, \underline{u}_\bullet) \to \tilde{S}$ the smooth family of stable maps constructed above.

We claim that $(\underline{\Sigma}_\bullet, \underline{u}_\bullet)$ is a universal smooth deformation of constant type. Suppose that $(\underline{\Sigma}'_\bullet, \underline{u}'_\bullet) \to S', \varphi'$ is another smooth deformation of fixed type. By the versal property of S, there exists a map $\psi : S' \to S$ and an isomorphism of $\underline{\Sigma}'_\bullet$ with the pull-back of $\underline{\Sigma}$. By the implicit function theorem again, there exists a smooth map $\tilde{\psi} : S' \to \tilde{S}$ so that $(\underline{\Sigma}', \underline{u}')$ is equal to the pull-back of $(\underline{\Sigma}, \underline{u})$ by ψ. Any two such maps inducing the identity on the central fiber $(\underline{\Sigma}, \underline{u})$ necessarily differ by smooth family of automorphisms of $\underline{\Sigma}$. Since $\tilde{S}$ is contained in a complement of $\operatorname{aut}(\underline{\Sigma}, \underline{u})$ the two maps must be equal, which proves the claim.

On the other hand, suppose that $(\underline{\Sigma}, \underline{u})$ is not stable, that is, some component Σ_i of $\underline{\Sigma}$ for which u_i is constant admits infinitesimal automorphisms. Let $(\underline{\Sigma}_\bullet, \underline{u}_\bullet)$ be a smooth deformation of constant type. The corresponding component $\Sigma_{\bullet,i}$ must admit infinite automorphisms for every fiber, and furthermore, the restriction $u_{\bullet,i}$ must be constant, since constant maps admit no deformations. Taking a non-constant automorphism φ_s with φ_0 the identity shows that the universal property cannot be satisfied. □

Let $M^{\mathrm{reg}}_{g,n,\Gamma}(X, d)$ denote the coarse moduli space of isomorphism classes of parametrized regular stable maps of combinatorial type Γ.

Theorem 4.1.12. *The universal deformations above provide* $M^{\mathrm{reg}}_{g,n,\Gamma}(X, d)$ *with the structure of a smooth orbifold of dimension* $(3g-3+n)+(1-g)\dim(X)+2(c_1(TX), d)$*, with tangent space at* $(\underline{\Sigma}, \underline{u})$ *equal to the quotient of the kernel* $\ker(\tilde{D}_{\underline{\Sigma},\underline{u}})$ *by the infinitesimal action of the space of automorphisms* $\operatorname{aut}(\underline{\Sigma}) \to \ker(\tilde{D}_{\underline{\Sigma},\underline{u}})$.

Proof. Any universal deformation $(\underline{\Sigma}_\bullet, \underline{u}_\bullet) \to S$ induces a map

$$S/\operatorname{Aut}(\underline{\Sigma}, \underline{u}) \to M^{\mathrm{reg}}_{g,n,\Gamma}(X, d), \ \ s \mapsto [\underline{\Sigma}_s, \underline{u}_s].$$

By Theorem 4.1.11, these are homeomorphisms onto their image and hence provide charts. Compatibility follows from universality. □

4.2. Sobolev spaces.

In preparation for the gluing construction in the next section we review the construction of Sobolev spaces. Sobolev spaces on domains in Euclidean are space are discussed in e.g. [1], and Sobolev spaces on Riemannian manifolds are discussed in e.g. [3] and [32].

Let (M, g) be a smooth compact Riemannian manifold. For integers $k \geq 0, p \geq 1$, the generalized Sobolev space $W^{k,p}(M, g)$ is defined as the completion of the space of smooth functions in the norm

$$\|f\|_{k,p} = \left(\sum_{j \leq k} \int_M |\nabla^j f|^p \right)^{1/p} \tag{21}$$

using the Levi-Civita connection ∇. The space $W^{k,p}(M, g)$ is independent of the choice of metric g. Let $W^{-k,p}(M)$ be the dual of $W^{k,p}(M)$. For g a continuous metric the Sobolev spaces $W^{k,p}(M)$ are defined for $k = -1, 0, 1$; for higher derivatives the definition involves derivatives of the metric.

If $E \to M$ is a Euclidean vector bundle equipped with a connection then similar formulas using covariant derivatives define a Sobolev space of (equivalence classes of almost everywhere defined) sections $W^{k,p}(E)$, independent of the choice of metric and connection. In particular, for a vector bundle E we denote by $\Omega^i(M,E)_{k,p}$ the space of i-forms with values in E of Sobolev class $W^{k,p}$. If g', ∇' are another metric and connection uniformly comparable to g, ∇ in all derivatives then the norm $\| \cdot \|'_{k,p}$ is comparable to $\| \cdot \|_{k,p}$, that is, there exist constants c, c' depending only on suprema of derivatives of $\nabla - \nabla' \in \Omega^1(M, \mathrm{End}(TM))$and $g - g' \in \Omega^0(M, \mathrm{Hom}(TM \otimes TM, \mathbb{R}))$ such that

$$c\| \cdot \|_{k,p} \leq \| \cdot \|'_{k,p} \leq c' \| \cdot \|_{k,p}.$$

In particular, suppose that g is continuous and M admits a decomposition $M = M_0 \cup M_1$ into smooth manifolds with boundary so that g is smooth on each piece. In this case,

$$\|f\|^p_{W^{k,p}(E)} = \|f|M_0\|^p_{W^{k,p}(E_0)} + \|f|M_1\|^p_{W^{k,p}(E_1)}. \tag{22}$$

Sobolev spaces with fractional order can be defined by complex interpolation

$$W^{\theta k + (1-\theta) l, p}(M) = [W^{k,p}(M), W^{l,p}(M)]_\theta.$$

If $p = 2$ then these interpolation spaces agree with the Besov spaces defined by real interpolation; this is the only case in which we will use fractional orders, so we will not distinguish between Besov and Sobolev spaces in the following.

For treatment of the Sobolev embedding theorems we refer to [1]. In particular, for $k - n/p > l$ we have a compact embedding

$$W^{k,p}(M) \to C^l(M). \tag{23}$$

Using Sobolev embedding multiplication of functions defines a continuous map

$$W^{k,p}(M) \times W^{l,p}(M) \to W^{m,p}(M) \tag{24}$$

for $m < k + l - n/p$, and also the boundary cases $m = k + l - n/p$ except $k = -l, k = m, l = m$. By [1, Theorem 7.34(a)] if M is a domain in Euclidean space and E is trivial then the constants in the Sobolev embedding theorem depends only on the dimensions of the cone in the cone condition for each piece. Suppose that $M = M_0 \cup M_1$ is a decomposition into pieces for which M_0, M_1 are domains and the bundle $E_j = E|M_j$ are trivializable, as in (22). The constants in the Sobolev embedding theorem depend only on the cone condition for M_0, M_1 and bounds for the difference in metrics and connections.

4.3. **The gluing construction.** We wish to prove, in analogy with the holomorphic situation, that a parametrized regular nodal pseudoholomorphic map $(\underline{\Sigma}, \underline{u})$ admits a partially smooth universal deformation if and only if $(\underline{\Sigma}, \underline{u})$ is stable. The main ingredient is the gluing construction, which produces from a smooth family of pseudoholomorphic maps of constant type, a partially smooth family of maps of varying type. Our basic principle is to avoid as much as possible the introduction of any new analysis beyond that in [32, Chapter 10]. Unfortunately this isn't completely possible: [32] does not discuss the case of varying complex structure, or differentiability of the evaluation maps with respect to the gluing parameter.

We first discuss the case that $\underline{\Sigma}$ is a single nodal curve. A *collection of gluing parameters* is an element $\underline{\delta} = (\delta_1, \ldots, \delta_m)$ in the bundle $\underline{I}$ of (4). In addition, the construction depends on the choice of a parameter κ which describes the width of the annulus on which the gluing of maps is performed. Namely, for the usual gluing profile the gluing region will be the annulus near the nodes given by

$$\mathcal{A}(\kappa, \delta) = \left\{ z \in \mathbb{C} | \quad |\delta|^{1/2}\kappa^{-1} \leq |z| \leq |\delta|^{1/2}\kappa \right\}.$$

The constant κ will be chosen very large, but with $|\delta|^{-1}\kappa^{-2} \gg 0$.

Given local coordinates near the nodes, let $\underline{\Sigma}(\underline{\delta})$ denote the glued curve from (3). Let $\underline{u} : \underline{\Sigma} \to X$ be a pseudoholomorphic map. Define a *approximately pseudoholomorphic* (pre-glued) maps

$$\underline{u}(\underline{\delta}) : \underline{\Sigma}(\underline{\delta}) \to X$$

as follows. Fix a smooth cutoff function

$$\beta_1 : \mathbb{C} \to [0, 1], \quad \beta_1(z) = \left\{ \begin{array}{cc} 0 & |z| \leq 1 \\ 1 & |z| \geq 2 \end{array} \right\}. \tag{25}$$

Near each node w_j let $i(j, \pm)$ denote the components on either side of w_j. In the neighborhoods $U_j^\pm$ (assuming they have been chosen sufficiently small) define maps

$$\xi_j^\pm : U_j \to T_{x_j}X, \quad u_{i(j,\pm)}(z) = \exp_{x_j}(\xi_i^\pm)$$

where $x_j = u(w_j)$ and $\exp_{x_j} : T_{x_j}X \to X$ denotes geodesic exponentiation using the metric defined by J. Define the *preglued map*

$$\underline{u}(\underline{\delta}) : \underline{\Sigma}(\underline{\delta}) \to X, \quad z \mapsto \begin{cases} \underline{u}(z) & z \notin U_j^\pm \\ \exp_{x_j}(\beta_1(\kappa_j |\delta_j|^{1/2})\xi_j^+(z)) & z \in U_j^\pm \end{cases}. \tag{26}$$

The constant κ_j represents the size of the region over which u_i transitions to u_j. The formula (26) also defines a map

$$\underline{u}_0(\underline{\delta}) : \underline{\Sigma} \to X$$

which is constant near the nodes.

We introduce the following Sobolev spaces. Assume that $\underline{\Sigma}$ is equipped with a metric which is the Fubini-Study metric near the nodes. Define on $\underline{\Sigma}(\underline{\delta})$ the C^0-metric g by the identification

$$\underline{\Sigma}(\underline{\delta}) = \underline{\Sigma} - \bigcup_{j,\pm} \varphi_j^\pm(B_{|\delta_j|^{1/2}}(0)) / \sim \tag{27}$$

and taking the metric induced by the metric on $\underline{\Sigma}$. The Sobolev spaces $W^{k,p}(\underline{\Sigma}(\underline{\delta}))$ are defined for $p \geq 1$ and real $k \in [-1, 1]$, as in Section 4.2. The constants in the

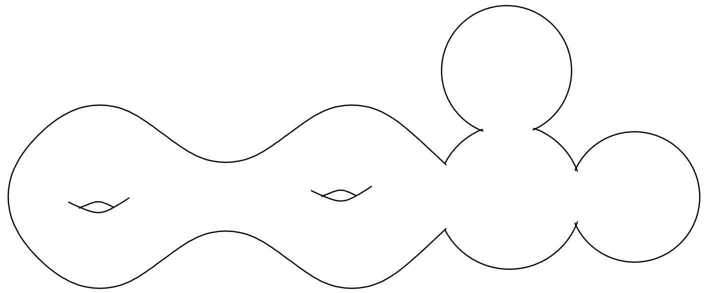

FIGURE 1. Continuous metric on a glued surface

Sobolev embeddings depend only on the dimensions of the cone in the cone condition and the bounds on the differences between the derivatives of the connections and metrics.

Consider the maps defined by parallel transport using the modified Levi-Civita connection,

$$\Phi^{\underline{u}}_{\underline{u}_0(\underline{\delta})} : \Omega^0(\underline{\Sigma}, \underline{u}_0^* TX) \to \Omega^0(\underline{\Sigma}, \underline{u}^* TX), \quad \Psi^{\underline{u}}_{\underline{u}_0(\underline{\delta})} : \Omega^{0,1}(\underline{\Sigma}, \underline{u}_0^* TX) \to \Omega^{0,1}(\underline{\Sigma}, \underline{u}^* TX).$$

Lemma 4.3.1. *The operator* $\Psi^{\underline{u}}_{\underline{u}_0(\underline{\delta})} \tilde{D}_{\underline{\Sigma},\underline{u}_0(\underline{\delta})} (\Phi^{\underline{u}}_{\underline{u}_0(\underline{\delta})})^{-1}$ *approaches the operator* $\tilde{D}_{\underline{\Sigma},\underline{u}}$ *as* $\underline{\delta} \to 0$.

Proof. Since $\underline{u}_0(\underline{\delta})$ approaches $\underline{u}$ in $W^{p,1}(\underline{\Sigma}, X)$, and Sobolev multiplication $W^{0,p} \times W^{1,p} \to W^{0,p}$. □

It follows from the lemma that for sufficiently small $\underline{\delta}$ there exists a right inverse $Q_{\underline{u}_0(\underline{\delta})}$ of $\tilde{D}_{\underline{u}_0(\underline{\delta})}$ with image the L^2-perpendicular of the kernel of $\tilde{D}_{\underline{u}_0(\underline{\delta})}$. Let $\| \cdot \|_{j,p,\delta}$ denote the Sobolev $W^{j,p}$-norm on $\Omega^0(u^*TX)$ defined using the metric (27).

Proposition 4.3.2 (Uniformly bounded error term). *Suppose that* $\underline{u} : \underline{\Sigma} \to X$ *is a stable map, and* $\underline{u}(\underline{\delta}) : \underline{\Sigma}(\underline{\delta}) \to X$ *is the pre-glued map defined in* (26), *defined for* $\underline{\delta}$ *sufficiently small. There is a constant* $c = c(\underline{\kappa}, \underline{\Sigma}, \underline{u})$ *and an* $\epsilon > 0$ *such that if*

$$\|\underline{\delta}\| < \epsilon, \quad \|\underline{\kappa}\| > 1/\epsilon, \quad |\delta_j|^2 \kappa_j < \epsilon, \quad j = 1, \ldots, m$$

then

$$\|\overline{\partial} \underline{u}(\underline{\delta})\|^2 \leq c \sum_{j=1}^{m} (|\delta_j|^{-1/2}/\kappa_j)^{-2/p}.$$

Proof. The error term $\overline{\partial}\underline{u}(\underline{\delta})$ can be estimated by terms of two types; those involving derivatives of the cutoff functions and those involving derivatives of the map ξ_j. The derivative of $\exp_{x_j}$ is approximately the identity near the node. The derivatives of β_1 grow like $1/\kappa_j |\delta_j|^{1/2}$, while the norm of $\xi_j^{\pm}$ is bounded by $\kappa_j |\delta_j|^{1/2}$ on the gluing region. Hence the term involving the derivatives of β_1 is bounded and supported on region of area $c\kappa_j^2 |\delta_j|$ for each node, hence the $0,p$ norm is bounded by $c(\kappa_j^2/|\delta_j|)^{-1/p}$.

The derivatives of $\xi_j^{\pm}$ are also uniformly bounded, and the area bound as before gives the required estimate. $\square$

Next, suppose that $\underline{u} : \underline{\Sigma} \to X$ is a parametrized regular map. Let $(\underline{\Sigma}_\bullet \to S, \varphi)$ denote the versal deformation of $\underline{\Sigma}$ of constant type, and $(\underline{\Sigma}_\bullet(\underline{\delta}) \to S(\underline{\delta}), \varphi(\underline{\delta})$ the versal deformation of $\underline{\Sigma}(\underline{\delta})$. The gluing construction (3) applies to the family $\underline{\Sigma}_\bullet$ produces a smooth map $S \to S(\underline{\delta})$, and in particular, a linearized map

$$D\#_{\underline{\delta}} : T_0 S \to T_0 S(\underline{\delta}). \tag{28}$$

Let $\tilde{D}_{\underline{\Sigma},\underline{u}}$ be the parametrized linear operator of (19). We claim that there is an approximate inverse

$$T_{\underline{\delta}} : \Omega^{0,1}(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX) \to T_0 S \oplus \Omega^0(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX)$$

of the parametrized linear operator $\tilde{D}_{\underline{\delta}} := \tilde{D}_{\underline{u}(\underline{\delta})}$ of the pre-gluing $\underline{u}(\underline{\delta})$.

The construction of the approximate right inverse depends on a carefully chosen cutoff function:

Lemma 4.3.3. [32, Section 10.3] *For any $\delta > 0, \kappa > 1$ there exists a function*

$$\beta_{2;\kappa,\delta} : \mathbb{R}^2 \to [0,1]$$

which satisfies

$$\beta_{2;\kappa,\delta}(z) = \begin{cases} 1 & |z| \le \sqrt{\delta\kappa} \\ 0 & |z| \ge \sqrt{\delta/\kappa} \end{cases}$$

and for all $\xi \in W^{1,p}(B_{\kappa_j|\delta_j|})$ satisfying $\xi(0) = 0$

$$\|(\nabla \beta_{2;\kappa,\delta})\xi\|_{W^{1,p}} \le c \log(\kappa^2)^{-1+1/p} \|\xi\|_{W^{1,p}}, . \tag{29}$$

Ideally one would like such an estimate for all functions ξ, but this is not possible for $p > 2$, see [32].

Define the approximate right inverse by composing the right inverse on the nodal curve for $\underline{u}_0$ with a cutoff and extension operator,

$$T_{\underline{\delta}} := P_{\underline{\delta}} Q_{\underline{u}_0(\underline{\delta})} K_{\underline{\delta}}.$$

(a) The *cutoff operator*

$$K_{\underline{\delta}} : \Omega^{0,1}(\underline{u}(\underline{\delta})^* TX)_{0,p,\delta} \to \Omega^{0,1}(\underline{u}_0(\underline{\delta})^* TX)_{0,p}$$

is defined by

$$(K_{\underline{\delta}}(\eta))(z) = \begin{cases} \eta(z) & z \notin B_{|\delta_j(z)|^{1/2}}(0) \\ 0 & \text{otherwise} \end{cases}.$$

We have

$$\|K_{\underline{\delta}}\eta\|_{0,p} \le \|\eta\|_{0,p,\delta}$$

by definition of the $0, p, \delta$ norm.

(b) The *extension operator*

$$P_{\underline{\delta}} : T_0 S \oplus \Omega^0(\underline{\Sigma}, \underline{u}_0(\underline{\delta})^* TX)_{1,p,\delta} \to T_0 S \oplus \Omega^0(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX)_{1,p,\delta}$$

is defined as follows. For each component Σ_j let Σ_i^* denote the complements of small balls around the nodes

$$\Sigma_i^* = \Sigma_i - \bigcup_j B_{\kappa_j |\delta_j(z)|^{1/2}}(w_j)$$

and

$$\pi_j : \Sigma_j^* \to \underline{\Sigma}(\underline{\delta}) \tag{30}$$

the inclusion into the glued curve. The inclusion induces a map

$$\pi_{j,*} \Omega_c^0(\Sigma_j^*, u_j^* TX) \to \Omega^0(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX)$$

on sections with compact support in Σ_j^*. Define

$$(P_{\underline{\delta}}(\zeta, \xi)) = (\zeta(\underline{\delta}), \xi(\underline{\delta}))$$

where $\zeta(\underline{\delta}) := D\#_{\underline{\delta}}\zeta$ is the push-forward of ζ under the gluing map (28) and

$$\xi(\underline{\delta}) = \begin{cases} \xi(z) & z \notin B_{|\delta_j(z)|^{1/2}}(0) \\ \sum_j \pi_{j,*} \beta_{2,\kappa_j,\delta_j}(z)(\xi_j(z) - \xi(w_j)) + \xi(w_j) & \text{otherwise} \end{cases} .$$

Fix a metric $\| \cdot \|_S$ on $T_0 S$ and define

$$\|(\zeta, \xi)\|_{1,p,\delta} = \left(\|\zeta\|_S^2 + \|\xi\|_{1,p,\delta}^2 \right)^{1/2}$$

Proposition 4.3.4 (Uniformly bounded right inverse)**.** *The map*

$$T_{\underline{\delta}} : \Omega^{0,1}(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX) \to T_0 S \oplus \Omega^0(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX)$$

satisfies

$$\|(\tilde{D}_{\underline{\delta}} T_{\underline{\delta}} - I)\eta\|_{0,p,\delta} \leq 1/2 \|\eta\|_{0,p,\delta}$$

and the operator norm of $T_{\underline{\delta}}$ is uniformly bounded in δ,

Proof. By construction $T_{\underline{\delta}}$ is an exact right inverse for $\tilde{D}_{\underline{\delta}}$ away from gluing regions, and in the gluing regions the variation of complex structure on the curve vanishes. In the gluing regions we have $D_{\underline{\delta}} = D_{x_j}$, the standard Cauchy-Riemann operator with values in $T_{x_j} X$. So

$$\begin{aligned} \tilde{D}_{\underline{\delta}} T_{\underline{\delta}} \eta - \eta &= \sum D_{x_j} \beta_2(z)(\xi_{i(j,\pm)}(z) - \xi(w_j)) \\ &= \sum (D_{x_j} \beta_2(z)) \xi_{i(j,\pm)}(z) \end{aligned}$$

since $D_{\underline{u}_0(\underline{\delta})} \xi = K_{\underline{\delta}} \eta = 0$ on $B_{|\delta_j|^{1/2}}(0)$. Since $p > 2$, the $0, p, \delta$-norm of the right hand side is controlled by the $0, p$ norm. By (29) we have

$$\begin{aligned} \|D_{\underline{\delta}} T_{\underline{\delta}} \eta - \eta\|_{0,p,\delta} &\leq \sum_j c |\log(\kappa_j^2)|^{-1+1/p} \|\xi_{i(j,\pm)}(z) - \xi(w_j)\|_{1,p} \\ &\leq \sum_j c |\log(\kappa_j^2)|^{-1+1/p} \left(\|\xi_{i(j,\pm)}(z)\|_{1,p} |\xi(w_j)| \right). \end{aligned}$$

The both terms are bounded by $\|K_{\underline{\delta}}\eta\|_{0,p}$, by the uniform bound on $Q_{\underline{\delta}}$, and hence $\|\eta\|_{0,p,\delta}$, by the uniform boundedness of $K_{\underline{\delta}}$. □

Define a right inverse $Q_{\underline{\delta}}$ to $\tilde{D}_{\underline{\delta}}$ by the formula

$$Q_{\underline{\delta}} = T_{\underline{\delta}}(\tilde{D}_{\underline{\delta}} T_{\underline{\delta}})^{-1} = \sum_{k \geq 0} T_{\underline{\delta}}(\tilde{D}_{\underline{\delta}} T_{\underline{\delta}} - I)^k.$$

The uniform bound on $T_{\underline{\delta}}$ from Lemma 4.3.4 implies a uniform bound on $Q_{\underline{\delta}}$.

We now repeat the construction of the moduli space locally as the zero set of a Fredholm section of a Banach vector bundle. Let

$$\mathcal{B}(\underline{\delta}) = S \times \mathrm{Map}(\underline{\Sigma}(\underline{\delta}), X)_{1,p}$$

and

$$\mathcal{E}(\underline{\delta}) \to \mathcal{B}(\underline{\delta}), \quad \mathcal{E}(\underline{\delta})_{j,u} = \Omega^{0,1}(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX)$$

denote the Banach vector bundle whose fiber over $(\underline{j}, \underline{u})$ is $\Omega^{0,1}(\underline{u}^* TX)$, the space of $(0,1)$-forms on $\underline{u}^* TX$ with respect to the complex structure j_s on $\underline{\Sigma}(\underline{\delta})$. We define a trivialization of $\mathcal{E}(\underline{\delta})$ near $\underline{u}(\underline{\delta}), j_0(\underline{\delta})$ as follows. We may write any map u_1 close to $\underline{u}(\underline{\delta})$ as

$$u_1 = \exp_{\underline{u}(\underline{\delta})}(\xi)$$

for some $\xi \in \Omega^0(\underline{\Sigma}(\underline{\delta}), \underline{u}_{\underline{\delta}}^* TX)$. Let

$$\Phi_{\underline{u}}(\xi) : \underline{u}(\underline{\delta})^* TX \to \exp_{\underline{u}(\underline{\delta})}(\xi)^* TX$$

denote parallel transport with respect to the Hermitian connection. Acting on the space of $(0,1)$-forms parallel transport induces an isomorphism

$$\Phi_{\underline{u}}(\xi)^{-1} : \Omega^{0,1}(\underline{\Sigma}(\underline{\delta}), \exp_{\underline{u}(\underline{\delta})}(\xi)^* TX; j(\underline{\delta})) \to \Omega^{0,1}(\underline{\Sigma}(\underline{\delta}), (\underline{u}_{\underline{\delta}})^* TX; \underline{j}(\underline{\delta})).$$

Furthermore, choose a coordinate chart for parameter space S,

$$\exp_{\underline{\Sigma}} : U \to S, U \subset T_s S$$

where U is a neighborhood of 0. We denote by $\exp_{\underline{\Sigma}}(\zeta, \underline{\delta})$ the Riemann surface obtained by applying with the gluing construction (3) to $\exp_{\underline{\Sigma}}(\zeta)$. We identify $\Omega^{0,1}(\exp_{\underline{\Sigma}}(\zeta, \underline{\delta}), u^* TX)$ with $\Omega^{0,1}(\underline{\Sigma}(\underline{\delta}), u^* TX)$ by composing the inclusion

$$\Omega^{0,1}(\exp_{\underline{\Sigma}}(\zeta, \underline{\delta}), u^* TX) \to \Omega^1(\exp_{\underline{\Sigma}}(\zeta, \underline{\delta}), u^* TX)_{\mathbb{C}} = \Omega^1(\underline{\Sigma}(\underline{\delta}); u^* TX)_{\mathbb{C}}$$

with the projection $\Omega^1(\underline{\Sigma}(\underline{\delta}); \underline{u}(\underline{\delta})^* TX)_{\mathbb{C}} \to \Omega^{0,1}(\underline{\Sigma}(\underline{\delta}, \underline{u}(\underline{\delta})^* TX)$. We denote by

$$\Phi_{\underline{\Sigma}}(\zeta)^{-1} : \Lambda^{0,1}(\exp_{\underline{\Sigma}}(\zeta), u^* TX) \to \Lambda^{0,1}(\underline{\Sigma}, \underline{u}^* TX)$$

parallel transport with respect to the connection induced by the trivialization. By composing $\Phi_{\underline{u}}(\xi)^{-1}$ and $\Phi_{\underline{\Sigma}}(\zeta)^{-1}$ we obtain an identification

$$\Phi^{-1}_{\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})} : \Omega^{0,1}(\exp_{\underline{\Sigma}(\underline{\delta})}(\zeta), \exp_{\underline{u}(\underline{\delta})}(\xi)^* TX) \to \Omega^{0,1}(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX).$$

Define

$$\begin{aligned} \mathcal{F}^{\delta}_{\underline{\Sigma}, \underline{u}} : T_{\underline{\Sigma}} M_{g,n,\Gamma} \times \Omega^0(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX) &\to \Omega^{0,1}(\underline{\Sigma}, \underline{u}^* TX) \\ (\zeta, \xi) &\mapsto \Phi_{\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})}(\zeta, \xi)^{-1}(\overline{\partial}_{\exp_{\underline{\Sigma}(\underline{\delta})}(\zeta)}(\exp_{\underline{u}(\underline{\delta})}(\xi))). \end{aligned}$$

Proposition 4.3.5 (Uniform quadratic bound). *Let $p > 2$. For every constant $c_0 > 0$ there exist constants $c, \delta_0 > 0$ such that if $\underline{u} \in \mathrm{Map}(\underline{\Sigma}, X)_{1,p}, \xi \in \Omega^0(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})^* TX)_{1,p}$*

$$\|\mathrm{d}\underline{u}\|_{0,p} \leq c_0, \quad \|\xi\|_{L_\infty} \leq c_0 \quad \|\zeta\| \leq c_0, \quad \delta < \delta_0$$

then

$$\|(D\mathcal{F}^{\delta}_{\underline{\Sigma}(\underline{\delta}),\underline{u}(\underline{\delta})}(\zeta,\xi) - \tilde{D}_{\underline{\Sigma}(\underline{\delta}),\underline{u}(\underline{\delta})})(\zeta',\xi')\|_{0,p,\delta} \leq c(\|\zeta,\xi\|_{1,p}(\|\zeta',\xi'\|_{1,p}).$$

Proof. Let $\tilde{\nabla}$ be the connection on $\Lambda^{0,1}T^{*,\mathrm{Vert}}\Sigma_{g,n} \otimes TX$ defined by combining the Hermitian connection on TX with the connection induced by the local trivialization on the space of $0,1$-forms. Let

$$\Phi_{\underline{\Sigma},x} : \Lambda^{0,1}T^*_z\underline{\Sigma}(\underline{\delta}) \otimes T_xX \to \Lambda^{0,1}T^*_z \exp_{\underline{\Sigma}}(\zeta) \otimes T_{\exp_x(\xi)}X$$

the resulting parallel transport map. Write

$$\Psi_{\underline{\Sigma}(\underline{\delta}),x}((\zeta,\xi),(\zeta',\xi'),\eta) = \tilde{\nabla}_t \Phi_{\underline{\Sigma}(\underline{\delta}),x}(\zeta + t\zeta', \xi + t\xi')\eta.$$

For ξ, η sufficiently small there exists a constant c such that

$$|\Psi_{\underline{\Sigma}(\underline{\delta}),x}(\zeta,\xi,\zeta',\xi')\eta| \leq c\|\xi,\zeta\|\|\xi',\zeta'\|\|\eta\| \tag{31}$$

where the norms on the right-hand side are any norms on the finite dimensional vector spaces $T_{\underline{\Sigma}}M_{g,n,\Gamma}$ and T_xX. This estimate is uniform in δ, because if z is on the neck then the variation in complex structure vanishes. Now differentiate the equation

$$\Phi_{\underline{\Sigma},\underline{u}}(\xi,\zeta)^{-1}\mathcal{F}_{\underline{\Sigma},\underline{u}}(\zeta,\xi) = \overline{\partial}_{\exp_{\underline{\Sigma}}(\zeta)}(\exp_u(\xi)))$$

with respect to (ζ',ξ') to obtain

$$\begin{aligned}\Psi_{\underline{\Sigma},\underline{u}}(\zeta,\xi;\zeta',\xi';\mathcal{F}_{\underline{\Sigma},\underline{u}}(\zeta,\xi)) + \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)(D\mathcal{F}_{\underline{\Sigma},\underline{u}}(\zeta,\xi)(\zeta',\xi')) = \\ \tilde{D}_{\exp_{\underline{\Sigma}}(\zeta),\exp_u(\xi)} D\exp_u(\zeta,\xi,\zeta',\xi').\end{aligned} \tag{32}$$

Using

$$|\mathcal{F}_{\underline{\Sigma},\underline{u}}(\zeta,\xi)| \leq c|\mathrm{d}\exp_u(\xi)||\zeta| < c|(|\mathrm{d}u| + |\nabla\xi|)(c + |\zeta|)$$

for ζ, ξ sufficiently small, the estimate (31) on Ψ produces an pointwise estimate

$$|\Phi^{-1}_{\underline{\Sigma},\underline{u}}\Psi_{\underline{\Sigma},\underline{u}}(\zeta,\xi,\zeta',\xi',\mathcal{F}_{\underline{\Sigma},\underline{u}}(\zeta,\xi))| \leq c(|\mathrm{d}u| + |\nabla\xi|)(c + |\zeta|)|(\xi,\zeta)||(\xi',\zeta')|.$$

Hence

$$\|\Phi^{-1}_{\underline{\Sigma},\underline{u}}\Psi_{\underline{\Sigma},\underline{u}}(\zeta,\xi,\zeta',\xi',\mathcal{F}_{\underline{\Sigma},\underline{u}}(\zeta,\xi))\|_{0,p} \leq c(1+\|\mathrm{d}u\|_{0,p}+\|\nabla\xi\|_{0,p}+\|\zeta\|_{0,p})\|(\xi,\zeta)\|_{L^\infty}\|(\xi',\zeta')\|_{L^\infty}.$$

It follows that

$$\|\Phi^{-1}_{\underline{\Sigma},\underline{u}}\Psi_{\underline{\Sigma},\underline{u}}(\zeta,\xi,\zeta',\xi',\mathcal{F}_{\underline{\Sigma},\underline{u}}(\zeta,\xi))\|_{0,p} \leq c\|(\xi,\zeta)\|_{1,p}\|(\xi',\zeta')\|_{1,p} \tag{33}$$

since the $W^{1,p}$ norm controls the L^∞ norm.

In what follows, we drop $\underline{\delta}$ from the notation, so that $\underline{\Sigma}$ means $\underline{\Sigma}(\underline{\delta})$ etc. We claim that there exists a constant $c > 0$ such that uniformly in $\underline{\delta}$,

$$\|\Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)^{-1}\tilde{D}_{\exp_\zeta(\underline{\Sigma}),\exp_u(\xi)} D\exp_u(\xi,\xi') - \tilde{D}_{\underline{\Sigma},\underline{u}}\|_{0,p} \leq c\|\zeta,\xi\|_{1,p}\|\zeta',\xi'\|_{1,p}. \tag{34}$$

Together with (33) this proves the proposition. To see (34), note that as in (20),

$$\begin{aligned}(35)\quad \tilde{D}_{\zeta,\xi}(\zeta',\xi') &:= D\mathcal{F}_{\underline{\Sigma},\underline{u}}(\zeta,\xi;\zeta',\xi') \\ &= \pi^{0,1}_{\underline{\Sigma}}(\nabla^{0,1}_{\exp_{\underline{\Sigma}}(\zeta)}\xi' - \frac{1}{2}J(u)\mathrm{d}uD_\zeta \exp_{\underline{\Sigma}}\zeta' - \frac{1}{2}J_u(\nabla_\xi J)_u\partial_{\exp_{\underline{\Sigma}}(\zeta)}(u)).\end{aligned}$$

Hence

$$\begin{aligned}\tilde{D}_{\zeta,\xi}(\zeta', D_\xi \exp_u(\xi')) = \pi^{0,1}_{\underline{\Sigma}}\nabla^{0,1}_{\exp_{\underline{\Sigma}}(\zeta)}(D_\xi \exp_u(\xi')) - \frac{1}{2}J_{u(\underline{\delta},\xi)}\mathrm{d}u(\underline{\delta},\xi)\zeta' \\ -\frac{1}{2}J_{u(\underline{\delta},\xi)}(\nabla_{D_\xi \exp_u(\xi')}J_{u(\underline{\delta},\xi)})\partial u(\underline{\delta},\xi).\end{aligned}$$

Hence

$$\tilde{D}_{\underline{\Sigma}(\zeta),u(\xi)}(D_\zeta \exp_{\underline{\Sigma}}(\zeta'), D_\xi \exp_u(\xi')) - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)\tilde{D}_{\underline{\Sigma},\underline{u}}(\zeta',\xi') = \pi^{0,1}_{\underline{\Sigma}}(\Pi_1+\Pi_2+\Pi_3)$$

where the three terms Π_1,Π_2,Π_3 are

$$\begin{aligned}\Pi_1 &= \pi^{0,1}_{\underline{\Sigma}(\zeta)}\nabla(D_\xi \exp_u(\xi')) - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)\pi^{0,1}_{\underline{\Sigma}}\nabla\xi' \\ \Pi_2 &= -\frac{1}{2}J_{u(\xi)}\mathrm{d}u(\xi)D_\zeta \exp_{\underline{\Sigma}}(\zeta') + \frac{1}{2}\Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)J_u\mathrm{d}u\zeta' \\ \Pi_3 &= -\frac{1}{2}J_{u(\xi)}\pi^{0,1}_{\underline{\Sigma}(\zeta)}(\nabla_{D_\xi \exp_u(\xi')}J_{u(\xi)})\partial u(\xi) + \frac{1}{2}\Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)J_u\pi^{0,1}_{\underline{\Sigma}}(\nabla_{\xi'}J_u)\partial u\end{aligned}$$

We write

$$\begin{aligned}(36)\quad &\pi^{0,1}_{\underline{\Sigma}(\zeta)}(\nabla(D_\xi \exp_u(\xi')) - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)\nabla\xi') \\ &= |\nabla(D_\xi \exp_u(\xi')) - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)\nabla\xi'| \\ &= |\nabla(D_\xi \exp_u(\xi')) - D\exp_u(\xi,\nabla\xi')| + |D\exp_u(\xi,\nabla\xi') - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)\nabla\xi'| \\ &\le c|\nabla\xi||\xi'| + c|\zeta,\xi||\nabla\xi'| + c|\mathrm{d}u||\xi||\xi'|.\end{aligned}$$

We write for the second term

$$\begin{aligned}(37)\quad &|J_{u(\xi)}\mathrm{d}u(\xi)D_\zeta \exp_{\underline{\Sigma}}(\zeta') - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)J_u\mathrm{d}u\zeta'| \\ &\le |J_{u(\xi)}\mathrm{d}u(\xi)D_\zeta \exp_{\underline{\Sigma}}(\zeta') - J_{u(\xi)}\Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)\mathrm{d}u\zeta'| + \\ &\quad |J_{u(\xi)}\Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)\mathrm{d}u\zeta' - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)J_u\mathrm{d}u\zeta'| \\ &\le c|\zeta,\xi|\zeta' + c|\zeta,\xi|\mathrm{d}u||\zeta'|.\end{aligned}$$

The third term can be estimated pointwise by

$$\begin{aligned}&|J_{u(\xi)}\pi^{0,1}_{\underline{\Sigma}(\zeta)}(\nabla_{D_\xi \exp_u(\xi')}J_{u(\xi)})\partial u(\xi) - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)J_u\pi^{0,1}_{\underline{\Sigma}}(\nabla_{\xi'}J_u)\partial u| \\ &\le |J_{u(\xi)}\pi^{0,1}_{\underline{\Sigma}(\zeta)}(\nabla_{D_\xi \exp_u(\xi')}J_{u(\xi)})\partial u(\xi) - \Phi_{\underline{\Sigma}}(\zeta)J_{u(\xi)}\pi^{0,1}_{\underline{\Sigma}}(\nabla_{D_\xi \exp_u(\xi')}J_{u(\xi)})\partial u(\xi) \\ &\quad +\Phi_{\underline{\Sigma}}(\zeta)J_{u(\xi)}\pi^{0,1}_{\underline{\Sigma}}(\nabla_{D_\xi \exp_u(\xi')}J_{u(\xi)})\partial u(\xi) - \Phi_{\underline{\Sigma},\underline{u}}(\zeta,\xi)J_u\pi^{0,1}_{\underline{\Sigma}}(\nabla_{\xi'}J_u)\partial u| \\ &\le c|\zeta||\mathrm{d}u + \nabla\xi||\xi'| + c(|\mathrm{d}u||\xi| + |\nabla\xi|)(|\xi'|).\end{aligned}$$

Combine these estimates and integrate. Apply $0,p,\underline{\delta}$-norms on $\mathrm{d}u$, $\nabla\xi,\nabla\xi'$ and the L^∞ norms on the other factors. The L^∞-norm is controlled uniformly by the $1,p,\underline{\delta}$-norm, by uniform Sobolev embedding as in Section 4.2. It follows (34). □

The same gluing procedure works in families. Let $\underline{\Sigma}_\bullet \to S$ be a smooth family of nodal curves of a fixed combinatorial type. We suppose that we fix identifications of each component of $\underline{\Sigma}_\bullet$, so that the variation of complex structure occurs in the complement of an open neighborhood of the nodal points. For each node $\{w_j^+, w_j^-\}$ we denote by $\Sigma_{i^\pm(j)}$ the component containing $w_j^\pm$. Let

$$\underline{I}_\bullet \to S, \quad \underline{I}_s = \bigoplus_{j=1}^m (T^*_{w_j}\Sigma_{i(j,-),s} \otimes T^*_{w_j}\Sigma_{i(j,+),s})$$

be the bundle consisting of the sum of tensor products of cotangent lines at the nodes. Recall that a *set of gluing parameters* for $\underline{\Sigma}_s$ is an element

$$\underline{\delta} = (\delta_1, \ldots, \delta_m) \in \underline{I}_s.$$

Since we are working locally near a fixed curve, we may fix an identification

$$\underline{I}_\bullet \cong S \times \mathbb{C}^m.$$

Let $\underline{u}_\bullet : \underline{\Sigma}_\bullet \to X$ a smooth family of J-holomorphic maps.

Theorem 4.3.6 (Gluing stable maps)**.** *Let $\underline{u} : \underline{\Sigma} \to X$ be a stable map and $(\underline{\Sigma}_\bullet \to S, \varphi)$ a parametrized regular deformation of $\underline{u}$. There exist constants $\epsilon, \epsilon_1 > 0$ such that*

(a) *For any $(\zeta_0, \xi_0) \in \ker(\tilde{D}_{\underline{u}(\underline{\delta})})$, there is a unique*

$$(\zeta_1, \xi_1) = \tilde{D}^*_{\underline{u}(\underline{\delta})}\eta_1$$

of norm at most ϵ such that the pair

$$\underline{\Sigma}_{\zeta_0,\xi_0,\underline{\delta}} := \exp_{\underline{\Sigma}(\underline{\delta})}(\zeta_0 + \zeta_1), \quad \underline{u}_{\zeta_0,\xi_0,\delta} := \exp_{\underline{u}(\underline{\delta})}(\xi_0 + \xi_1)$$

is pseudoholomorphic, and

(b) *if (Σ_1, u_1) is a J-holomorphic map within ϵ_1 of $(\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta}))$ for some $\underline{\delta}$, then (Σ_1, u_1) is equal to $(\underline{\Sigma}_{\zeta_0,\xi_0,\underline{\delta}}, \underline{u}_{\zeta_0,\xi_0,\underline{\delta}})$ for some $(\zeta_0, \xi_0, \underline{\delta})$.*

Proof. The first statement is an application of the implicit function theorem, using the uniform error bound from Proposition 4.3.2, uniformly bounded right inverse from Proposition 4.3.4, and uniform quadratic estimate from Proposition 4.3.5. To prove the second statement, write

$$(\Sigma_1, u_1) = \exp_{\underline{\Sigma}(\underline{\delta})}(\zeta), \quad u_1 = \exp_{\underline{u}(\underline{\delta})}\xi$$

and decompose $(\zeta, \xi) = (\zeta_0, \xi_0) + (\zeta_1, \xi_1)$ according to the splitting

$$\ker \tilde{D}_{\underline{\Sigma}(\underline{\delta}),\underline{u}(\underline{\delta})} \oplus \operatorname{im} \tilde{D}^*_{\underline{\Sigma}(\underline{\delta}),\underline{u}(\underline{\delta})}.$$

For ϵ_1 sufficiently small, (ζ_0, ξ_0) has norm at most ϵ. The implicit function theorem shows that (ζ_1, ξ_1) is the unique element in the image of $\tilde{D}^*_{\underline{\Sigma}(\underline{\delta}),\underline{u}(\underline{\delta})}$ so that u_1 is pseudoholomorphic. □

The gluing construction thus produces from a deformation $(\underline{\Sigma}_\bullet, \varphi) \to S$ and a parametrized regular stable map $\underline{u} : \underline{\Sigma} \to X$ a family of nodal pseudoholomorphic maps over $\ker(\tilde{D}_{\underline{\Sigma},\underline{U}}) \times B_\epsilon(0, \mathbb{C}^m)$ equipped with a canonical identification $\#\varphi$ of the central fiber $(\#\underline{\Sigma})_0$ with $\underline{\Sigma}$. Let $\operatorname{aut}(\underline{\Sigma})^\perp$ be a complement of $\operatorname{aut}(\underline{\Sigma})$ in $\ker(\tilde{D}_{\underline{\Sigma},\underline{u}})$, and $\#S$ a neighborhood of 0 in $\operatorname{aut}(\underline{\Sigma})^\perp \times \mathbb{C}^m$. Restriction to $\#S$ defines a *glued family of holomorphic maps*

$$(\#\underline{\Sigma}_\bullet, \#\underline{u}_\bullet) \to \#S. \tag{38}$$

Theorem 4.3.7. *If $(\underline{\Sigma}_\bullet, \varphi)$ is a versal deformation of a nodal surface $\underline{\Sigma}$ and $u : \underline{\Sigma} \to X$ is a pseudoholomorphic map, then*

(a) $(\#\underline{\Sigma}_\bullet, \#\underline{u}_\bullet, \#\varphi)$ *is a partially smooth versal deformation of* $(\underline{\Sigma}, \underline{u})$.
(b) *If* $(\underline{\Sigma}_\bullet, \varphi)$ *is the minimal versal deformation constructed above, then* $(\#\underline{\Sigma}_\bullet, \#\underline{u}_\bullet)$ *is universal iff* $(\underline{\Sigma}, \underline{u})$ *is stable.*
(c) *If* $(\underline{\Sigma}, \underline{u})$ *is unstable then it does not admit a partially smooth universal deformation.*

Sketch of Proof. Suppose that $(\underline{\Sigma}^1_\bullet \to S^1, \underline{u}^1_\bullet \to S^1, \varphi^1)$ is a partially smooth deformation of $(\underline{\Sigma}, \underline{u})$. Since $\underline{\Sigma}_\bullet$ is versal, $\underline{\Sigma}^1_\bullet$ is obtained by pull-back of $\underline{\Sigma}_\bullet$ by a partially smooth map $\psi : S^1 \to S$. By definition the family $\underline{\Sigma}^1_s, \underline{u}^1_s \to S^1$ converges to the central fiber in the Gromov topology as s converges to the base point $0 \in S^1$. The exponential decay estimates in [32] imply that for s sufficiently close to 0, $\underline{\Sigma}^1_s, \underline{u}^1_s$ is within ϵ_1 of a pre-glued curve $\underline{\Sigma}(\underline{\delta}), \underline{u}(\underline{\delta})$, for s sufficiently close to 0. The surjectivity statement in Theorem 4.3.6 then produces a partially smooth map $\#\psi : S^1 \to \ker(\tilde{D}_{\underline{\Sigma},\underline{u}})$ such that $(\underline{\Sigma}^1_\bullet, \underline{u}^1_\bullet)$ is the pull-back of $\#\psi$. To prove the second statement, suppose that $\varphi, \varphi' : \underline{\Sigma}^1_\bullet \to \psi^* \underline{\Sigma}_\bullet$ are isomorphisms of families inducing the identity on the central fiber. Necessarily the difference between φ, φ' is an automorphism of the rational components with at most two special points. If $\underline{u}$ is non-constant on these components then the difference between φ, φ' must be constant, hence the claim. The last claim is by the same argument as in the constant type case Theorem 4.1.11. □

Remark 4.3.8. If $(\underline{\Sigma}, \underline{u})$ is stable, then $(\#\underline{\Sigma}_\bullet, \#\underline{u}_\bullet)$ is a universal C^1 deformation for deformations compatible with the exponential gluing profile.

4.4. **Charts.** Given a partially smooth universal deformation $(\underline{\Sigma}_\bullet \to S, \underline{u}_\bullet \to S, \varphi)$ of a parametrized regular stable map $(\underline{\Sigma}, \underline{u})$, define a map

$$\#_{\underline{\Sigma},\underline{u}} : \#S \to \overline{M}_{g,n}(X, d), \quad s \mapsto [(\underline{\Sigma}, \underline{u})]. \tag{39}$$

We wish to prove the following generalization of Theorem 2.1.5:

Theorem 4.4.1. *Let $(\underline{\Sigma}, \underline{u})$ be a parametrized regular stable map. Two fibers of the partially smooth universal deformation* (39) *are isomorphic if and only if they are related by an automorphism of $(\underline{\Sigma}, \underline{u})$. Furthermore,* (39) *is a partially smooth universal deformation of any of its fibers.*

The proof of the corresponding property of universal deformations of stable curves Theorem 2.1.5 depends strongly on the analyticity of the deformation. One has no such properties for partially smooth universal deformations of pseudoholomorphic maps. The proof of Theorem 4.4.1 takes a huge detour, in which one constructs an non-canonical differentiable structure on the universal deformation via the evaluation maps. This makes it possible to apply the implicit function theorem for differentiable manifolds, to obtain the first claim in Theorem 4.4.1.

The technique of proof is to note that any family of pseudoholomorphic maps induces a family of stable marked curves, via the rigidification construction and the choice of auxiliary collection of hypersurfaces. For such a family of stable curves, we already know the corresponding statement, by Section 2.2. However, in order to show that the resulting family of marked curves is contained in a universal deformation requires a study of the differentiability properties of the evaluation maps.

Let $(\underline{\Sigma}_\bullet, \underline{u}_\bullet) \to S$ be a partially smooth family of nodal pseudoholomorphic maps. Define an *evaluation map*

$$\mathrm{ev} : \underline{\Sigma}_\bullet \to X, \quad z \mapsto \underline{u}(z).$$

If z is a smooth point in its fiber and S has a smooth structure, then $\underline{\Sigma}_\bullet$ is smooth at z and it makes sense to ask what regularity properties the map ev has at z. Suppose that $(\#\underline{\Sigma}_\bullet, \#\underline{u}_\bullet) \to \#S$ is the family constructed by gluing in the previous section using the exponential gluing profile and a deformation of constant type of a stable map $(\underline{\Sigma}, \underline{u})$.

Theorem 4.4.2 (Differentability of evaluation maps in the exponential gluing profile)**.** *Using the exponential gluing profile and any smooth system of local coordinates on a neighborhood U of the nodes, the map* $\mathrm{ev} : \#\underline{\Sigma}_\bullet \to X$ *is differentiable on the complement of U.*

Proof. First we indicate the changes necessary in [32, Section 10.6] necessary to show that the derivative of the gluing map with respect to ξ, ζ is well-defined. For simplicity, we assume that there is a single gluing parameter δ. Let

$$\underline{u}_\bullet(\delta) : \underline{\Sigma}_\bullet(\delta) \to X$$

denote the pre-glued family. By definition

$$\overline{\partial}_{\exp_{\underline{\Sigma}_s(\delta)}(\zeta_s)} \exp_{\underline{u}_s(\delta)}(\xi_s) = 0.$$

Differentiating with respect to $v \in T_s S$ gives

$$\tilde{D}_s(v + \nabla_v \zeta, E_1 \partial_v \underline{u} + E_2 \nabla_v \xi) = 0$$

where

$$L_v \exp_u(\xi) = E_1(\mathfrak{u}, \xi) \frac{d}{dt} u + E_2(u, \xi) \nabla_v \xi.$$

The elliptic estimate for $\tilde{D}_s$ implies

$$\| \nabla_v \zeta, \nabla_v \xi \|_{1,p,\delta} \le c(\kappa |\delta|^2)^{-1/p} \| v, \partial_v \underline{u} \|_{0,2}.$$

In particular, $\mathrm{ev}(\delta)$ approaches $\mathrm{ev}(0)$ in the C^1-topology as $\delta \to 0$.

Next we take the derivative with respect to the gluing parameter. Let $\underline{u} : \underline{\Sigma} \to X$ be a stable map, $\underline{u}(\delta) : \underline{\Sigma}(\delta) \to X$ denote the pre-glued map. By definition

$$\overline{\partial}_{\exp_{\underline{\Sigma}(\delta)}(\zeta(\delta))}(\exp_{\underline{u}(\delta)}(\xi(\delta))) = 0.$$

We fix a trivialization of the universal curve near $\underline{\Sigma}(\delta)$, so varying δ changes the complex structure in the gluing region only. Differentiating with respect to δ gives

$$\tilde{D}_{\underline{\Sigma}(\delta),\underline{u}(\delta)}(D\exp_{\underline{\Sigma}(\delta),\zeta(\delta)}(\frac{d}{d\delta}\underline{\Sigma}(\delta), \nabla_\delta \zeta(\delta)), D\exp_{\underline{u}(\delta),\xi(\delta)}(\frac{d}{d\delta}\underline{u}(\delta), \nabla_\delta \xi(\delta)) = 0.$$

We write

$$(\nabla_\delta \zeta(\delta), \nabla_\delta \xi(\delta)) = (\hat{\zeta}(\delta), \hat{\xi}(\delta)) + T_\delta \nabla_\delta \eta(\delta)$$

where

$$\eta(\delta) = (\tilde{D}_\delta T_\delta)^{-1} \tilde{D}_\delta(\zeta(\delta), \xi(\delta)).$$

Now write

$$\begin{aligned} &\tilde{D}_{\underline{\Sigma}(\delta),\underline{u}(\delta)} \left(D_2 \exp_{\underline{\Sigma}(\delta),\zeta(\delta)} \times D_2 \exp_{\underline{u}(\delta),\xi(\delta)} \right) T_\delta \nabla_\delta \eta_\delta \\ &= -\tilde{D}_{\underline{\Sigma}(\delta),\underline{u}(\delta)}(D_1 \exp_{\underline{\Sigma}(\delta),\zeta(\delta)}(\frac{d}{d\delta}\underline{\Sigma}(\delta)), D_1 \exp_{\underline{u}(\delta),\xi(\delta)}(\frac{d}{d\delta}\underline{u}(\delta)) \end{aligned}$$

plus terms involving $\hat{\xi}_\delta$. From (26) we have near the node

$$\begin{aligned} \frac{d}{d\delta}\overline{\partial}_{\underline{\Sigma}(\delta)}\underline{u}(\delta) &= \tilde{D}_{\underline{\Sigma}(\delta),\underline{u}(\delta)}(\frac{d}{d\delta}\underline{\Sigma}(\delta), \frac{d}{d\delta}\underline{u}(\delta)) \\ &= \frac{d}{d\delta}\overline{\partial}\exp_x(\beta_1(\kappa\sqrt{\varphi}\xi(z)) \\ &= D\exp_x(\beta_1(\kappa\sqrt{\varphi}|z|)\xi(z), d\beta_1(\kappa\sqrt{\varphi}|z|)|z|\xi(z)\frac{d}{d\delta}\sqrt{\varphi}\kappa). \end{aligned}$$

This has a pointwise bound

$$|\frac{d}{d\delta}\overline{\partial}_{\underline{\Sigma}(\delta)}\underline{u}(\delta; z)| \le C|\sqrt{\varphi}\kappa||\frac{d}{d\delta}\sqrt{\varphi}\kappa| \tag{40}$$

and is supported on a region of area $\pi\kappa^2\varphi$. If φ is the exponential gluing profile then

$$\begin{aligned} \mathrm{d}\varphi^{1/2} &= \mathrm{d}(e^{1/\delta} - e)^{1/2} \\ &= (e^{1/\delta} - e)^{-1/2} e^{1/\delta}\delta^{-2}\mathrm{d}\delta \\ &= (e^{-1/\delta} - e^{-2/\delta+1})^{-1/2}\delta^{-2}\mathrm{d}\delta. \end{aligned}$$

Hence $\sqrt{\varphi}\frac{d}{d\delta}\sqrt{\varphi}$ satisfies a pointwise estimate

$$\sqrt{\varphi}\frac{d}{d\delta}\sqrt{\varphi} \le C\log(\varphi)^2. \tag{41}$$

Integrating and using the pointwise estimates (40),(41) gives

$$\|\frac{d}{d\delta}\overline{\partial}_{\underline{\Sigma}(\delta)}\underline{u}(\delta)\|_{0,p} \le C(\kappa\sqrt{\varphi})^{2/p}\log(\varphi)^2$$

which approaches 0 exponentially as δ does. Now as $\delta \to 0$,

$$\tilde{D}_{\underline{\Sigma}(\delta),\underline{u}(\delta)} \to \tilde{D}_{\underline{\Sigma}(\delta),\underline{u}(\delta)}$$

by the uniform quadratic estimate (4.3.5). Furthermore,

$$\|D_1 \exp_{\underline{\Sigma}(\delta),\zeta(\delta)} - I\| < 1/2, \quad \|D_1 \exp_{\underline{u}(\delta),\xi(\delta)} - I\| < 1/2$$

since $\zeta(\delta), \xi(\delta)$ are exponentially small. The operator

$$\tilde{D}_{\underline{\Sigma}(\delta),\underline{u}(\delta)} \left(D_2 \exp_{\underline{\Sigma}(\delta),\zeta(\delta)} \times D_2 \exp_{\underline{u}(\delta),\xi(\delta)} \right) T_\delta$$

is invertible, for δ sufficiently small, since T_δ is an approximate inverse and the exponential maps are approximately the identity. It follows that $\nabla_\delta \eta_\delta$ and hence $\nabla_\delta \zeta_\delta$, $\nabla_\delta \xi_\delta$ are exponentially small. In particular, if

$$\operatorname{ev}(\delta) : \underline{\Sigma} \times \ker(\tilde{D}_{\underline{\Sigma},\underline{u}}) \to X$$

is given by

$$(z, \zeta, \xi) \mapsto (\#\underline{u})_{\xi,\zeta,\delta}(z)$$

then for all z away from the nodes, $\zeta, \xi \in \ker(\tilde{D}_{\underline{\Sigma},\underline{u}})$ and $i = 1, \ldots, m$, we have

$$\lim_{\delta \to 0} \left(\frac{\partial}{\partial \delta_i} \operatorname{ev}(\delta)\right)(z, \zeta, \xi) = 0.$$

It follows that $D\operatorname{ev}$ has a continuous limit as $\delta \to 0$ away from the nodes, given by projecting (ζ, ξ, δ) onto (ζ, ξ) and then differentiating the boundary evaluation map. □

Remark 4.4.3. Presumably, one can show that the evaluation maps are smooth in the exponential gluing profile; perhaps this will appear in the work of Hofer et al.

Next we discuss injectivity of the gluing families in Theorem 4.4.1. Let $\underline{u} : \underline{\Sigma} \to X$ be a marked nodal map. A real hypersurface $Y \subset X$ is *transverse* to $\underline{u}$ if $\underline{u}$ meets Y transversally in a single point $u(z)$.

Definition 4.4.4. We say that $\underline{Y}, \underline{u}$ are *compatible* if

(a) each Y_j intersects u_j transversally in a single point $z_j \in \underline{\Sigma}$;
(b) the map $\ker(\tilde{D}_{\underline{\Sigma},\underline{u}}) \to \times_{j=1}^m T_{\underline{u}(z_j)} X / T_{\underline{u}(z_j)} Y_j, \quad \xi \mapsto \xi(z_j) \mod TY_j$ is an injection;
(c) the curve $\underline{\Sigma}$ marked with the additional points $z_1, \ldots, z_m$ is stable.
(d) if some automorphism of $(\underline{\Sigma}, \underline{u})$ maps z_i to z_j then Y_i is equal to Y_j.

Let $(\underline{\Sigma}_\bullet, \underline{u}_\bullet, \underline{z}_\bullet) \to S$ an n-marked family of maps. Given any family $\underline{Y} = (Y_1, \ldots, Y_m)$ of hypersurfaces, we define a *rigidified family* of $n+m$-marked nodal surfaces

$$\underline{\Sigma}_\bullet^{\underline{Y},\underline{u}} := (\underline{\Sigma}_\bullet, (z_{1,\bullet}, \ldots, z_{n+m,\bullet})) \to S, \quad \underline{u}_s(z_{n+i,s}) \in Y_i. \tag{42}$$

The automorphism group $\operatorname{Aut}(\underline{\Sigma}, \underline{u})$ of $(\underline{\Sigma}, \underline{u})$ acts on $\underline{\Sigma}^{\underline{Y},\underline{u}}$ permuting the additional marked points; this induces a homomorphism

$$\operatorname{Aut}(\underline{\Sigma}, \underline{u}) \to \operatorname{Aut}(\underline{\Sigma}^{\underline{Y},\underline{u}}) \times S_m.$$

Proposition 4.4.5. *Let $(\underline{\Sigma}, \underline{u})$ be a parametrized regular stable map, and $(\underline{\Sigma}_\bullet, \underline{u}_\bullet) \to S$ the partially smooth universal deformation constructed in* (75).

(a) *If* $\underline{Y}$ *is compatible with* $\underline{u}$*, then the rigidified family* $\underline{\Sigma}_{\bullet}^{\underline{Y},\underline{u}}$ *of* (42) *is a partially smooth deformation of* $\underline{\Sigma}^{\underline{Y},\underline{u}}$*, which defines an immersion of* S *into the parameter space for the universal deformation of the central fiber.*
(b) *There exists a collection* $\underline{Y}$ *compatible with* $\underline{u}$.

Proof of Theorem 4.4.1. (a) Let $\underline{\Sigma}_{1,\bullet} \to S_1$ be the universal deformation of $\underline{\Sigma}^{\underline{Y},\underline{u}}$. By Definition 5.10.3, the family $\underline{\Sigma}_{\bullet}^{\underline{Y},\underline{u}}$ induces a map $S \to S_1$ whose differential at 0 is an injection. The claim follows from versality. (b) The proposition implies that the map

$$S/\operatorname{Aut}(\underline{\Sigma}^{\underline{Y},\underline{u}}) \to \overline{M}_{g,n+m}, \quad [s] \mapsto [\underline{\Sigma}_s^{\underline{Y},\underline{u}}]$$

is an orbifold immersion near 0, and in particular, injective. It follows that

$$S/\operatorname{Aut}(\underline{\Sigma},\underline{u}) \to \overline{M}_{g,n}(X,d), \quad s \mapsto [\underline{\Sigma}_s,\underline{u}_s] \tag{43}$$

must also be an immersion, hence injection in a neighborhood of 0. By the inverse function theorem, the map (43) is a homeomorphism of $S/\operatorname{Aut}(\underline{\Sigma},\underline{u})$ onto its image in $\overline{M}_{g,n}(X,d)$. □

Proof of Proposition. Given a regular stable map $(\underline{\Sigma}, \underline{z} = (z_1,\ldots,z_n), \underline{u} : \underline{\Sigma} \to X)$, choose points $z_1,\ldots,z_{n'}$ on the unstable components of $\underline{\Sigma}$, so that all components become stable, and the set $\{z_1,\ldots,z_{n'}\}$ is stable under the action of $\operatorname{Aut}(\underline{\Sigma},\underline{u})$. Denote by

$$\underline{\Sigma}_1 = (\underline{\Sigma}, \underline{z} \cup \underline{z}'), \quad \underline{u}_1 : \underline{\Sigma}_1 \to X$$

the stable map with the additional marked points. Let $\underline{\Sigma}_{\bullet,1} \to S_1$ denote the universal deformation of $\underline{\Sigma}_1$. By universality, the family $\underline{\Sigma}_{\bullet}^{\underline{Y},\underline{u}}$ is induced by a map $\psi : S \to S_1$.

We successively add marked points until ψ is an embedding. Let $\underline{\Sigma}_{\bullet,0}, \underline{u}_{\bullet,0}$ denote the family of maps given by restricting $\underline{\Sigma}_{\bullet}, \underline{u}_{\bullet}$ to $\psi^{-1}(0)$. This family consists of a those marked stable maps such that the additional marked points are the same as those for $\underline{\Sigma}^{\underline{u},\underline{Y}}$. Thus, the map ψ is an embedding if and only if it is a submersion and the fibers are point. Suppose that ψ has positive dimensional fibers. Choose an additional marked point $z_{n+n'+1} \in \underline{\Sigma}$ such that $d\,\mathrm{ev}_{n+n'+1}$ is non-trivial on the family $\underline{\Sigma}_{0,\bullet}, \underline{u}_{0,\bullet}$. Since $\underline{u}$ is pseudoholomorphic, $d\underline{u}(z_{n+n'+1})$ is rank two at $z_{n+n'+1}$. Let $Y_{n+n'+1} \subset X$ be a codimension two submanifold containing $\underline{u}(z_{n+n'+1})$ such that

(a) $\underline{u}$ is transverse to $Y_{n+n'+1}$ at $z_{n+n'+1}$, *and*
(b) $Y_{n+n'+1}$ is transversal to $\mathrm{ev}_{n+n'+1}$ at $\underline{\Sigma}, \underline{u}$.

Suppose $z_{n+n'+1}$ has orbit $z_{n+n'+1}, z_{n+n'+2}, \ldots, z_{n+n'+k}$ under the group $\operatorname{Aut}(\underline{\Sigma},\underline{u})$. Repeating the same hypersurface for each marking $n + n' + j, j = 1,\ldots,k$ gives a collection invariant under the action of automorphisms. The map ψ_1 for the new family has the property that the fiber of ψ_1 over 0 has dimension at least two less than that of ψ. It follows that the procedure terminates after adding a finite number of markings. □

Theorem 4.4.6. *For any $J \in \mathcal{J}(X)^G$, the partially smooth universal deformations of parametrized regular stable maps provide the coarse moduli space $\overline{M}^{\mathrm{reg}}_{g,n}(X,J,d)$ with the structure of a partially smooth topological orbifold, equipped with partially smooth evaluation maps* $\mathrm{ev} : \overline{M}^{\mathrm{reg}}_{g,n}(X,d) \to X^n$ *and partially smooth forgetful map* $f_j : \overline{M}_{g,n}$. *For $J \in \mathrm{Map}([0,1],\mathcal{J}(X))$ is a one-parameter family of almost complex structures, then the parametrized regular moduli space $\overline{M}^{\mathrm{reg}}_{g,n}(X,d,J_t)$ admits the structure of a partially smooth cobordism between $\overline{M}^{\mathrm{reg}}_{g,n}(X,d,J_0)$ and $\overline{M}_{g,n}(X,d,J_1)$, equipped with partially evaluation maps* $\mathrm{ev} : \overline{M}^{\mathrm{reg}}_{g,n}(X,d,J_t) \to X^n$ *and partially smooth forgetful maps $f_j : \overline{M}^{\mathrm{reg}}_{g,n}(X,d,J_t) \to \overline{M}^{\mathrm{reg}}_{g,n-1}$.*

Proof. For any partially smooth universal deformation $(\underline{\Sigma}_\bullet, \underline{u}_\bullet) \to S$ of a stable map $(\underline{\Sigma}, \underline{u})$, the induced map $S \to \overline{M}_{g,n}(X,d)$ are orbifold charts by Theorem 4.4.1. Compatibility follows from universality. The results for the parametrized moduli space $\overline{M}_{g,n}(X,J_t,d)$ are similar and left to the reader. □

In order to apply localization one needs to know that the fixed point sets admit tubular neighborhoods. For this it is helpful to know that $\overline{M}^{\mathrm{reg}}_{g,n}(X,d)$ admits (non-canonically) a differential structure. In order to obtain compatible charts, we construct the local coordinates inductively as in Definition 2.2.3, starting with the strata of highest codimension.

Proposition 4.4.7. *For any system of local coordinates near the nodes constructed inductively as above, the universal deformations constructed using the exponential gluing profile equip $\overline{M}^{\mathrm{reg}}_{g,n}(X,d)$ with the structure of a C^1-orbifold.*

Proof. We claim that the charts induced by the universal deformations are C^1-compatible, assuming they are constructed from the same system of local coordinates near the nodes. Given two sets of hypersurfaces $\underline{Y}_1, \underline{Y}_2$, define $\underline{Y} = \underline{Y}_1 \cup \underline{Y}_2$. The family $\underline{\Sigma}_\bullet^{\underline{Y},\underline{u}} \to S$ admits a proper étale forgetful map

$$\underline{\Sigma}_\bullet^{\underline{Y},\underline{u}} \to \underline{\Sigma}_\bullet^{\underline{Y}_1,\underline{u}} \times_S \underline{\Sigma}_\bullet^{\underline{Y}_2,\underline{u}}.$$

This map is a C^1-diffeomorphism, by construction, and a bijection, since the fiber to the product consists exactly of reorderings of one set of additional marked points induced by the action of $\mathrm{Aut}(\underline{\Sigma}, \underline{u})$. Thus the fiber products have partially smooth structures, that is, the orbifold charts are compatible. □

4.5. **Boundary structure.** The boundary structure of $\overline{M}_{g,n}(X,d)$ is similar to that of $\overline{M}_{g,n}$.

(a) if $g > 0$, there is a subset

$$\iota_{g-1,n+2} : D_{g-1,n+2}(X,d) \to \overline{M}_{g,n}(X,d)$$

equipped with an isomorphism

$$\varphi_{g-1,n+2} : D_{g-1,n+2}(X,d) \to \mathrm{ev}^{-1}_{n+1,n+2}(\Delta) \subset \overline{M}_{g-1,n+2}(X,d)$$

with $\iota_{g-1,n+2}$ resp. $\varphi_{g-1,n+2}$ an immersion resp. isomorphism of orbifolds over the regular locus. The inclusion is obtained by identifying the last two marked points. The forgetful morphism

$$f : D_{g-1,n+2}(X,d) \to D_{g-1,n+2}$$

is a fibration over a subset of formal codimension two, namely the locus where no collapsing of unstable components occurs.

(b) for each splitting $g = g_1 + g_2, \{1,\ldots,n\} = I_1 \cup I_2$ with $j = 1,2$, a divisor

$$\iota_{g_1+g_2,I_1\cup I_2} : D_{g_1+g_2,I_1\cup I_2}(X,d_1,d_2) \to \overline{M}_{g,n}(X,d)$$

corresponding to the formation of a separating node, splitting the surface into pieces of genus g_1, g_2 with markings I_1, I_2 and degrees d_1, d_2, equipped with an isomorphism

$$\varphi_{g_1+g_2,I_1\cup I_2} : D_{g_1+g_2,I_1\cup I_2}(X,d_1,d_2) \to \overline{M}_{g_1,|I_1|+1}(X,d_1) \times_X \overline{M}_{g_2,|I_2|+1}(X,d_2) \tag{44}$$

(except in the cases $I_1 = I_2 = \emptyset$ and $g_1 = g_2$ in which case there is an additional automorphism) with $\iota_{g_1+g_2,I_1\cup I_2}$ resp. $\varphi_{g_1+g_2,I_1\cup I_2}$ an immersion resp. isomorphism of orbifolds over the regular locus. If $2g_j + |I_j| \geq 2$ (that is, each component is stable) there is a forgetful morphism

$$f : D_{g_1+g_2,I_1\cup I_2}(X,d_1,d_2) \to D_{g_1+g_2,I_1\cup I_2}$$

is a fibration over a subset of codimension two, namely the locus where no collapsing of unstable components occurs.

4.6. **Equivariant Gromov-Witten invariants for convex varieties.** Let X be a compact symplectic manifold.

Definition 4.6.1. $J \in \mathcal{J}(X)^G$ is *convex* if D_u is surjective for every J-holomorphic $\underline{u} : \mathbb{P}^1 \to X$.

Proposition 4.6.2. *If J is convex, then every genus zero stable map $\underline{u} : \underline{\Sigma} \to X$ is regular.*

Proof. By openness of the regularity condition in families of stable maps. □

Thus the convexity condition avoid the multiple cover problem etc.

Lemma 4.6.3. *If (X,J) is homogeneous variety for* $\mathrm{Aut}(X)$*, then J is convex.*

Proof. Since TX is generated by global sections, see e.g. [15]. □

The motivating examples of Hori and Vafa [23, Appendix] are therefore convex.

We restrict from now on to genus zero, and assume that J is convex. Let

$$\delta_{g_1+g_2,I_1\cup I_2}(X,d_1,d_2) \in H^2_G(\overline{M}_{0,n}(X,d))$$

denote the dual class to $D_{g_1+g_2,I_1\cup I_2}(X,d_1,d_2)$. Since f is a fibration over a subset of codimension two, we have

$$f^*\delta_{g_1+g_2,I_1\cup I_2} = \sum_{d_1+d_2=d} \delta_{g_1+g_2,I_1\cup I_2}(X,d_1,d_2). \tag{45}$$

For any cohomology classes $\alpha \in H_G(X,\mathbb{Q})^n$ and $\beta \in H(\overline{M}_{0,n},\mathbb{Q})$, integration defines an *equivariant Gromov-Witten invariant*

$$\langle \alpha;\beta\rangle_{X,d} = \int_{[\overline{M}_{0,n}(X,d)]} \mathrm{ev}^*\alpha \wedge f^*\beta \in H_G(\mathrm{pt},\mathbb{Q}).$$

More generally, define a vector space

$$QH_G(X,\mathbb{Q}) := H_G(X,\mathbb{Q}) \otimes \Lambda[[q]]$$

where $\Lambda[[q]]$ is the Novikov ring of formal Laurent series with fractional powers in a formal variable q,

$$\Lambda[[q]] = \left\{ \sum_{i=0}^{\infty} a_i q^{d_i} \right\}$$

with d_i real and $d_i \to \infty$ as $i \to \infty$. Equivariant Poincaré duality (9) extends to this setting and we define composition maps

$$\mu^n : QH_G(X,\mathbb{Q})^n \otimes H(\overline{M}_{0,n+1},\mathbb{Q}) \to QH_G(X,\mathbb{Q})$$

by

$$(\mu^n(\alpha_1,\ldots,\alpha_n;\beta),\alpha_0) = \sum_{d\in H_2(X)} q^{([\omega],d)}\langle \alpha_0,\ldots,\alpha_n;\beta\rangle_{X,d}.$$

Theorem 4.6.4. *Suppose that (X,ω) is a compact symplectic G-manifold equipped with a convex almost complex structure $J \in \mathcal{J}(X)^G$. The maps μ^n form a cohomological field theory with values in $QH_G(X,\mathbb{Q})$.*

Proof. By Equations (45) and (44). □

By Proposition 4.4.7, the fixed point sets for elements of G acting on $\overline{M}_{g,n}(X,d)$ have tubular neighborhoods. This implies that the localization formula (10) holds for the G action on $\overline{M}_{g,n}(X,d)$. The structure of the fixed point sets is described in [28].

We remark that if $\overline{M}_{0,n}$ is equipped with a C^1-tructure using the same gluing profile as $\overline{M}_{0,n}(X,d)$, then f is differentiable away from the collapsing locus and a submersion. However, using a softer gluing profile on $\overline{M}_{0,n}$ as $\overline{M}_{0,n}(X,d)$, then f is differentiable globally but not a submersion near the boundary.

5. Gauged Gromov-Witten theory

In this section we construct a coarse moduli space of stable vortices, and associated vortex invariants in the case that X is convex.

5.1. **Traces on cohomological field theories.** Let Σ be a compact Riemann surface. A *family of markings on* Σ *over a base manifold* S is a collection of sections $\underline{z}_\bullet = (z_{1,\bullet}, \ldots, z_{n,\bullet})$ of sections of $\Sigma \times S \to S$. We define isomorphism of families to be equality. The functor from S to families of markings over S is represented by a coarse moduli space $M_n(\Sigma)$ of n-tuples of distinct points on Σ, *not* up to automorphisms of Σ. We can compactify $M_n(\Sigma)$ by allowing the points to come together, as in the moduli space of stable curves. In fact this is a special case of the Deligne-Mumford construction but we prefer to think of it in a slightly different way.

Definition 5.1.1. A Σ*-rooted stable marked nodal curve* is a nodal curve $\hat{\Sigma} = \Sigma_0 \cup \ldots \cup \Sigma_m$ together with an isomorphism $\phi_0 : \Sigma_0 \to \Sigma$, and distinct, non-singular points $z_1, \ldots, z_n$, such that each of the other components have at least three special (marked or singular) points. Two Σ-rooted curves $\hat{\Sigma}_1, \hat{\Sigma_2}$ are isomorphic if there exists an isomorphism of nodal curves $\hat{\Sigma}_1 \to \hat{\Sigma}_2$, inducing the identity on Σ, and mapping each marking $z_{1,i}$ to its counterpart on $z_{i,2}$. Families and deformations of rooted stable marked nodal curves are defined similarly.

In particular, the markings on the principal components of two isomorphic Σ-rooted curves should be equal. Let $\overline{M}_n(\Sigma)$ denote the coarse moduli space of isomorphism classes of Σ-rooted stable marked curves. Convergence of a sequence of rooted stable curves can be defined as in Definition 2.1.2, by adding the requirement that the identifications $\Sigma_0 \to \Sigma$ converge uniformly in all derivatives. $\overline{M}_n(\Sigma)$ has canonically the structure of a compact complex orbifold. One could prove this by extending the deformation theory of Section 2.1. More directly, $\overline{M}_n(\Sigma)$ is isomorphic (non-canonically) to a fiber of the forgetful morphism $\overline{M}_{g,n+j} \to \overline{M}_{g,j}$ over Σ with j additional markings chosen to fix the automorphism group, with $j = 3$ in genus zero and $j = 1$ in genus one. We have forgetful morphisms $f_j : \overline{M}_n(\Sigma) \to \overline{M}_{n-1}(\Sigma)$ which forget the j-th marking and collapse any unstable components. For curves Σ without automorphisms, $\overline{M}_n(\Sigma)$ is isomorphic to the fiber of the forgetful morphism $\overline{M}_{g,n} \to \overline{M}_{g,0}$ over $[\Sigma] \in \overline{M}_{g,0}$, where $g = \text{genus}(\Sigma)$.

The boundary structure is similar to that for the usual Deligne-Mumford spaces. For each subset $I \subset \{1, \ldots, n\}$ there is a boundary divisor

$$\iota_I : D_I \to \overline{M}_n(\Sigma)$$

where the markings for $i \in I$ have bubbled off onto an (unparametrized) sphere bubble, and a homeomorphism

$$\varphi_I : D_I \cong \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1}(\Sigma).$$

In particular, for any $\beta \in \overline{M}_n(\Sigma)$, the pull-back $\iota_I^* \beta$ has a Kunneth decomposition

$$\iota_I^* \beta = \sum_{j \in J} \beta_{1,j} \otimes \beta_{2,j} \tag{46}$$

for some index set J and classes $\beta_{1,j} \in H(\overline{M}_{0,|I|+1})$ and $\beta_{2,j} \in \overline{M}_{n-|I|+1}(\Sigma)$.

Definition 5.1.2. A Σ-*cohomological trace* on a CohFT V is a collection of *correlators*

$$\langle\ ;\rangle_n : V^n \times H^\bullet(\overline{M}_n(\Sigma)) \to \Lambda[[q]]$$

satisfying a splitting axiom

$$\langle \alpha; \beta \wedge \delta_I \rangle_n = \langle \alpha_i, i \notin I, \mu^{|I|}(\alpha; \cdot); \cdot \rangle_{n-|I|+1}(\iota_I^* \beta)$$

where the $\cdot$'s denote insertion of the Kunneth components of β and $\mu^\bullet$ are the composition maps of (7).

That is, with β as in (46),

$$\langle \alpha; \beta \wedge \delta_I \rangle_n = \sum_{j \in J} \langle \alpha_i, i \notin I, \mu^{|I|}(\alpha; \beta_{1,j}); \beta_{2,j} \rangle_{n-|I|+1}.$$

Let $\mathrm{Tr}(V)$ denote the space of traces on V.

5.2. **Pseudoholomorphic sections.** Let Σ be a compact Riemann surface, and $\pi : P \to \Sigma$ a smooth principal G-bundle. Given any left G-manifold F we have a left action of G on $P \times F$ given by $g(p, f) = (pg^{-1}, gf)$ and we denote by $P(F) = (P \times F)/G$ the quotient, that is, the associated fiber bundle with fiber F.

Let X denote a compact Hamiltonian G-manifold with symplectic form ω and moment map $\Phi : X \to \mathfrak{g}^*$. The action of G on X induces an action on $\mathcal{J}(X)$ by conjugation, and we denote by $\mathcal{J}(X)^G$ the invariant subspace.

let $\psi : \Sigma \to BG$ be a classifying map for $P \to \Sigma$. Sections $u : \Sigma \to P(X)$ are in one-to-one correspondence with lifts of ψ to X_G. The degree $\deg(u)$ of the section u is defined to be the degree $\deg(u) \in H_2^G(X, \mathbb{Z})$ of the corresponding lift. The projection of $\deg(u)$ onto $H_2^G(\mathrm{pt}, \mathbb{Z}) = H_2(BG, \mathbb{Z})$ is the first Chern class of P; in particular, since $\dim(\Sigma) = 2$ the degree $\deg(u)$ determines the topological type of P. An equivalent definition of degree is

$$\deg(u) = \pi_* u_* [\Sigma]$$

where $u_* : H_2(\Sigma) \to H_2(P(X))$ is the push-forward and π_* the composition of the maps $H_2(P(X)) \to H_2^G(P \times X) \to H_2^G(X)$. This definition extends immediately to maps $u : \hat{\Sigma} \to P(X)$ where $\hat{\Sigma}$ is an arbitrary (possibly nodal surface). If u maps into a single fiber $P(Z)_z \cong X$ then the degree $\deg(u)$ is the degree of the corresponding map to X, composed with the inclusion $H_2(X) \to H_2^G(X)$. If u is nodal section consisting of an honest section and a collection of sphere bubbles in the fibers, then the degree $\deg(u)$ is the sum of the degree of the section and the degree of the bubbles.

We denote by $\mathcal{A}(P)$ the space of smooth connections on P, and by $P(\mathfrak{g}) := (P \times \mathfrak{g})/G$ the adjoint bundle. For any $A \in \mathcal{A}(P)$, we denote $F_A \in \Omega^2(\Sigma, P(\mathfrak{g}))$ the curvature of A. Any connection $A \in \mathcal{A}(P)$ induces a map of spaces of almost complex structures

$$\mathcal{J}(X)^G \to \mathcal{J}(P(X)), \quad J \mapsto J_A$$

by combining the almost complex structure on X and Σ using the splitting defined by the connection. Let $\Gamma(\Sigma, P(X))$ denote the space of sections of $P(X)$. We denote by

$$\overline{\partial}_A : \Gamma(\Sigma, P(X)) \to \Omega^{0,1}(\Sigma, (\cdot)^* T^{\mathrm{vert}} P(X)) \tag{47}$$

the Cauchy-Riemann operator defined by J_A.

The bundle $P(X)$ can be made into a symplectic fiber bundle as follows. By the Cartan construction the equivariant symplectic form $\omega \in \Omega^2_G(X)$ descends to a closed two-form $\omega_A \in \Omega^2(P(X))$, by the formula

$$\pi^* \omega_A = \omega + \mathrm{d}(\Phi, A). \tag{48}$$

In general, ω_A is closed but not symplectic. The symplectic perpendicular $TP^{\mathrm{vert}}(X)^{\omega_A}$ of the vertical space in $P(X)$ with respect to ω_A defines a symplectic connection on $P(X)$,

$$TP(X) = TP^{\mathrm{vert}}(X) \oplus TP^{\mathrm{vert}}(X)^{\omega_A}.$$

equal to the connection induced by A, that is, the pull-back of $TP^{\mathrm{vert}}(X)^{\omega_A}$ to $T(P \times X)$ is the subspace $\ker(A) \oplus \{0\}$. We denote by E_A the *twisted energy*

$$E_A(u) = \int_\Sigma |\nabla_A u|^2.$$

Define the *twisted symplectic action* by the formula

$$D_A(u) = \int_\Sigma u^* \omega_A.$$

Pseudoholomorphic sections then satisfy the energy-action relation

$$E_A(u) = D_A(u) - (F_A, u^*\Phi).$$

In order to apply Gromov compactness for pseudoholomorphic sections, we wish to equip $P(X)$ with the structure of a symplectic manifold. To carry this out requires adding the pull-back of a form on Σ to the fiber-wise symplectic form on $P(X)$. Namely let $\omega_\Sigma \in \Omega^2(\Sigma)$ be a volume form on Σ, and for any $c > 0$ let $\omega_{A,c} = \omega_A + c\pi^*\omega_\Sigma$. For c sufficiently large $\omega_{A,c}$ is symplectic, and the almost complex structure J_A determined by $J \in \mathcal{J}(X)^G$ and the connection is compatible with $\omega_{A,c}$, since $\dim(\Sigma) = 2$. The almost complex structure J_A determined by $J \in \mathcal{J}(X)^G$ and the connection is automatically compatible with $\omega_{A,c}$. We denote by $g_{A,c}$ the metric determined by $J_A, \omega_{A,c}$ on $P(X)$. The following is then a corollary of Gromov compactness in [32, Chapter 5], choosing c sufficiently large so that $\omega_{A,c}$ is symplectic:

Theorem 5.2.1. *Suppose that (A_α) is a sequence of connections, and u_α a sequence of J_{A_α}-holomorphic sections of $P(X)$ of constant degree. Suppose that A_α converges uniformly in all derivatives to a limiting connection A_∞. Then the sequence u_α converges to a stable map $u_\infty : \hat{\Sigma} \to P(X)$.*

5.3. **Symplectic vortices.** A *gauged section* of P is a pair (A, u) where $A \in \mathcal{A}(P)$ and $u : \Sigma \to P(X)$ is a section. The Yang-Mills-Higgs *energy* of a gauged section (A, u) is given by

$$E(A, u) = \frac{1}{2} \int_\Sigma \left(|\mathrm{d}_A u|^2 + |F_A|^2 + |u^* P(\Phi)|^2 \right) \mathrm{Vol}_\Sigma .$$

The *action* of a pair (A, u) is pairing of the degree with the class $[\omega_G] \in H^2(X_G)$,

$$D(A, u) = (d(u), [\omega_G]) = ([\Sigma], u^*[\omega_G]).$$

Let $\mathcal{A}(P, X)$ be the space of *gauged pseudoholomorphic maps* consisting of pairs

$$\mathcal{A}(P, X) = \{(A, u) \in \mathcal{A}(P) \times \Gamma(\Sigma, P(X)), \quad \overline{\partial}_A u = 0\}.$$

Suppose Vol_Σ is the volume form determined by a choice of metric on Σ. The energy and action are related by

$$E(A, u) = D(A, u) + \int_\Sigma |\overline{\partial}_A u|^2 \mathrm{Vol}_\Sigma + \frac{1}{2} \int |F_A / \mathrm{Vol}_\Sigma + u^* \Phi|^2 \mathrm{Vol}_\Sigma \tag{49}$$

see [7, Proposition 2.2]. In particular, for any $(A, u) \in \mathcal{A}(P, X)$ the energy-action relation (49) simplifies to

$$E(A, u) = D(A, u).$$

The space $\mathcal{A}(P, X)$ has a closed two-form induced from the sum of the symplectic form on the affine space of connections and the space of maps to X. Namely, let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be an invariant inner product, and

$$\Omega^1(P(\mathfrak{g}))^2 \to \mathbb{R}, \quad (a_1, a_2) \mapsto \int_\Sigma B(a_1 \wedge a_2)$$

the symplectic form on the affine space of connections $\mathcal{A}(P)$. On the other hand, choose a two-form $\mathrm{Vol}_\Sigma \in \Omega^2(\Sigma)$ and define a formal 2-form

$$\ker(D\overline{\partial}_A)^2 \to \mathbb{R}, \quad (v_1, v_2) \mapsto \int_\Sigma \mathrm{Vol}_\Sigma(u^* \omega(v_1, v_2)). \tag{50}$$

(50) is the restriction of a closed two-form on the space of all sections $\Sigma \to P(X)$, and so restricts to a closed two-form on $\mathcal{A}(P, X)$, where smooth. If X is Kähler, then the space of pseudoholomorphic sections is a formally complex manifold, and this can be used to show non-degeneracy of (50), where smooth. In general, we know of no argument which implies that (50) is non-degenerate. Choose a constant $\epsilon > 0$ and consider the summed two-form

$$((a_1, v_1), (a_2, v_2)) \to \int_\Sigma B(a_1 \wedge a_2) + \epsilon^{-1} \mathrm{Vol}_\Sigma(u^* \omega)(v_1, v_2).$$

Let $\mathcal{G}(P)$ denote the group of gauge transformations

$$\mathcal{G}(P) = \{a : P \to P, a(pg) = a(p)g, \quad \pi \circ a = \pi\}.$$

The Lie algebra of $\mathcal{G}(P)$ is the space of sections $\Omega^0(\Sigma, P(\mathfrak{g}))$ of $P(\mathfrak{g})$. The action of $\mathcal{G}(P)$ on $\mathcal{A}(P, X)$ has generating vector fields given by the covariant derivative and infinitesimal action

$$\xi_{\mathcal{A}(P,X)}(A, u) = (-\mathrm{d}_A \xi, \xi_X(u)) \in \Omega^1(\Sigma, P(\mathfrak{g})) \times \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X)), \quad \xi \in \Omega^0(\Sigma, P(\mathfrak{g})).$$

We say that a pair (A, u) is *stable* if it has finite automorphism group.

The stability terminology for vortices is somewhat problematic because of a conflict between the stability terminology for bundles and for curves. In particular (A, u) is *strictly polystable* if it satisfies the vortex equations but has infinite automorphism group. On the other hand, when we consider nodal vortices later, we cannot really call them *semistable* since we allow infinite automorphism group, but require that the bubble components on which the map is constant are stable (at least three special points) rather than only semistable (at least two special points).

The action preserves the formal two-form (50) and has formal moment map given by the curvature plus pull-back of the moment map for X,

$$\mathcal{A}(P, X) \to \Omega^2(\Sigma, P(\mathfrak{g})), \quad (A, u) \mapsto F_A + \epsilon^{-1} \operatorname{Vol}_\Sigma u^* P(\Phi).$$

We say that a gauged map (A, u) is an *ϵ-vortex* if it satisfies the moment map condition

$$F_A + \epsilon^{-1} \operatorname{Vol}_\Sigma u^* \Phi = 0.$$

Let $M(P, X, d)_\epsilon$ be the quotient of the zero level set of the moment map by gauge transformations, the *coarse moduli space of ϵ-vortices* of degree $d \in H_2^G(X)$,

$$M(P, X, d)_\epsilon := \mathcal{A}(P, X, d) /\!\!/ \mathcal{G}(P) = \{F_A + \epsilon^{-1} \operatorname{Vol}_\Sigma u^* \Phi = 0, d(u) = d\} / \mathcal{G}(P).$$

We define a metric on $M(P, X, d)_\epsilon$ by

$$\begin{aligned} \operatorname{dist}([A_0, u_1], [A_1, u_1])^2 &= \inf_{g \in \mathcal{G}(P)} \operatorname{dist}((gA_0, gu_0), (A_1, u_1))^2 \\ &= \inf_{g \in \mathcal{G}(P)} \int_\Sigma \|A_1 - gA_0\|^2 + \operatorname{dist}(gu_0(z), u_1(z))^2. \end{aligned}$$

The topology defined by the metric is the same as that defined by uniform convergence in all derivatives, by elliptic regularity [8, Section 3]. For any constants $c_1, c_2 > 0$, the subset of $M(P, X, d)_\epsilon$ defined by $E(A, u) < c_1$ and $\sup |\nabla_A u| < c_2$ is compact, by [8, Section 3.2].

A *smooth family of vortices* on Σ is a smooth family $(A_s, u_s), s \in S$ of vortices over a smooth parameter space S. A *deformation* of (A, u) is a smooth family $(A_\bullet, u_\bullet)$ together with an isomorphism (gauge transformation) relating (A_0, u_0) with (A, u). We wish to know when a vortex (A, u) admits a universal deformation; as in the case of pseudoholomorphic maps, we prove that a universal deformation exists if a certain regularity hypothesis is satisfied.

We give the spaces of connections and sections the structure of Banach manifolds by taking completions with respect to Sobolev norms $\| \cdot \|_{p,k}$. For $p > 2$, define

$$\mathrm{d}_{A,u,\epsilon} : \Omega^1(\Sigma, P(\mathfrak{g}))_{1,p} \oplus \Omega^0(\Sigma, u^* TP(X))_{1,p} \to \Omega^2(\Sigma, P(\mathfrak{g}))_{0,p}$$

$$\mathrm{d}_{A,u,\epsilon}(a, \xi) := \mathrm{d}_A a + \epsilon^{-1} \operatorname{Vol}_\Sigma L_\xi \Phi.$$

Consider (abusing notation) the operator

$$\mathrm{d}^*_{A,u,\epsilon} : \Omega^1(\Sigma, P(\mathfrak{g}))_{1,p} \oplus \Omega^0(\Sigma, u^* TP(X))_{1,p} \to \Omega^2(\Sigma, P(\mathfrak{g}))_{0,p}$$

$$\mathrm{d}^*_{A,u,\epsilon}(a,\xi) = \mathrm{d}_A * a + \epsilon^{-1}\operatorname{Vol}_\Sigma L_{J\xi}\Phi.$$

It is shown in [8, Section 4] that if (A,u) is stable then the set

$$S_{A,u} = \{(A+a, \exp_u(\xi)), (a,\xi) \in \ker \mathrm{d}^*_{A,u,\epsilon}\} \tag{51}$$

is a slice for the gauge group action near (A,u). Define

$$\begin{aligned}\mathcal{F}^\epsilon_{A,u} : \Omega^1(\Sigma, P(\mathfrak{g}))_{1,p} \oplus \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{1,p} \\ \to (\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g}))_{0,p} \oplus \Omega^{0,1}(\Sigma, u^*T^{\mathrm{vert}}P(X))_{0,p}\end{aligned} \tag{52}$$

by

$$(a,\xi) \mapsto \left(F_{A+a} + \epsilon^{-1}\operatorname{Vol}_\Sigma \exp_\xi(u)^*\Phi, \mathrm{d}^*_{A,\epsilon}(a,\xi), \overline{\partial}_{A+a}\exp_u(\xi)\right).$$

Let

$$\Omega^1(\Sigma, P(\mathfrak{g}))_{1,p} \to \Omega^1(\Sigma, u^*TX)_{1,p}, \quad a \mapsto a_X$$

denote the map induced by the infinitesimal action. The linearization of the parametrized Cauchy-Riemann operator is

$$D_{A,u}(a,\xi) = (\nabla_A\xi + J(\nabla_\xi J)\mathrm{d}u + a_X)^{0,1}.$$

Here $0,1$ denotes projection on the $0,1$-component. Let $\tilde{D}_{A,u}$ denote the combined operator

$$\tilde{D}_{A,u} = (\mathrm{d}_{A,u,\epsilon}, \mathrm{d}^*_{A,u,\epsilon}, D_{A,u}), \tag{53}$$

which is the linearization of $\mathcal{F}^\epsilon_{A,u}$ at (A,u). We say that (A,u) is *regular* if the operator $\tilde{D}_{A,u}$ is surjective.

Theorem 5.3.1. *Any regular vortex with smooth domain (A,u) has a smooth universal deformation $(A_\bullet, u_\bullet) \to S$. The universal deformation has the properties*

(a) *Two pairs (A_{s_j}, u_{s_j}) are isomorphic if and only if they are related by the action of* $\operatorname{Aut}(A,u)$.
(b) *Possibly after shrinking S, the family $(A_\bullet, u_\bullet)$ is a universal deformation of any of its fibers.*

Proof. This is an application of the implicit function theorem. Let $\mathcal{A}(P,X)_{1,p}$ denote the spaces of pairs (A,u) of Sobolev class $1,p$. As explained in [8], the kernel of $\mathrm{d}^*_{A,u,\epsilon}$ is a slice for the action of the gauge group near (A,u). That is, there exists a constant $c > 0$ such that if $\|(a,\xi)\| < c$ then there exists a unique element $\xi \in \mathfrak{g}^\perp_{A,u}$ such that $\mathrm{d}^*_{A,u,\epsilon}\exp(\xi)(A+a, \exp_u(\xi)) = 0$. If (A,u) is stable then we obtain a local homeomorphism

$$\left(\ker(\mathrm{d}^*_{A,u,\epsilon}) \cap \{F_A + \operatorname{Vol}_\Sigma u^*\Phi = 0, \overline{\partial}_A u = 0\}\right) / \operatorname{Aut}(A,u) \to M(P,X)_\epsilon. \tag{54}$$

From the implicit function theorem, we obtain a family of vortices $(A_\bullet, u_\bullet) \to S$ over a neighborhood S of 0 in $\ker(\tilde{D}_{A,u})$. By [8, Theorem 3.1], $(A_\bullet, u_\bullet)$ is a smooth family. Given any other family $(A'_\bullet, u'_\bullet) \to S'$ of stable vortices with $(A'_0, u'_0) = (A,u)$, the implicit function theorem provides a smooth map $S' \to S$ so that $(A'_\bullet, u'_\bullet)$ is obtained from (A,u) by pull-back. Universality follows from finiteness of $\operatorname{Aut}(A,u)$.

The first property of the universal deformation is a consequence of the slice condition; the second property follows from the fact that $\ker(\tilde{D}_{A,u}) \to \ker(\tilde{D}_{A_s,u_s})$ is an isomorphism for sufficiently small s. $\square$

Theorem 5.3.2. *The universal deformations above provide $M^{\mathrm{reg}}(P,X)_\epsilon$ with the structure of a smooth orbifold with tangent space isomorphic to the kernel of $\tilde{D}_{A,u}$, and dimension of the component of degree $d \in H_2^G(X)$ is given by*

$$\dim(M^{\mathrm{reg}}(P,X)_\epsilon) = \mathrm{Ind}(\tilde{D}_{A,u}) = \dim(X) + 2(c_1^G(TX), d) - 2(g-1)\dim(G).$$

For any constants $c_1, c_2 > 0$, the subset of the moduli space with $\sup|\mathrm{d}_A u| < c_1, E(A,u) < c_2$ is compact.

Proof. Compactness is [8, Theorem 3.2]. $\square$

Lemma 5.3.3. *Suppose that J is integrable. If $D_{A,u}$ is surjective and (A,u) is stable, then (A,u) is a regular ϵ-vortex.*

Proof. Suppose that (A,u) is as in the theorem. By the moment map condition, the cokernel of $\mathrm{d}_{A,u}^*$ in $\Omega^0(\Sigma, P(\mathfrak{g}))$ is isomorphic to the infinitesimal stabilizer of (A,u), and similarly for the cokernel of $\mathrm{d}_{A,u}$. Hence both are trivial. Since $D_{A,u}$ is surjective, so is $\tilde{D}_{A,u}$. $\square$

An *n-marked* symplectic vortex is a vortex $[A,u]$ together with n-tuple $\underline{z} = (z_1,\ldots,z_n)$ of distinct points on Σ. Let $M_n(P,X)_\epsilon$ denote the moduli space of isomorphism classes of n-marked ϵ-vortices, up to gauge transformation, and $M_n(\Sigma,X)_\epsilon)$ the union over types of bundles P. The moduli space $M_n(\Sigma,X)_\epsilon$ is homeomorphic to the product

$$M_n(\Sigma,X)_\epsilon \cong M(\Sigma,X)_\epsilon \times M_n(\Sigma)$$

where $M_n(\Sigma)$ denotes the configuration space of n-tuple of distinct points on Σ. The metric on $M(\Sigma,X)_\epsilon$ defined above generalizes naturally to a metric on $M_n(\Sigma,X)_\epsilon$, in which convergence is equivalent to uniform convergence in all derivatives and convergence of the marked points. For any constants $c_1,c_2,c_3 > 0$, the subset of $M_n(\Sigma,X,d)_\epsilon$ defined by $E(A,u) < c_1$, $\sup|\nabla_A u| < c_2$, and $\mathrm{dist}(z_i,z_j) > c_3$ is compact, by combining compactness of Σ with compactness for unmarked vortices. The formal tangent space to $M_n(\Sigma,X)_\epsilon$ is the sum of the formal tangent space to $M(\Sigma,X)_\epsilon$, and the sum of the tangent spaces $T_{z_i}\Sigma$.

A *framed ϵ-vortex* is a collection $(A,u,\underline{z},\underline{\phi})$ where $(A,u,\underline{z})$ is a marked ϵ-vortex and $\underline{\phi} = (\phi_1,\ldots,\phi_n)$ are trivializations of the fibers of P at $z_1,\ldots,z_n$, that is, each $\phi_j : P_{z_j} \to G$ is a G-equivariant isomorphism. Since the principal bundle is trivial on the sphere bubbles, we could instead require the framing to be at the marked points. However, we prefer to think of the framing as existing at the attaching points. Let $M_n^{\mathrm{fr}}(P,X)_\epsilon$ denote the moduli space of isomorphism classes of framed n-marked stable ϵ-vortices. Evaluation together with the framings define a map

$$\mathrm{ev}^{\mathrm{fr}} : M_n^{\mathrm{fr}}(P,X)_\epsilon \to X^n, \quad (A,\underline{u},\underline{z}) \mapsto (\phi_1(\underline{u}(z_1)),\ldots,\phi_n(\underline{u}(z_n)))$$

The metric on $M_n(P,X)_\epsilon$ generalizes naturally to a metric on $M_n^{\mathrm{fr}}(P,X)_\epsilon$, in which convergence is equivalent to uniform convergence in all derivatives and convergence for marked points and framings. For any constants $c_1, c_2, c_3 > 0$, the subset of $M_n(P,X,d)_\epsilon$ defined by $E(A,u) < c_1$, $\sup|\nabla_A u| < c_2$, and $\mathrm{dist}(z_i, z_j) > c_3$ is compact, by combining compactness of G with compactness for marked vortices. The formal tangent space to $M_n^{\mathrm{fr}}(P,X)_\epsilon$ is the sum of the formal tangent space to $M_n(P,X)_\epsilon$, and the sum of the tangent spaces to the space of framings, equal to a copy of the Lie algebra $\mathfrak{g}$ for each marking. Forgetting the framings defines a map

$$\pi : M_n^{\mathrm{fr}}(P,X)_\epsilon \to M_n(P,X)_\epsilon.$$

Since the action of the gauge group admits slices (54), the map π is a topological principal orbifold G-bundle.

Suppose that the action of G^n on $M_n^{\mathrm{fr,reg}}(P,X)_\epsilon$ is free, so that π is a topological principal G-bundle. Let

$$\psi : M_n^{\mathrm{fr,reg}}(P,X)_\epsilon \to EG^n$$

be a classifying map for the bundle $\pi : M_n^{\mathrm{fr,reg}}(P,X)_\epsilon \to M_n^{\mathrm{reg}}(P,X)_\epsilon)$. Combining ψ with the evaluation map $\mathrm{ev}^{\mathrm{fr}}$ gives rise to a G^n-equivariant map

$$\mathrm{ev}^{\mathrm{fr}} \times \psi : M_n^{\mathrm{fr,reg}}(P,X)_\epsilon \to X^n \times EG^n$$

which induces a map

$$\mathrm{ev} : \ M_n^{\mathrm{reg}}(P,X)_\epsilon \to (X^n \times EG^n)/G^n = X_G^n.$$

In particular, pull-back by ev induces a map in equivariant cohomology

$$\mathrm{ev}^* : H_G(X,\mathbb{Z})^n \to H(M_n^{\mathrm{reg}}(P,X)_\epsilon, \mathbb{Z}).$$

More generally, if the action of G^n is only locally free then as in (18) we obtain a map

$$\mathrm{ev}^* : H_G(X,\mathbb{Q})^n \to H(M_n^{\mathrm{reg}}(P,X)_\epsilon, \mathbb{Q}).$$

5.4. **Reducible vortices.** We say that a polystable vortex (Σ, A, u) is *reducible* if it is fixed by a one-parameter subgroup of gauge transformations, that is, it is not stable. Suppose that ζ is the infinitesimal gauge transformation fixing the polystable vortex. Fix a base point $z \in \Sigma_0$ and a trivialization of the fiber $P_z \to G$. Let $\zeta(z) \in \mathrm{Aut}(P_z) \cong G$ denote the evaluation of ζ at z and G_ζ the one-parameter subgroup generated by $\zeta(z)$.

Lemma 5.4.1. *If $(\underline{\Sigma}, A, \underline{u})$ is fixed by ζ then*

(a) *Parallel transport by A defines a reduction P_ζ of structure group of P to G_ζ, with connection A_ζ;*
(b) *the image of u is contained in the fixed point set $P(X)^\zeta = P_\zeta(X^{\zeta(z)})$, and so defines a section u_ζ of $P_\zeta(X^{\zeta(z)})$;*
(c) *the datum $(\Sigma, A_\zeta, u_\zeta)$ is a stable ϵ G_ζ-vortex.*

Proof. Let P, z, G_ζ be as above, and $\phi_z : P_z \cong G$ the given trivialization of the fiber at the base point. Parallel transport of $\phi_z^{-1}(G^\zeta)$ using A defines a reduction $P_\zeta \subset P$, since the holonomies of A are contained in G^ζ, and restriction of A to P_ζ defines the connection A_ζ. Necessarily, u is fixed by ζ. □

Repeated application of the lemma shows the following, which justifies the use of the terminology reducible:

Lemma 5.4.2. *If (Σ, A, u) is a vortex then there exists a reduction P_1 of P of structure group from G to G_1 preserved by A so that u takes values in $P_1(X^{G_1})$ and the corresponding datum (Σ, A_1, u_1) is stable as a G_1-vortex.*

Next we show that in good cases, reducibles occur for a discrete set of values of the vortex parameter. First we show:

Lemma 5.4.3. *Suppose that the image of u is contained in a component C of a fixed point set X^ζ for some $\zeta \in \mathfrak{g}$ with $(\Phi(C), \zeta) \neq 0$. Then there is a discrete set $Z(C) \subset (0, \infty)$ such that if u is part of an ϵ-vortex (A, u), then $\epsilon \in Z(C)$.*

Proof. Pair $F_A = -\epsilon^{-1} \operatorname{Vol}_\Sigma u^* P(\Phi)$ with ζ. Since (Φ, ζ) is constant on C, F_A is constant. As in e.g. Atiyah-Bott [2], (F_A, ζ) lies in a finite quotient of Λ_ζ, where Λ_ζ is the coweight lattice of G_ζ. Hence $\epsilon^{-1}(\Phi(C), \zeta)$ lies in this lattice as well, which can happen for only a discrete set of values of ϵ. □

Consider the orbit-type stratification

$$X = \bigcup_K X^K \tag{55}$$

where

$$X^K := G \times_{N(K)} X_K, \quad X_K = \{x \in X | G_x = K\}$$

as K ranges over all conjugacy classes of subgroups of G. The following is well-known:

Proposition 5.4.4. (a) *The orbit-type decomposition* (55) *is finite.*
 (b) *Each X^K is a smooth submanifold whose relative boundary in X is contained in the union of orbit-type strata $X_{K'}$ with* $\dim(K') > \dim(K)$.
 (c) *There is a unique open stratum X_0.*

Corollary 5.4.5. *There exists a discrete set of values $Z \subset (0, \infty)$ such that if $\epsilon \notin Z$, then every polystable vortex is stable.*

Proof. Since the orbit-type decomposition of X is finite, and each component of X^ζ is equal to the closure of some X_H, there are only finitely many possibilities for X^ζ. □

5.5. **Transversality.** We describe how to achieve transversality through suitable Hamiltonian perturbations in certain special situations. Later in Section 5.13.1, we will describe how to avoid transversality, again in this special case, by relaxing the vortex equations.

We say that X has *holomorphic orbit-type stratification* if the strata in (55) are holomorphic. This holds if G is abelian, but not in general, because the orbit-type decomposition for the action of $G_{\mathbb{C}}$ and for G may be different.

Our Hamiltonian perturbations will be supported on the principal orbit-type stratum X_0. Define a space of *admissible Hamiltonian perturbations*

$$\mathcal{H}_c(\Sigma, X) = \Omega^1(\Sigma, C_c^\infty(X_0)^G).$$

That is, $\mathcal{H}_c(\Sigma, X)$ is the space of one-forms with values in smooth, compactly supported, invariant functions on X_0. Let

$$\mathcal{V}(P, X) = \Omega^1(\Sigma, P(X))$$

denote the space of one-forms with values in vector fields. Associating a Hamiltonian vector field to any invariant function and taking the associated vector field on the associated fiber bundle defines a map

$$\mathcal{H}_c(\Sigma, X) \to \mathcal{V}(P, X), \quad K \mapsto V_K.$$

We say that (A, u) is a *K-perturbed* ϵ-vortex iff

$$F_A + \epsilon^{-1} \operatorname{Vol}_\Sigma u^*\Phi = 0, \qquad J_{A,u}(\mathrm{d}u + V_K)J_\Sigma = 0.$$

We denote by $D_{A,u,K}$ the perturbed linearized operator. If J is integrable, this is given by

$$D_{A,u,J,K}(a, \xi) = (\nabla_A \xi + V_K + a_X)^{0,1}.$$

We say that a pair (A, u) is *J, K-regular* if the operator

$$\tilde{D}_{A,u,J,K} = (\mathrm{d}_{A,u,\epsilon}, \mathrm{d}^*_{A,u,\epsilon}, D_{A,u,J,K}), \tag{56}$$

is surjective. Let $M(P, X, J, K)_\epsilon$ denote the moduli space of K-perturbed ϵ-vortices and $M^{\mathrm{reg}}(P, X, J, K)_\epsilon$ the regular locus. The proof of following is similar to that of Theorem 5.5.1 and will be omitted.

Theorem 5.5.1. *$M^{\mathrm{reg}}(P, X, J, K)^{1,p}_\epsilon$ is a smooth manifold with tangent space isomorphic to the kernel of $\tilde{D}_{A,u,J,K}$, and dimension of the component of degree $d \in H_2^G(X)$ is given by*

$$\dim(M^{\mathrm{reg}}(P, X, J, K)_\epsilon) = \operatorname{Ind}(\tilde{D}_{A,u,J,K}) = \dim(X) + 2(c_1^G(TX), d) - 2(g-1)\dim(G).$$

Furthermore, any element of $M^{\mathrm{reg}}(P, X, J, K)_\epsilon$ can be represented by a smooth pair (A, u). For any constants $c_1, c_2 > 0$, the subset of the moduli space with $\sup |\mathrm{d}_A u| < c_1, E(A, u) < c_2$ *is compact.*

We claim that for generic Hamiltonian perturbations, the moduli space $M(P, X, J, K)_\epsilon$ is smooth, under a suitable technical hypothesis which will always be fulfilled in the situations we consider. Let $P(X_0) \subset P(X)$ denote the associated fiber bundle. Let

$M^e(P,X,J,K)_\epsilon \subset M(P,X,J,K)_\epsilon$ denote the locus of pairs (A,u) such that the image of u contains at least one point in $P(X)^e$.

Theorem 5.5.2. *There exists a subset $\mathcal{H}_c(\Sigma,X)^{\mathrm{reg}} \subset \mathcal{H}_c(\Sigma,X)$ of Baire second category such if $K \in \mathcal{H}_c(\Sigma,X)^{\mathrm{reg}}$, then every element of $M^e(P,X,J,K)_\epsilon$ is regular.*

Proof. For any compact invariant subset C of the principal orbit-type stratum in X let $\mathcal{H}_c(\Sigma,X,C) \subset \mathcal{H}_c(\Sigma,X)$ be the Banach space of one-forms with values in functions with compact functions with support contained in a neighborhood of C with finite Floer-style norm

$$\|H\|_{\underline{\rho}} = \sum_k \rho_k^{-1} \sup(D^k H)$$

where $\underline{\rho} = (\rho_k)_{k=0}^\infty \in C^\infty(X)^G$ is a sequence of functions equal to 1 on C, and compactly supported in the principal orbit-type stratum, and approaching zero uniformly as $k \to \infty$. Consider the universal moduli space

$$M^{\mathrm{univ},C}(P,X,J)_\epsilon \subset \mathcal{H}_c(\Sigma,X) \times \mathcal{A}(P)_{1,p} \times \mathrm{Map}(\Sigma,P(X))_{1,p}.$$

consisting of triples (A,u,K) such that (A,u) is a K-perturbed ϵ-vortex, and the image of u meets $P(C)$. Let $\tilde{D}^{\mathrm{univ}}_{A,u,K}$ denote the universal linearized operator. Suppose that $\zeta = (\phi,\psi,\eta)$ lies in the cokernel of $\tilde{D}^{\mathrm{univ}}_{A,u,K}$. Since u meets $P(C)$, the stabilizer of (A,u) is trivial, which implies that $\phi = \psi = 0$. As usual we have $\tilde{D}^*_{A,u}\eta = 0$ and $(Y,\eta) = 0$ for every Hamiltonian perturbation K. The first equation implies in particular that $D_u^*\eta = 0$ and η pairs trivially with the vector field-valued one-forms $a_X^{0,1}$, that is, η vanishes on the generating vector fields for the action. Suppose that for some point z with $u(z) \in P(X)^e$, we have $\eta(z) \neq 0$. There exists $K \in \Omega^1(\Sigma, C^\infty(X)^G)$ such that $(Y(w),\eta(w)) \neq 0$. Using a cutoff function we find a Hamiltonian perturbation Y that pairs non-trivially with η, which is a contradiction. Hence η must vanish on the set of good points. Hence η vanishes everywhere, by unique continuation. Exhausting the principal orbit-type stratum by a countable sequence of compact sets completes the proof. □

The following describes one situation where we may always obtain transversality by Hamiltonian perturbation.

Definition 5.5.3. We say that $X = (X,\omega,\Phi)$ is *generic* iff for all $x \in X$ and rational $\xi \in \mathfrak{g}$ such that ξ fixes x, that is, $\xi_X(x) = 0$, the pairing $(\Phi(x),\xi)$ is non-zero.

To give some feeling for this definition, we state

Lemma 5.5.4. (a) *A Hamiltonian $U(1)$-action (X,ω,Φ) is generic iff $U(1)$ acts locally freely on the zero level set $\Phi^{-1}(0)$.*
(b) *Any Hamiltonian $U(1)$-action on a compact symplectic manifold (X,ω) can be made generic by a perturbing the moment map by a generic constant.*

Proof. (a) The condition in Definition (5.5.3) says that $0 \notin \Phi(X^{U(1)})$. (b) follows easily from (a). □

For actions of tori of higher dimension, the condition is never satisfied if we drop the rationality assumption on ξ, since there always exist fixed points of the maximal torus. We expect that one can always attain genericity, for example, by taking the product of the original manifold with a generalized flag variety G/T with generic Hamiltonian structure. However, we have not checked the details.

Theorem 5.5.5. *Suppose that $J \in \mathcal{J}(X)^G$ is integrable, $\pi_1([G,G]) = 1$, the orbit-type decomposition of X is holomorphic, and X is generic in the sense of Definition 5.5.3. Then there exists a finite set $Z \subset \mathbb{R}$ of vortex parameters such that if $\epsilon \notin Z$, then every vortex is stable and meets the principal orbit-type stratum.*

Proof. Suppose that (A,u) is a vortex such that the image of u is contained in $P(X - X_0)$. By assumption $P(X^i)$ is a holomorphic, so there exists a unique value of i and a dense open subset Σ' of Σ such that $u(\Sigma') \subset P(X^i)$. Let $g \in \mathcal{G}(P|\Sigma')$ be such that $g(z)u(z) \in P(X_i), z \in \Sigma'$. The section gu is holomorphic with respect to J_{gA} and extends to a section with values in $P(\overline{X}_i)$. Replacing (A,u) with the vortex (gA, gu), we consider the equation $F_A = -\epsilon^{-1}\operatorname{Vol}_\Sigma u^*P(\Phi)$. The projection $\pi_{\mathfrak{g}_i}(F_A)$ onto $P(\mathfrak{g}_i)$ is constant, since $P(\Phi)$ is constant on $P(\overline{X}_i)$. By Atiyah-Bott [2], $\pi_{\mathfrak{g}_i}(F_A)$ lies in a finite quotient of Λ_i, where Λ_i is the coweight lattice of G_i. Hence $\epsilon^{-1}\pi_{\mathfrak{g}_i}(u^*P(\Phi))$ lies in this lattice as well, which can happen for only a discrete set of values of ϵ, as long as the projection is non-zero which follows from the assumption in Definition 5.5.3. If (A,u) is not stable, a similar argument holds, using the discussion in Section 5.4. □

5.6. **Polystable nodal vortices.** To compactify $M_n(P,X,d)_\epsilon$ we allow bubbling in the fibers of $P(X)$.

Definition 5.6.1. A *nodal gauged marked (pseudoholomorphic) map* consists of a datum $(\underline{\Sigma}, A, \underline{u}, \underline{z})$ where

(a) $\underline{\Sigma}$ is a connected nodal curve consisting of $\Sigma = \Sigma_0$ as *principal component* together with a number of projective lines $\Sigma_1, \ldots, \Sigma_k$. We denote by $w_1^\pm, \ldots, w_k^\pm$ the nodes. For each $i = 1, \ldots, k$, we denote by $w_i^0 \in \Sigma$ the attaching point to the principal component.
(b) a gauged pseudoholomorphic $(A,u) \in \mathcal{A}(P,X)$ on Σ.
(c) for each non-principal component Σ_i, a pseudoholomorphic map $u_i : \Sigma_i \to P(X)_{w_i}$.
(d) $\underline{z} = (z_1, \ldots, z_n) \in \underline{\Sigma}$ are distinct, smooth points of $\underline{\Sigma}$.

A *nodal ϵ-vortex* is a stable nodal gauged map such that the principal component is an ϵ-vortex. We say that a nodal vortex $(\underline{\Sigma}, A, \underline{u}, \underline{z})$ is *polystable* if each sphere bubble Σ_i on which u_i is constant has at least three marked or singular points, and *stable* if it has finite automorphism group. Note that there is no condition for points on the principal component. In particular, nodal gauged maps with no markings can be polystable.

An *isomorphism* of nodal ϵ-vortices $(\underline{\Sigma}, A, \underline{u}, \underline{z}), (\underline{\Sigma}', A', \underline{u}', \underline{z}')$ consists of an automorphism of the domain, acting trivially on the principal component, and a corresponding automorphism of the principal bundle (in particular, a gauge transformation over the principal component) mapping $(A, \underline{u})$ to $(A', \underline{u}')$ and mapping the markings $\underline{z}$ to $\underline{z}'$. In particular, the markings on the principal component Σ must be equal.

The *combinatorial type* $\Gamma(\underline{\Sigma}, A, \underline{u}, \underline{z})$ of a gauged nodal map is a rooted graph whose vertices represent the components of $\underline{\Sigma}$, whose finite edges represent the nodes, semi-infinite edges represent the markings, and whose root vertex represents the principal component.

A *smooth deformation of constant type* of a nodal vortex $(A, \underline{\Sigma}, \underline{u}, \underline{z})$ consists of a family $(A_\bullet, \underline{\Sigma}_\bullet, \underline{u}_\bullet, \underline{z}_\bullet) \to S$ of nodal vortices together with an identification φ of the central fiber $(A_0, \underline{\Sigma}_0, \underline{u}_0, \underline{z}_0)$ with $(A, \underline{\Sigma}, \underline{u}, \underline{z})$.

Remark 5.6.2. The terminology *polystable* is borrowed from the vector bundle case. In that situation, a bundle is *stable* if it is flat and has only central automorphisms; polystable if it is a direct sum of stable bundles of the same slope. Any flat bundle is automatically polystable; a bundle is *semistable* if it is grade equivalent to a polystable bundle. In particular, the moduli space of stable bundles is definitely *not* compact, and we feel that the vortex terminology should include this fact as a special case. Of course, we could restrict to the case that polystability is the same as stability. But this would rule out any possibility of wall-crossing, since it precisely the failure of equality of stability with polystability which creates the phenomenon, and we would have nothing to write about in this paper.

As for the case of vortices with smooth domain, we will prove that a nodal vortex admits a versal deformation under a certain regularity hypothesis. For a stable ϵ-vortex $(A, \underline{u}) = (\hat{\Sigma}, (A, u), (v_1, \ldots, v_n))$, consisting of a principal component (A, u) and bubbles $v_j : \mathbb{P}^1 \to P(X)_{w_j}$, let $\tilde{D}^\epsilon_{A,\underline{u}}$ denote the operator

$$\begin{aligned} (57) \quad & \Omega^1(\Sigma, P(\mathfrak{g}))_{1,p} \oplus \Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X))_{1,p} \oplus \bigoplus \Omega^0(\Sigma, v_j^* TP(X)_{z_j})_{1,p} \\ & \to (\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g}))_{0,p} \oplus \Omega^{0,1}(\Sigma, u^* TX)_{0,p} \oplus \mathfrak{g} \oplus \bigoplus T_{v_j(w_j)} P(X)_{z_j} \end{aligned}$$

given by the operator $\tilde{D}_{A,u}$ on the principal component, the linearized Cauchy-Riemann operator $\tilde{D}_{v_j}$ on the bubbles, and the difference operator on the fibers over the nodes. We say that (A, u) is *regular* if $\tilde{D}^\epsilon_{A,\underline{u}}$ is surjective. The following is the analog of Theorem 5.3.1:

Theorem 5.6.3. *Any stable regular nodal vortex $(\underline{\Sigma}, A, \underline{u})$ has a smooth universal deformation of fixed type $(A_\bullet, \underline{\Sigma}_\bullet, \underline{u}_\bullet) \to S$, such that:*

(a) *Two fibers are isomorphic if and only if they are related by the action of* $\mathrm{Aut}(\underline{\Sigma}, A, \underline{u})$*;*
(b) *Possibly after shrinking S, the family is a universal deformation of constant type of any of its fibers.*

The proof is similar to the case of smooth domain, and left to the reader.

We denote by $M_{n,\Gamma}(P,X,J,d)_\epsilon$ of the moduli space of isomorphism classes of polystable ϵ-vortices of combinatorial type Γ of degree $d \in H_2^G(X,\mathbb{Z})$, and $M_\Gamma(P,X,d)_\epsilon^{\mathrm{reg}}$ the regular locus. The proof of the following is similar to that of Theorem 5.5.1:

Theorem 5.6.4. *The universal deformations provide* $M_\Gamma(P,X,d)_\epsilon^{\mathrm{reg}}$ *with the structure of a smooth orbifold of dimension* $\dim(X)+2(c_1^G(TX),d)-2\dim(G)-2\#\operatorname{Edge}(\Gamma)$. *Furthermore, any element can be represented by a smooth pair. For any constants* $c_1,c_2>0$*, the subset of the moduli space such that* $\sup|\mathrm{d}_A u|<c_1$ *and* $E(A,u)<c_2$ *is compact.*

Let $\overline{M}_n(P,X,J,d)_\epsilon$ denote the union over combinatorial types,

$$\overline{M}_n(P,X,J,d)_\epsilon = \bigcup_\Gamma M_\Gamma(P,X,J,d)_\epsilon .$$

We define a variant of the Gromov topology on $\overline{M}_n(P,X,J,d)_\epsilon$ as follows.

Definition 5.6.5. Suppose that (A_α,u_α) is a sequence of polystable nodal ϵ-vortices on Σ with values in X, and $(A,\underline{u})$ is a polystable nodal ϵ-vortex on $\underline{\Sigma}$ with values in X. We say that $(\Sigma,A_\alpha,u_\alpha)$ *Gromov converges* to $(\underline{\Sigma},A,\underline{u})$ if there exist a sequence g_α of gauge transformations such that

(a) A_α converges uniformly to A_∞;
(b) $u_{0,\alpha}$ converges to u_∞ uniformly on compact subsets of the complement of the bubbling set $Z_0 \subset \Sigma_0$;
(c) for every bubbling component Σ_i of $\underline{\Sigma}$, there exists a sequence $\epsilon_{i,\alpha}\to\infty$ and maps $\phi_{i,\alpha}: B_{\epsilon_{i,\alpha}(w_i)} \to \Sigma_i$ such that $u_\alpha\circ\phi_{i,\alpha}$ converges uniformly on compact subsets of the complement of the bubbling set $Z_i\subset\Sigma_i$ to $u_{i,\infty}$.
(d) for any bubble point w_j on Σ_i, the energy lost
$$m(w_j) := \lim_{\epsilon\to 0}\lim_{\alpha\to\infty} E(u_\alpha\circ\phi_{i,\alpha}; B_\epsilon(z_j))$$
is equal to the sum of the energies on the components of $\underline{u}$ attached to w_j.
(e) for any bubble point w_j on Σ_i, $\phi_{\alpha,i}^{-1}\circ\phi_{\alpha,j}$ converges to w_j uniformly on compact sets in a neighborhood of w_i.
(f) if z_i is contained in Σ_j, then $z_i = \lim_{\alpha\to\infty}\phi_{j,\alpha}^{-1}(z_{i,\alpha})$.

The extension to convergence of polystable ϵ-vortices is left to the reader. A subset C of $\overline{M}(\Sigma,X,J)$ is *Gromov closed* if any convergent sequence in C has limit point in C, and *Gromov open* if its complement is closed.

Remark 5.6.6. We do not require that A_α converge to A uniformly in all derivatives; in fact, we do not even know if we require this stronger definition, whether the moduli spaces $\overline{M}(P,X)_\epsilon$ are compact.

We will need the following alternative, less canonical description of Gromov convergence of polystable vortices. Suppose that $\underline{\Sigma}_0$ is a nodal marked curve,

$\underline{u}_0 : \underline{\Sigma}_0 \to P(X)$ a pseudoholomorphic section with respect to a connection A_0. Suppose that $\underline{\Sigma}_0$ is stable. Any nearby curve $\underline{\Sigma}_1$ is obtained by the gluing construction using local coordinates $z_1^\pm, \ldots, z_m^\pm$ and a set of gluing parameters $\epsilon_1, \ldots, \epsilon_m$, by identifying neighborhoods via $z_i^+ \sim \delta_i / z_i^-$. Let Σ_0' denote the smooth curve obtained by removing small balls around the nodes of $\underline{\Sigma}_0$. The gluing construction identifies Σ_0' with a subset of $\underline{\Sigma}_1$.

Definition 5.6.7. We say that $\underline{\Sigma}_1, A_1, \underline{u}_1$ is *ϵ-close* to $\underline{\Sigma}_0, A_0, \underline{u}_0$ if

$$\underline{u}_1 | \Sigma_0' = \exp_{\underline{u}_0 \Sigma_0'} \xi_1$$

for some $\xi_1 \in \Omega^0(\Sigma_0', (u_0')^* TX)$ and

(a) $\|\xi_1\|_{L^2} < \epsilon$;
(b) $\|A_1 - A_0\|_{L^2} < \epsilon$; and
(c) $E(\underline{u}_1 | \underline{\Sigma_1} - \Sigma_0') < E(\underline{u}_0 | \underline{\Sigma_0} - \Sigma_0') + \epsilon, \quad \sum_{i=1}^m |\delta_i|^2 < \epsilon.$

Let $B_\epsilon(\underline{\Sigma}_0, \underline{u}_0)$ denote the space of stable maps ϵ-close to $\underline{\Sigma}_0, \underline{u}_0)$, and define

$$\rho_\epsilon(\underline{\Sigma}_1, \underline{z}_1, A_1, \underline{u}_1) = \sum |t_j|^2 + \sum |\delta_i|^2 + \|A_1 - A_0\|^2 + \|\xi_0'\|^2 + |E(\underline{u}_1 | \underline{\Sigma_1} - \Sigma_0') - E(\underline{u}_0 | \underline{\Sigma_0} - \Sigma_0')|. \tag{58}$$

More generally, suppose that $\underline{\Sigma}_0$ has unstable components. We suppose, for simplicity, that there is a single unstable component Σ_i with a single special (necessarily nodal) point. By assumption, the map u_i is non-trivial on Σ_i. In particular, it's differential $\mathrm{d}u_i$ is non-zero almost everywhere. Choose points additional marked points $z_1', z_2' \in \Sigma_i$ at which $\mathrm{d}u_i$ is non-vanishing, and transverse (locally defined) hypersurfaces $H_1, H_2 \subset P(X)$. Requiring $u_i(z_i') \in H_i, i = 1, 2$ fixes the parametrization of the component Σ_i. The discussion is then the same as before.

Remark 5.6.8. The advantage of the functions ρ_ϵ over the ones defined on [32, p. 134] is that they are smooth on the regular locus in any stratum. Furthermore, $\rho_\epsilon^{-1}[0, \epsilon/2)$ is properly contained in the subset of ϵ-close maps.

Lemma 5.6.9. *Suppose that $(\underline{\Sigma}_\alpha, A_\alpha, \underline{u}_\alpha)$ and $(\underline{\Sigma}_1, A_1, \underline{u}_1)$ are ϵ-close to $(\underline{\Sigma}, A, \underline{u})$. Then $(\underline{\Sigma}_\alpha, A_\alpha, \underline{u}_\alpha)$ Gromov converges to $\underline{\Sigma}_1, \underline{u}_1$ if and only if*

(a) *the gluing parameters $\epsilon_{i,\alpha}$ for $\underline{\Sigma}_\alpha$ converge to the parameters $\epsilon_{1,\alpha}$ for $\underline{\Sigma}_1$,*
(b) *A_α converges uniformly to A, and*
(c) *ξ_α converges to ξ_1 uniformly on its domain.*

Proof. Similar to the proof of Lemma 4.1.5, using the exponential decay and energy quantization results of Section 5.7. □

As in Proposition 4.1.6, the Gromov open sets form a topology for which any convergent sequence is Gromov convergent. Furthermore, any convergent sequence has a unique limit. The following is essentially a result of Mundet [34]:

Theorem 5.6.10. *For any constant* $C > 0$ *and* $J \in \mathcal{J}(X)^G$, *the union of components* $\overline{M}_n(P, X, J, d)_\epsilon$ *for which* $([\omega_G], d) < C$ *is a compact, Hausdorff space.*

Proof. The argument is a combination of Uhlenbeck and Gromov compactness. The bound on the degree implies a bound on the energy, by (49). Suppose that (A_α, u_α) is a sequence of symplectic ϵ-vortices with bounded energy (as for example, a sequence of ϵ-vortices with fixed degree, by the energy-action relation.) Since X is compact, the curvature F_{A_α} is pointwise bounded. By Uhlenbeck compactness, see e.g. [42] there exists a connection A_∞ such that after gauge transformation $A_\alpha \to A_\infty$ weakly in $W^{1,p}$ and strongly in C^0, for any $p \geq 2$.

If $A_\alpha \to A_\infty$ uniformly in all derivatives, the theorem would be a consequence of Theorem 5.2.1. If $\sup |\mathrm{d}u_\alpha|$ is bounded then the bootstrapping argument in [8, Section 3.2] shows that A_α, u_α converges in all derivatives to a limiting vortex A_∞, u_∞. More generally, suppose that $|\mathrm{d}u_\alpha(z_\alpha)| \to \infty$ for some sequence z_α. By compactness of Σ, we may assume after passing to a subsequence $z_\alpha \to z$ for some $z \in \Sigma$. We may assume, after gauge transform, that the connections A_α are in local Coulomb gauge with respect to some trivialization of P near z, see Section 5.8 below. After passing to a subsequence and rescaling by a sequence c_α, the sequence $u_\alpha(c_\alpha \cdot)$ converges uniformly in all derivatives to a pseudoholomorphic sphere $v : \mathbb{P}^1 \to P(X)_z$. Energy quantization for pseudoholomorphic spheres in $P(X)$ implies there exists $\hbar > 0$ such that any pseudoholomorphic sphere has energy at least $\hbar$. It follows from the energy bound that hence bubbles form at a finite set of fibers of $P(X)$.

Suppose that bubbling occurs at a point $z \in \Sigma$, and $E > \hbar$ is the total energy lost at z. To construct the first bubble in the bubble tree, one considers the sequence of maps v_α given by rescaling at z by δ_α, defined by $E(v_\alpha) = E - \hbar/2$. By Proposition 5.8.1, v_α converges away from a finite set of points uniformly in all derivatives to a pseudoholomorphic curve $v : \mathbb{P}^1 \to P(X)_z$, and if v is constant then there is bubbling at at least two points. The same proposition shows that the bubble v connects to the principal component; repeated application of soft rescaling constructs the limiting sequence, as in the case of pseudoholomorphic maps in [32, Chapter 5]. That $\overline{M}_n(\Sigma, X)_\epsilon$ is Hausdorff is another application of [32, Lemma 5.6.6]. □

Remark 5.6.11. Mundet [34] uses rather the results of [26], which requires only C^0 convergence of the sequence A_α for exponential decay and bubbles connect.

A *framed stable* ϵ*-vortex* consists of a stable vortex together with framings at the attaching points of the bubbles

$$\phi_i : P_{\hat{z}_i} \to G, \quad i = 1, \ldots, n.$$

The gauge group $\mathcal{G}(P)$ acts naturally on the space of framings, by evaluation at the attaching points. The definition of isomorphism of framed nodal ϵ-vortices is similar to the unframed case in Definition 5.6.1. Let $\overline{M}_n^{\mathrm{fr}}(P, X)_\epsilon$ denote the coarse moduli space of gauge equivalence classes of stable framed ϵ-vortices. Since G and

$\overline{M}_n(P,X,d)$ are compact, so is $\overline{M}_n^{\rm fr}(P,X,d)$, for any degree $d \in H_2^G(X,\mathbb{Z})$. The group G^n acts on $\overline{M}_n^{\rm fr}(P,X)_\epsilon$ by changing the framings at the attaching points. Let

$$\pi : \overline{M}_n^{\rm fr}(P,X)_\epsilon \to \overline{M}_n(P,X)_\epsilon$$

denote the map forgetting the framings.

5.7. **Exponential decay.** This section and that following provide the technical details necessary for the compactness result Theorem 5.6.10. Let Σ be a not necessarily compact Riemann surface; we have in mind a long cylinder.

Definition 5.7.1. Let $f \in C^\infty(\Sigma)$. An *f-vortex* is a pair (A,u) solving the equations

$$F_A + f \operatorname{Vol} u^*\Phi = 0, \quad \overline{\partial}_A u = 0.$$

We wish to show that if A and f are sufficiently small and A is in Coulomb gauge with respect to a smooth connection A_0, then all derivatives of A, u on a compact set are controlled by e.g. the C^0 norm of A and the C^1 norm of u on a larger open set.

Lemma 5.7.2 (Bootstrapping for Vortices in Coulomb Gauge)**.** *Let X be a compact Hamiltonian G-manifold equipped with an invariant almost complex structure J. Let $U \subset \mathbb{R}^2$ be an open set and $K \subset U$ a compact subset. For all integers $k > 0, p > 2$, there exists constants c_1, c_2, c_3 such that for all f-vortices (A,u) on U, if*

$$\sup |A| < c_1, \quad {\rm d}_{A_0}^* A = 0 \quad \sup |{\rm d}_A u| < c_2, \quad \|f\|_{k,p} < c_3, \quad \|A\|_{W^{k,p}(U)} \le c_4$$

then there exists a constant $c = c(c_1,c_2,c_3,c_4,K,U,k,p)$ such that

$$\|A,u\|_{W^{k+1,p}(K)} \le c\|A,u\|_{W^{k,p}(U)}.$$

In addition, for the case $k=0$ we have

$$\|A\|_{W^{1,p}(K)} \le c\|A\|_{W^{0,p}(U)} + c\|u\|_{W^{1,p}(U)}.$$

Proof. Consider first the case $X = \mathbb{R}^{2n}$. The elliptic estimate for the pseudoholomorphic section proved in [8, Lemma 3.3] gives for $k > 0$,

$$\|u\|_{k+1,p} \le c\|u\|_{k,p} + \|u\|_{1,\infty}.$$

We carry out a similar estimate for the connection A, for any $k \ge 0$. Let $\rho : \mathbb{R} \times S^1$ be a cutoff function with compact support, equal to 1 on K. The connection ρA satisfies

$$\begin{aligned} {\rm d}(\rho A) &= ({\rm d}\rho)A + \rho {\rm d}A \\ &= ({\rm d}\rho)A - \rho[A,A]/2 - \rho f \operatorname{Vol} u^*\Phi \end{aligned}$$

and

$${\rm d}^*(\rho A) = *({\rm d}\rho) \wedge *A + \rho * [A_0, *A].$$

It follows that for $\|A\|_{k,p} \le 1$,

$$\|F_{\rho A} + f \operatorname{Vol}(\rho u)^*\Phi\|_{k,p}\|_{k+1,p} \le c\|A\|_{k,p}$$

and

$$\|\mathrm{d}^*(\rho A)\|_{k,p} \leq c\|A\|_{k,p}.$$

By the elliptic estimate for the $\mathrm{d} + \mathrm{d}^*$,

$$\|\rho A\|_{k+1,p} \leq c(\|A\|_{k,p} + \|u\|_{k,p})$$

To prove the claim for arbitrary X, choose r sufficiently small so that on any ball $B_r(z)$, the image of u lies in a Darboux chart on X, and cover X by a finite number of Darboux charts and Σ by a finite number of balls so that the image of each ball lies in a Darboux chart. Summing the estimates for the restriction of u to each ball proves the claim. □

Lemma 5.7.3 (Uniform mean value inequality)**.** *Let Σ be a Riemann surface (possibly with boundary), X a compact symplectic manifold, $P \to \Sigma$ the trivial G-bundle, $P(X)$ the associated fiber bundle. Fix $c > 0$, and for any connection $A \in \Omega^1(\Sigma, P(\mathfrak{g}))$. For any $c_1 > 0$, there exist constant $\delta > 0$ such that if $\|A\|_{C_2} < c_1$ then*

$$E_A(u|B_r(z)) \leq \delta \implies |\mathrm{d}_A u(z)|^2 \leq (8/\pi r^2) E_A(u|B_r(z)).$$

.

Proof. The computation on [32, p.90] uses only an estimate on the second derivative of the almost complex structure (hence the C^2-norm of the connection) and the metric (determined by the choice of c above and the C^0-norm of the connection. □

Let $z \in \Sigma$ be a point, and fix a trivialization of P in a neighborhood of z. Let $\mathcal{A}(\delta, \epsilon)$ denote a small annulus around z, conformally equivalent to the cylinder $[\log(\delta), \log(\epsilon)] \times S^1$.

Lemma 5.7.4 (Uniform exponential decay)**.** *Let X be a compact Hamiltonian G-manifold equipped with an invariant almost complex structure J. There exists constants $\kappa, c_0, c_1, c_2, c_3, c_4$ such that for all $\delta, \epsilon > 0$, all $A \in \Omega^1(\mathcal{A}(\delta, \epsilon), \mathfrak{g})$ and all $\underline{u} : \mathcal{A}(\delta, \epsilon) \to X$ pseudoholomorphic with respect to J_A, if*

$$\epsilon/\delta > c_0, \quad \|A\|_{C_2} < c_1, \quad E(u; \mathcal{A}(\delta, \epsilon)) < c_2$$

then

(a) *the twisted energy*

$$E_A(u; T) := E_A(u|A(e^T\delta, e^{-T}\epsilon))$$

satisfies an exponential decay estimate

$$E_A(u; T) < c_3 e^{-\kappa T} E_A(u; 0).$$

(b) *The distance between $u(z)$ and $u(z')$ for $z, z' \in A(e^T\delta, e^{-T}\epsilon)$ satisfies the exponential decay estimate*

$$\sup_{z,z'} \mathrm{dist}(u(z), u(z')) \leq c_4 e^{-\kappa T} E_A(u; 0)^{1/2}$$

Furthermore, if (A, u) *satisfies*

$$F_A + f \operatorname{Vol} u^*\Phi = 0 \quad \mathrm{d}^*A = 0$$

then there exist constants c_5, c_6 *such that if* $\|f\|_{C_2} < c_5$ *and* $\|A\|_{C_0} < c_6$*, then the same conclusions hold.*

We do *not* claim that the constant κ can be chosen arbitrarily close to 1 as in [32].

Proof. We may assume that the energy and C^2 norm of A are sufficiently small so that the mean value theorem applies, so that the twisted action is well-defined. Since u is J_A-holomorphic,

$$\begin{aligned} E_A(u;T) &\leq D_A(u;T) - D_A(u;-T) \\ &\leq c_7(l_A(T)^2 - l_A(-T)^2) \\ &\leq c_8 \int_{S^1} |\partial_t u(T,\theta)|_A^2 \mathrm{d}\theta + c_2 \int_{S^1} |\partial_t u(-T,\theta)|_A^2 \mathrm{d}\theta \\ &\leq -c_8 \frac{\mathrm{d}}{\mathrm{d}T} E_A(u;T) \end{aligned}$$

for some constants c_7, c_8 depending on the previous constants. Thus

$$\frac{d}{dt} E_A(u;T) \leq -(1/c_8) E_A(u;T)$$

which implies

$$E_A(u;T) \leq c_3 \exp(-\kappa T) E_A(u;0)$$

where $\kappa = 1/c_8$ which proves the third claim. Applying the mean value inequality completes the proof of the second claim. The third claim uses 5.7.2 to obtain a C^2 bound on A from the C^0 bound on A and the bound on $\mathrm{d}_A u$. □

5.8. **Bubbles connect.** Let (A_α, u_α) be a sequence of vortices, with A_α converging weakly in $W^{1,p}$ and strongly in C^0 to a limiting connection A_∞. Let $z_0 \in \Sigma$ be a point at which bubbling occurs, and m_0 the energy lost at the point z. That is,

$$m_0 = \lim_{\epsilon \to 0} \lim_{\alpha \to \infty} E(A_\alpha, u_\alpha | B_\epsilon(z)).$$

After passing to a subsequence, we may assume that the limit m_0 exists. We also have

$$m_0 = \lim_{\epsilon \to 0} \lim_{\alpha \to \infty} E_{A_\alpha}(u_\alpha | B_\epsilon(z)).$$

since the energy of the connection on $B_\epsilon(z)$ approaches zero. Fix a trivialization of P in a neighborhood U of z_0, small enough that bubbling occurs at no other point in z_0. Given a local coordinate z near z_0, suppose that $|\mathrm{d}_{A_\alpha} u_\alpha|$ takes its maximum at $z = z_\alpha$. After the re-parametrizations $z \mapsto z - z_\alpha$ we may assume that $z_\alpha = 0$ for all α, see [32, p.101]. Choose δ_α so that $E_{A_\alpha}(u_\alpha | B_{\delta_\alpha}(z)) = m(z) - c$ where $c \leq \min(c_2, \hbar/2)$, and c_2 is the constant of Lemma 5.7.4. Let $v_\alpha(z) = u_\alpha(\underline{\delta}_\alpha z)$. We denote by $\hat{A}_\alpha$ the re-scaled connection on the re-scaled bundle $\hat{P}$, obtained from P

by pull-back under v_α. The pair $v_\alpha, \hat{A}_\alpha$ is a vortex for the pull-back of the volume form, which tends to zero in all derivatives as $\alpha \to \infty$.

Proposition 5.8.1 (Bubbles connect). (a) *The sequence (A_α, u_α) converges in all derivatives on compact subsets of $U - \{z_0\}$ to a limiting vortex (A_∞, u_∞).*

(b) *There exists a finite subset $Z = \{z_1, \ldots, z_l\}$ of $\mathbb{C}$ such that the sequence v_α converges uniformly in all derivatives on $\mathbb{C} - Z$ to a pseudoholomorphic map $v_\infty : \mathbb{C} \to P(X)_{z_0}$ and the limits*

$$m_j := \lim_{\epsilon \to 0} \lim_{\alpha} E_{\hat{A}_\alpha}(v_\alpha; B_\epsilon(z_j))$$

are greater than the energy quantization constant $\hbar$ for $P(X)_{z_0}$ for $j = 1, \ldots, l$.

(c) *The total energy lost at z_0 is related to the energy of v by*

$$m_0 := E(v) + \sum_{s_j \in Z_1} m_j.$$

(d) *We have $u(z) = v(\infty)$ in $P(X)_{z_0}$.*

(e) *If v is constant then $l \geq 2$.*

Proof. (a) After a sequence of gauge transformations, we may assume that A_α is in Coulomb gauge with respect to the trivial connection in a neighborhood of z_0. The claim now follows from convergence for sequences of vortices in Coulomb gauge with bounded first derivative. (b) follows by a similar argument, since the re-scaled connections are also in Coulomb gauge. Proof of (c): As in [32, p. 103], divide the annulus $A(\underline{\delta}_\alpha, \epsilon_\alpha)$ into three pieces,

$$A(\underline{\delta}_\alpha, \epsilon_\alpha) = A(\underline{\delta}_\alpha, e^T \delta_\alpha) \cup A(e^T \delta_\alpha, e^{-T} \epsilon_\alpha) \cup A(e^{-T} \epsilon_\alpha).$$

Consider the sequence $w_\alpha : B_{1/\epsilon_\alpha} \to X, \tilde{A}_\alpha \in \Omega^1(B_{1/\epsilon_\alpha}, \mathfrak{g})$ defined by

$$w_\alpha(z) = u_\alpha(\epsilon_\alpha z), \quad \tilde{A}_\alpha(z) = A_\alpha(\epsilon_\alpha z).$$

Since the energy is conformally invariant,

$$\begin{aligned} \lim_{\alpha \to \infty} E_{\tilde{A}_\alpha}(w_\alpha, A(e^{-T}, e^T)) &= \lim_{\alpha \to \infty} E_{A_\alpha}(u_\alpha, A(e^{-T}\epsilon_\alpha, e^T \epsilon_\alpha)) \\ &\leq \lim_{\alpha \to \infty} E_{A_\alpha}(u_\alpha, A(\underline{\delta}_\alpha, e^T \epsilon_\alpha)) \\ &\leq \hbar/2. \end{aligned}$$

Since the connections A_α are in Coulomb gauge near z_0, the connections $\tilde{A}_\alpha$ are in Coulomb gauge as well. Since there is not enough energy for bubbling, by passing to a subsequence, we obtain that $w_\alpha, \tilde{A}_\alpha$ converge uniformly on compact subsets of $\mathbb{C} - \{0\}$ to a pseudoholomorphic map $w_\infty : \mathbb{C} \to P(X)_{z_0}$, and the trivial connection. Hence, for every $T > 0$,

$$\lim_{\alpha \to \infty} E_{A_\alpha}(u_\alpha; A(e_T \epsilon_\alpha, \epsilon_\alpha)) = 0.$$

By the exponential decay estimate (5.7.4) there exists a constant $c_4 > 0$ such that

$$\lim_{\alpha \to \infty} E_{A_\alpha}(u_\alpha; A(e^T \delta_\alpha, e^{-T} \epsilon_\alpha)) \leq c_4 e^{-T} c.$$

Hence

$$\lim_{\alpha\to\infty} E(u_\alpha; A(\underline{\delta}_\alpha, e^T\delta_\alpha)) \geq (1 - c_4 e^{-T})c/2.$$

This implies that

$$\lim_{T\to\infty}\lim_{\alpha\to\infty} E_{A_\alpha}(u_\alpha, A(\underline{\delta}_\alpha, e^T\delta_\alpha)) = c/2.$$

Hence, as in [32, p. 102],

$$\lim_{R\to\infty}\lim_{\alpha\to\infty} E_{A_\alpha}(u_\alpha; B_{R\delta_\alpha}) = m_0. \tag{59}$$

By the definition of δ_α,

$$E_{\hat{A}_\alpha}(v_\alpha; B_1) \geq m_0 - \hbar/2.$$

Hence all the bubble points $z_1, \ldots, z_l$ lie in a ball of radius 1 around $z = 0$. Hence for any $s. > 1$,

$$\begin{aligned} m(z) &= \lim_{R\to\infty}\lim_{\alpha\to\infty} E_{\hat{A}_\alpha}(v_\alpha; B_R) \\ &= \lim_{R\to\infty}\lim_{\alpha\to\infty} E_{\hat{A}_\alpha}(v_\alpha; B_R - B_s) + \lim_{\alpha\to\infty} E_{\hat{A}_\alpha}(v_\alpha; B_s) \\ &= \lim_{R\to\infty} E(v; B_r - B_s) + \lim_{\alpha\to\infty} E_{\hat{A}_\alpha}(v_\alpha; B_s) \\ &= E_{\hat{A}_\alpha}(v; \mathbb{C} - B_s) + \lim_{\epsilon\to 0}\lim_{\alpha\to\infty} E_{\hat{A}_\alpha}(v_\alpha; B_s - \cup_{j=1}^l B_\epsilon(z_j)) + \sum m_j \\ &= E(v) + \sum_{j=1}^l m_j \end{aligned}$$

Proof of (d): By definition, there exists a sequence $\epsilon_\alpha \to 0$ such that

$$\lim_{\alpha\to\infty} E_{A_\alpha}(u_\alpha | B_{\epsilon_\alpha}(z)) \to m(z).$$

Map the annulus $\mathcal{A}(\delta_\alpha, \epsilon_\alpha)$ onto the annulus $[\log(\underline{\delta}_\alpha), \log(\epsilon_\alpha)] \times S^1$. By the uniform mean value inequality (5.7.3), on the subset $[\log(\underline{\delta}_\alpha)+1, \log(\epsilon_\alpha)-1] \times S^1$ the twisted derivatives $\mathrm{d}_{A_\alpha} u_\alpha$ are uniformly bounded. The exponential decay lemma 5.7.4 shows exponential decay of the energy on this region. Since the connections A_α are in Coulomb gauge, we also have exponential decay of the first derivative, and so

$$\mathrm{dist}(v_\infty(\infty), u_\infty(0)) \leq \lim_{\alpha\to\infty} \mathrm{dist}(u_\alpha(\epsilon_\alpha), u_\alpha(\underline{\delta}_\alpha)) = 0$$

by the exponential decay estimate in Lemma 5.7.4.

Proof of (e): If v is constant and $1 < R_1 < R_2$ then

$$\lim_{\alpha\to\infty} E_{\hat{A}_\alpha}(v_\alpha, A(R_1, R_2)) = E(v; A(R_1, R_2)) = 0.$$

Hence $\lim_{\alpha\to\infty} E(v_\alpha; B_R)$ is independent of $R > 1$. By (59), this implies that

$$\lim_{\alpha\to\infty} E_{\hat{A}_\alpha}(v_\alpha; B_R) = m_0$$

for every $R > 1$. Since $E_{\hat{A}_\alpha}(v_\alpha) \geq m_0 - \hbar/2$, this implies that bubbling occurs on the boundary of the unit disk. Hence there is a point z_j with $|z_j| = 1$. Since bubbling also occurs at $z = 0$, this proves $l \geq 2$. □

5.9. **The gluing construction.** In this section we prove the analogs of the results in Section 5.9, in particular,

Theorem 5.9.1. *Any regular stable marked nodal vortex* $(\underline{\Sigma}, A, \underline{u}, \underline{z})$ *admits partially smooth universal deformation.*

The proof depends on a gluing theorem which combines the gluing construction for pseudoholomorphic curves as in [32] and one for connections as in [12]. We explain the construction for a single bubble only. Technically the gluing theorem is complicated by the fact that one naturally wants to use $W^{k,2}$ for connections and $W^{1,p}, p > 2$ for pseudoholomorphic maps, which are the standard frameworks for the individual problems. (One could imagine instead using $W^{k,2}$ for $k \geq 2$ for pseudoholomorphic maps, but this would require a substantially more complicated gluing procedure than the one in [32].)

The construction depends on the choice of *gluing parameter* δ, determining the glued curve, as well as a *radius parameter* κ which describes the radius of the annulus on which the gluing is performed. We assume that κ is large but $|\delta|^{-1/2}\kappa^{-3} \gg 0$. We divide the region near the node $|z| \leq |\delta|^{1/2}\kappa^3$ according to powers of κ,

$$\mathcal{A}_i(\kappa, \delta) = \{z | |\delta|^{1/2}\kappa^i \leq |z| \leq |\delta|^{1/2}\kappa^{i+1}\}.$$

The region $\mathcal{A}_3$ will be used to make the connection go to zero. The regions $\mathcal{A}_{-2}, \mathcal{A}_2$ will be used to make the sections equal to their value at the node. The regions $\mathcal{A}_{-1}, \mathcal{A}_0, \mathcal{A}_1$ will be used for the construction of the uniformly bounded right inverse. Let (A, u_+) be a vortex on Σ and $u_- : \mathbb{P}^1 \to P(X)_z$ a sphere bubble in the fiber over $w_+ \in \Sigma$ with $x = u_-(\infty) = u_+(w_+)$. We define an *approximate vortex* $(A(\delta, \kappa), u(\delta, \kappa))$ on $\Sigma(\delta)$ as follows. The construction depends on the choice of two cutoff functions, one for the connection which must be carefully chosen, and one for the section which may be less carefully chosen.

Define a *preglued* connection and section as follows. We may assume, after gauge transformation, that the connection A vanishes at the node $z \in \Sigma$. Let A_0 denote the trivial connection over the product bundle $\mathbb{C} \times G$. For κ sufficiently small so that $|\delta|^{-1/2}\kappa^{-3} \gg 0$, define the *glued connection*

$$A(\kappa, \delta) = \beta_1(|\delta|^{1/2}\kappa^3 z)(A - A_0) + A_0.$$

To glue the sections, consider the exponential map

$$T_x P(X) \to P(X), \quad \zeta(z) \mapsto \exp_x(\zeta(z))$$

defined by a metric on $P(X)$. For r sufficiently small, define

$$\zeta_\pm : B_r(0) \to T_x P(X), \quad u_\pm(z) = \exp_x(\zeta_\pm(z)).$$

Define the pre-glued section on the gluing region by

$$u(\kappa, \delta; z) = \exp_x(\beta_1(z/|\delta|^{1/2}\kappa^2)\zeta_+(z) + \beta_1(z/|\delta|^{1/2}\kappa^{-2}))\zeta_-(\delta z)), \quad |\delta|^{1/2}\kappa^{-2} \leq |z| \leq |\delta|^{1/2}\kappa^2. \tag{60}$$

and elsewhere let $u(\kappa, \delta; z) = \underline{u}(z)$, using the identification of $\underline{\Sigma}^\delta$ with $\underline{\Sigma}$ away from the gluing region.

We will show that $(A, u) = (A(\kappa, \delta), u(\kappa, \delta))$ is an approximate solution to the vortex equations, that is, $\mathcal{F}^\epsilon(A, u)$ is small in a suitable Sobolev norm. Let g^δ denote the C^0 metric on the glued surface in (27). Equip TX with the trivialization near x given by the exponential map

$$\exp_x : T_x X \to X$$

and let ∇_{TX} be a connection on TX induced from this trivialization in a neighborhood of x. Using this connection and metric, define Sobolev spaces

$$\Omega(\Sigma^\delta, P^\delta(\mathfrak{g}))_{k,p}, \qquad \Omega^{0,1}(\Sigma^\delta, u^{\delta,*} T^{\text{vert}} P(X))_{k,p}$$

are well-defined for $p \geq 2$ and $k \in [-1, 1]$. Sobolev spaces Define

$$\mathcal{H}_\delta := (\Omega^0 \oplus \Omega^2)(\Sigma^\delta, P^\delta(\mathfrak{g}))_{-\frac{1}{2},2} \oplus \Omega^{0,1}(\Sigma^\delta, u^{\delta,*} T^{\text{vert}} P(X))_{0,3,\delta}$$

with norm

$$\|(\phi, \psi, \eta)\|^2_\delta = \|\phi\|^2_{0,3,\delta} + \|\psi\|^2_{-\frac{1}{2},2} + \|\eta\|^2_{0,3,\delta} \tag{61}$$

and

$$\mathcal{I}_\delta := \Omega^1(\Sigma^\delta, P^\delta(\mathfrak{g}))_{1/2,2,\delta} \oplus \Omega^0(\Sigma^\delta, u^{\delta,*} T^{\text{vert}} P(X))_{1,3,\delta}$$

with norm

$$\|(a, \xi)\|^2_{\underline{\delta}} = \|a\|^2_{\frac{1}{2},2} + \|\eta\|^2_{1,3,\delta}.$$

The fractional order Sobolev spaces are defined by complex or real interpolation, see Section 4.2. The choice of norms can be explained as follows.

(a) We wish to glue a connection on the principal component with trivial connection on the bubbles. There is no "matching condition" at the nodes as there is for pseudoholomorphic maps, and we need the "carefully chosen" cutoff function used in the construction of the approximate right inverse has small norm. To achieve this we take a Sobolev norm on the connection $W^{k,2}$ with $k \leq 1$.
(b) For pseudoholomorphic maps the space of $W^{1,p}$ maps is not available for $p > 2$ only, since we wish the maps to be continuous. The carefully chosen second cutoff function does not satisfy a uniform $W^{1,p}$ bound. Ideally we would like to work with $W^{1,2}$ for connections and $W^{1,p}$ for sections.
(c) Unfortunately, Sobolev multiplication $W^{1,2} \times W^{0,2} \to W^{0,2}$ for the connection times the cutoff function that would be necessary to obtain a uniformly bounded right inverse fails in this borderline case. This forces us to work with connections worse than $W^{1,2}$.
(d) We cannot work with $W^{0,2}$ connections, which is the critical norm in two dimensions, since Sobolev multiplication $W^{0,2} \times W^{0,2} \to W^{-1,2}$ appearing in the definition of curvature fails in this borderline case as well.

(e) We choose to work with a fractional norm, in the case $p = 2$ the various interpolation methods give the same result. We choose $W^{\frac{1}{2},2}$ for convenience, so that curvature takes values in $W^{-\frac{1}{2},2}$. As explained in the previous section, these spaces are well-defined for continuous metric on the surface.

(f) Unfortunately connections in this Sobolev class are not continuous. However, this poses no problem as the space of connections is linear, and one has continuous Sobolev multiplication $W^{\frac{1}{2},2} \times W^{1,3} \to W_{0,3}$ needed for the definition of the twisted Cauchy-Riemann operator (47).

For the use of fractional Sobolev spaces in a related context see [36].

Lemma 5.9.2 (Uniform error bound)**.** *There exists a constant $c > 0$ such that if*

$$|\delta| < c, \quad \kappa > 1/c, \quad |\delta|^2\kappa < c$$

then the pair $(A(\kappa,\delta), u(\kappa,\delta)) \in \mathcal{A}(P) \times \Gamma(P(X))$ satisfies

$$\|(F_{A(\kappa,\delta)} + \epsilon^{-1}(u(\kappa,\delta))^*\Phi, \overline{\partial}_{A(\kappa,\delta)}u(\kappa,\delta))\|_\delta \le c_0 f(\delta,\kappa) \tag{62}$$

where

$$f(\delta,\kappa) = (|\delta|\kappa^2)^{1/3}. \tag{63}$$

Proof. The expression $\overline{\partial}_{A(\kappa,\delta)}u(\kappa,\delta)$ can be estimated by terms of two types: those involving derivatives of the cutoff functions and those involving derivatives of $\zeta^\pm$. The derivative of $\exp_{x_j}$ is approximately the identity near the node. The derivative of $\rho(z/|\delta|^{1/2}\kappa^2)$ is on the order of $1/\kappa^2|\delta|^{1/2}$, while the norm of $\zeta^\pm$ is on the order of $\kappa^4|\delta|$ on the gluing region. Hence the term involving the derivative of ρ is bounded and supported on region of area $|\delta|\kappa^2$, hence the $0,3$ norm is bounded by $(|\delta|\kappa^2)^{1/3}$. The derivatives of $\zeta^\pm$ are also uniformly bounded, and the area bound as before gives the required estimate.

The expression $F_{A^\delta} + \epsilon^{-1}u^{\delta,*}\Phi$ can be expressed as a sum of terms of two types: those involving difference of curvatures and those involving differences of pull-backs of Φ. For the first type, it follows by the same reasoning as before and the assumption $A(x) = 0$ that the term involving the derivative of the cutoff function has $W^{0,2}$-norm bounded by $c(|\delta|\kappa^2)^{1/2}$. A easier estimate bounds the difference arising from the quadratic term in the curvature and the difference arising from the difference in moment maps. □

Lemma 5.9.3. *The constants in the Sobolev embeddings are independent of δ.*

Proof. Σ^δ can be written as a union of manifolds with boundary, each of which is a diffeomorphic to a domain in $\mathbb{R}^2$ satisfying the cone condition [1, Chapter 4] with dimension independent of δ, and metric uniformly comparable to the flat metric. By the discussion in Section 4.2, it suffices to show that the difference in connections is uniformly bounded. This holds by the assumption that the connection on TX near $u(z)$ is induced from a trivialization. □

In preparation for the construction of the uniformly bounded right inverse we introduce an intermediate family $(A_0(\kappa,\delta), u_0(\kappa,\delta))$ of gauged pseudoholomorphic maps on the nodal curve $\underline{\Sigma} = (\Sigma, \mathbb{P}^1)$ by

$$A_0(\kappa,\delta;z) = \begin{cases} A(\kappa;\delta;z), & \text{if } |z| \geq |\delta|^{1/2}\kappa^2, \\ A^0(0), & \text{if } |z| \leq |\delta|^{1/2}\kappa^2 \end{cases} \tag{64}$$

$$u_{+,0}(\kappa,\delta;z) = \begin{cases} u(\kappa;\delta;z), & \text{if } |z| \geq |\delta|^{1/2}, \\ u(0), & \text{if } |z| \leq |\delta|^{1/2} \end{cases} \tag{65}$$

$$u_{-,0}(\kappa,\delta;z) = \begin{cases} u(\kappa,\delta;z\delta), & \text{if } |z| \geq |\delta|^{-1/2}, \\ x, & \text{if } |z| \leq |\delta|^{-1/2}. \end{cases} \tag{66}$$

We identify the bundle $\Omega^0(\underline{u}_0(\delta,\kappa)^* T^{\mathrm{vert}} P(X))$ with $\Omega^0(\underline{u}^* T^{\mathrm{vert}} P(X))$ by parallel transport. Recall from (53) the linearized operator associated to a vortex.

Lemma 5.9.4. *The operator $\tilde{D}_{A_0(\kappa,\delta),\underline{u}_0(\kappa,\delta)}$ converges in the operator norm to $\tilde{D}_{(A,\underline{u})}$ as $\delta \to 0, \kappa \to \infty, \delta/\kappa^2 \to 0$.*

Proof. The section $u_0(\kappa,\delta)$ converges in the $W^{1,3}$ norm to u as $\kappa^2/|\delta|^{1/2} \to \infty$, and $A_0(\kappa,\delta)$ converges in $W^{0,2}$ to A. It follows that $\mathrm{d}_{A_0(\kappa,\delta)}$ converges to d_A, and $\mathrm{Vol}_\Sigma\, L.u_{+,0}(\delta,\kappa)^*\Phi$ converges to $\mathrm{Vol}_\Sigma\, L.u_+^*\Phi$. Hence $\mathrm{d}_{A_0(\kappa,\delta),u(\kappa,\delta),\epsilon}$ converges to $\mathrm{d}_{A,u,\epsilon}$, and similarly for the $\mathrm{d}^*_{A,u,\epsilon}$, and $D_{A_0(\kappa,\delta),u_0(\kappa,\delta)}$ converges to $D_{A,u}$. □

Proposition 5.9.5 (Uniformly bounded right inverse)**.** *Suppose that (A,u) is a regular nodal ϵ-vortex with a single node. There exists an approximate right inverse T_δ of the parametrized linear operator $\tilde{D}_\delta := \tilde{D}_{A(\kappa,\delta),u(\kappa,\delta)}$ of the approximate solution $(A(\kappa,\delta), u(\kappa,\delta))$, that is, a map $T_\delta : \mathcal{I}_\delta \to \mathcal{H}_\delta$ such that*

$$\|(\tilde{D}_\delta T_\delta - I)\eta\|_\delta \leq \frac{1}{2}\|\eta\|_\delta$$

and the operator norm of T_δ is uniformly bounded in δ,

$$\|T_\delta \eta\|_\delta \leq c\|\eta\|_\delta.$$

Given such an approximate inverse, we obtain a uniformly bounded right inverse Q_δ to $\tilde{D}_\delta$ by the formula

$$Q_\delta = T_\delta(\tilde{D}_\delta T_\delta)^{-1} = \sum_{k\geq 0} T_\delta(\tilde{D}_\delta T_\delta - I)^k.$$

Proof. By the regularity assumption, $\tilde{D}_{A,\underline{u}}$ is surjective when restricted to the space of vectors $(a,\underline{\xi})$ such that $\xi_0(0) = \xi_1(\infty)$ and $a(w) = 0$. By Lemma 5.9.4, $\tilde{D}^0_\delta := \tilde{D}_{A_0(\kappa,\delta),\underline{u}_0(\kappa,\delta)}$ is surjective for sufficiently small κ, δ, when restricted to the same space.

The approximate right inverse is constructed by composing a cutoff operator K_δ, right inverse Q_δ, and gluing operator R_δ, as follows.

(a) The first step is the same as for pseudoholomorphic curves: In order to apply the right inverse on the nodal curve, we multiply the given data by a cutoff function: Define

$$K_\delta : \Omega^{0,1}(\Sigma, u^{\delta,*} T^{\mathrm{vert}} P(X))_{0,3,\delta} \to \Omega^{0,1}(\underline{\Sigma}, u_\delta^{0,*} T^{\mathrm{vert}} P(X))_{0,3}$$

by

$$(K_\delta(\eta))(z) = \begin{cases} \eta(z) & z \notin B_{|\delta|^{\frac{1}{2}}}(0) \\ 0 & \text{otherwise} \end{cases}$$

and using the canonical identification of Σ with $\underline{\Sigma}$ away from the gluing region. We have

$$\|K_\delta(\eta)\|_{0,3} \leq \|\eta\|_{0,3,\delta}$$

using (61).

(b) The gluing operator

$$R_\delta : \Omega^1(\Sigma_0, P(\mathfrak{g}))_{\frac{1}{2},2} \oplus \Omega^0(\underline{\Sigma}, u_\delta^{0,*} TX)_{1,3} \to \Omega^1(\Sigma_0, P(\mathfrak{g}))_{\frac{1}{2},2} \oplus \Omega^0(\Sigma^\delta, u^{\delta,*} T^{\mathrm{vert}} PX)_{1,3,\delta} \tag{67}$$

is defined as follows. As in [32, Section 10.4] choose a cutoff function

$$\beta_{2,\kappa,\delta} : \mathbb{R}^2 \to [0,1], \quad \beta_{2,\kappa,\delta}(z) = \begin{cases} 1 & |z| \leq \sqrt{\delta}\kappa \\ 0 & |z| \geq \sqrt{\delta}/\kappa \end{cases}$$

with the following two properties:

$$\|(\nabla \beta_2)\xi\|_{1,p} \leq \frac{c}{\log(|\kappa|^{-2})^{1-1/p}} \|\xi\|_{1,p}, \quad \forall \xi \in W^{1,p}(B_{\kappa|\delta|^{1/2}}), \xi(0) = 0 \tag{68}$$

and

$$\|\nabla \beta_3\|_{0,2} \leq C \log(|\kappa|^{-2})^{-1/2}. \tag{69}$$

For each component $\Sigma_\pm$, where Σ_+ is the principal component, let $\Sigma_\pm^*$ denote the complements of small balls around the nodes

$$\Sigma_\pm^* = \Sigma_\pm - B_{\kappa^2 |\delta|^{1/2}}(w_\pm).$$

Let $\pi_\pm : \Sigma_\pm^* \to \Sigma^\delta$ denote the inclusions. These induce maps of sections with compact support in Σ_j^*.

$$\pi_{\pm,*} \Omega_c^0(\Sigma_\pm^*, u_\pm^* TX) \to \Omega^0(\Sigma(\delta), u^{\delta,*} TX). \tag{70}$$

Define

$$R_\delta(\xi) = \begin{cases} \xi(z) & z \notin B_{|\delta|^{1/2}\kappa^2}(0) \\ \sum_\pm \pi_{\pm,*} \beta_{2,\kappa,\delta}(z)(\xi_\pm(z) - \xi(w_\pm)) + \xi(w_\pm) & \text{otherwise} \end{cases}.$$

Define

$$T_\delta := (I \times R_\delta) Q_\delta (I \times I \times K_\delta),$$

that is, if

$$(a, \xi) = Q_\delta(\phi, \psi, K_\delta(\eta))$$

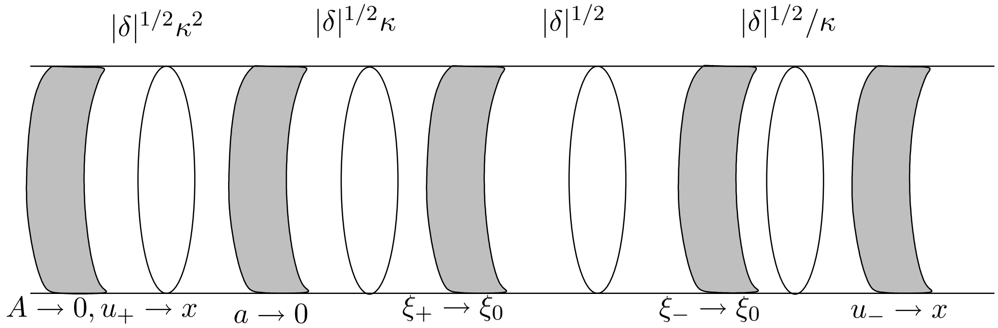

FIGURE 2. Gluing regions

then

$$T_\delta(\phi, \psi, \eta) = (a, R_\delta(\xi)).$$

We emphasize we are using the right inverse for the pair (64) which has simple form near the node.

In the proof of the corresponding result for pseudoholomorphic curves in [32], the operator T_δ is an exact inverse for $\tilde{D}_\delta$ away from the gluing region. This is no longer true in our case because of the terms

$$\phi - \epsilon^{-1} L_\xi u_\delta^* \Phi \operatorname{Vol}_\Sigma \tag{71}$$

$$\psi - \epsilon^{-1} L_{J\xi} u_\delta^* \Phi \operatorname{Vol}_\Sigma \tag{72}$$

$$\eta - D_{A,u}\xi - a_X^{0,1}(u_\delta) \tag{73}$$

which are all supported on the region $\Sigma_- - B_{|\delta|^{1/2}\kappa}(x)$ and the gluing region. Ignoring the gluing region for the moment, on $\Sigma_- - B_{|\delta|^{1/2}\kappa}(x)$ we have

$$\begin{aligned} \|\phi - \epsilon^{-1} L_\xi u_\delta^* \Phi \quad \operatorname{Vol}_\Sigma \quad & \|_{\Omega^0(\Sigma_- - B_{\kappa|\delta|^{1/2}}(x), P(\mathfrak{g}))_{-\frac{1}{2},2}} \\ \leq \quad & \kappa \|\phi - \epsilon^{-1} L_\xi u_\delta^* \Phi \operatorname{Vol}_\Sigma \||_{\Omega^0(\Sigma_- - B_{\kappa|\delta|^{1/2}}(x), P(\mathfrak{g}))_{0,3,\delta}} \\ \leq \quad & \kappa \|\xi\|_{\Omega^2(\Sigma_- - B_{\kappa|\delta|^{1/2}}(x), T^{\mathrm{vert}} P(X))_{0,3,\delta}} \end{aligned}$$

since

$$\mathrm{d}\operatorname{Vol}_{g^\delta}(z) \leq \kappa \mathrm{d}\operatorname{Vol}_g(z), \quad z \in \Sigma_- - B_{\kappa|\delta|^{1/2}}(x).$$

A similar computation gives a bound on the second term (72). For the third term (73) the change in the metric on on $T^{0,1}(\Sigma, T^{\mathrm{vert}} P(X))$ cancels the change in the volume form and we have

$$\begin{aligned} \|a_X^{0,1}(u_\delta)\|_{\Omega^{0,1}(\Sigma_- - B_{\kappa|\delta|^{1/2}}(x), T^{\mathrm{vert}}(P(X)))_{0,3,\delta}} \quad \leq \quad & \|a_X^{0,1}(u_\delta)\|_{\Omega^{0,1}(\Sigma_- - B_{\kappa|\delta|^{1/2}}(x), T^{\mathrm{vert}}(P(X)))_{0,3}} \\ \leq \quad & (\kappa|\delta|^{1/2})^{1/6} \|a\|_{\Omega^{0,1}(\Sigma_- - B_{\kappa|\delta|^{1/2}}(x), T^{\mathrm{vert}}(P(X)))_{0,2}} \end{aligned}$$

using Hölder's inequality, which approaches zero as $\kappa|\delta|^{1/2}$ does. In the gluing region the linearized operator is

$$\tilde{D}_\delta = \mathrm{d} \times \mathrm{d}^* \times \overline{\partial}_{A_0,x}$$

the operator for the trivial connection and map. The difference $\tilde{D}_\delta T_\delta \zeta - \zeta$ restricted to the gluing region has first component

$$(\tilde{D}_\delta T_\delta \zeta - \zeta)_1 = (\mathrm{d} \oplus \mathrm{d}^*)(\beta_{2,\kappa,\delta}(z) a_+(\cdot) - (\phi, \psi) \in (\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g})).$$

The terms involving the derivatives of the cutoff functions is bounded by $C \log(\kappa^{-2})^{-1/2}$, by (69) and Sobolev multiplication (24), and approaches zero as $\kappa \to 0$. The terms involving a cancel with ϕ, ψ away from the gluing region on the principal component. The integral over the gluing region vanishes in the limit $\kappa \to 0$ by definition of the cutoff function. On the bubble, the term $\mathrm{Vol}_\Sigma L_{\xi_-} u_-^* P(\Phi)$ approaches zero as $\delta \to 0$, since the norm of the embedding $W^{0,3,\delta} \to W^{0,2}$ vanishes in the limit for functions in $\Sigma_- - B_{|\delta|^{1/2}}(0)$. The second component of the difference $\tilde{D}_\delta T_\delta \zeta - \zeta$ is supported in the gluing region, where it is equal to

$$(\overline{\partial} \beta(\varphi^{\pm,-1}(z)))\xi_\pm(z) \in \Omega^{0,1}(\Sigma^\delta, u^{\delta,*} T^{\mathrm{vert}} P(X))$$

since $D_{u_0(\delta)} \xi = K_\delta \eta = 0$ on $B_{|\delta|^{1/2}}(0)$. Using (68) we have

$$\|D_\delta T_\delta \eta - \eta\|_{0,3,\delta} \le \sum_\pm C |\log(\kappa^{-2})|^{-2/3} (\|\xi_\pm(z) - \xi_\pm(z_\pm)\|_{1,3}.$$

Using the uniform bound on Q_δ, the difference is bounded by $C \log(\kappa^{-2})^{-2/3} \|K_\delta \eta\|_{0,3,\delta}$, and hence by $C \log(\kappa^{-2})^{-2/3} \|\eta\|_{1,3,\delta}$, by the uniform bounded on K_δ.

The proof of the uniform bound on T_δ is similar and will be omitted. □

Recall the Fredholm map $\mathcal{F}^\epsilon$ of (52) cutting out the moduli space.

Proposition 5.9.6 (Uniform quadratic bound)**.** *There exist constants $c, C > 0$ such that if*

$$|\delta| < c, \quad \kappa > 1/c, \quad |\delta|\kappa^2 < c$$

then the map $\mathcal{F}^\epsilon_{A(\kappa,\delta),u(\kappa,\delta)}$ satisfies a quadratic bound

$$\|D\mathcal{F}^\epsilon_{A(\kappa,\delta)+a,\exp_{u(\kappa,\delta)}\xi}(a_1,\xi_1) - \tilde{D}^\epsilon_{A,u}(a_1,\xi_1)\|_\delta \le C\|a,\xi\|_\delta \|a_1,\xi_1\|_\delta.$$

Proof. The non-linear part of the curvature $\|[a,a_1]\|_{-\frac{1}{2},2}$ is quadratic and independent of δ, by uniformity of the Sobolev embedding (24). The other term appearing in the first vortex equation satisfies

$$\|L_{\xi_1} \exp_{u(\kappa,\delta)}(\xi_0)^* \Phi\|_{-\frac{1}{2},2} \le C \|\xi_0\|_{1,3,\delta} \|\xi_1\|_{0,3,\delta}$$

for some constant C independent of δ, using that the $W^{-\frac{1}{2},2}$ norm is controlled by the $W^{0,2}$ norm, and the $W^{1,3,\delta}$ norm controls the $W^{1,3}$ norm uniformly. The non-linear terms in the Cauchy-Riemann equation are estimated as in [32, Section 3.5, Lemma 10.3.1]; note that we are fixing the complex structure on the curve Σ_0, which avoids the more complicated analysis we gave in the previous section. The second vortex equation also involves a term of mixed type

$$(a+a_1)_X^{0,1}(\exp_u(\xi+\xi_1)) - a_X^{0,1}(\exp_u(\xi)). \tag{74}$$

By [1, Theorem 4.39], the constant in Sobolev multiplication depends only on the dimensions of the cone in the cone condition, which is independent of δ for δ sufficiently small. It follows that the difference (5.10.2) has $0,3,\delta$-norm bounded by $C\|a_1\|_{\frac{1}{2},2}\|\xi+\xi_1\|_{1,3,\delta} + C\|a\|_{\frac{1}{2},2}\|\xi_1\|_{1,3,\delta}$ for some constant C independent of δ. □

Theorem 5.9.7 (Gluing Stable Vortices)**.** *Let $(\underline{\Sigma}, A, \underline{u})$ be a regular stable nodal ϵ-vortex.*

(a) *(Gluing) There exists a constant $C > 0$ such that if $(a,\xi) \in \ker \tilde{D}_{A,u}$ and $\|a,\xi\| < C$ and $\underline{\delta} = (\underline{\delta}_1,\ldots,\delta_k)$ a collection of gluing parameters and κ a real parameter with*
$$\|\delta\| < C, \quad \kappa > 1/C$$
then there exists a unique η of norm less than the function $f(\delta,\kappa)$ of (63) *such that if $(a_1,\xi_1) = Q_{A(\kappa,\delta),u(\kappa,\delta)}\eta$ then*
$$(\underline{\Sigma}, A, \underline{u})(\underline{\delta}, a, \xi) := (A + a + a_1, \exp_u(\xi+\xi_1))$$
is an ϵ-vortex.

(b) *(Smooth dependence) The gluing $(\underline{\Sigma}, A, \underline{u})(\delta, a, \xi)$ depends smoothly on a, ξ.*

(c) *(Surjectivity) The map # induces a surjection of a neighborhood of* 0 *in* $\ker(\tilde{D}_{\underline{\Sigma},A,\underline{u}}) \times \mathbb{C}^m$ *onto an open neighborhood of $(\underline{\Sigma}, A, \underline{u})$ in $\overline{M}_n(\Sigma, X)_\epsilon$.*

Proof. The first two claims are an application of the implicit function theorem, using Lemmas 5.9.2, 5.9.5, and 5.9.6. To prove surjectivity, write

$$(A_1, u_1) = (A(\kappa,\delta) + a, \exp_{u(\kappa,\delta)}\xi),$$

and decompose

$$(a,\xi) = (a_0,\xi_0) + (a_1,\xi_1)$$

according to the decomposition

$$\ker \tilde{D}_{A(\kappa,\delta),u(\kappa,\delta)} \oplus \operatorname{im} \tilde{D}^*_{A(\kappa,\delta),u(\kappa,\delta)}.$$

For ϵ_1 sufficiently small, (a_0,ξ_0) has norm at most ϵ. The implicit function theorem shows that (a_1,ξ_1) is the unique element in the image of $\tilde{D}^*_{A(\kappa,\delta),u(\kappa,\delta)}$ so that u_1 is pseudoholomorphic. The exponential decay estimates and elliptic bootstrapping in Section 5.7 imply that any stable nodal vortex in a sufficiently small neighborhood of $(\underline{A}, \underline{u})$ satisfies the above estimate. □

The gluing construction of Theorem 5.9.7 produces out of a family $(\underline{\Sigma}_\bullet, A_\bullet, \underline{u}_\bullet) \to S$ of parametrized regular family of nodal vortices of constant type and fixed principal component a family of nodal vortices

$$(\#\underline{\Sigma}_\bullet, \#A_\bullet, \#\underline{u}_\bullet) \to \#S. \tag{75}$$

where $\#S = S \times B_\epsilon(0,\mathbb{C}^m)$.

Theorem 5.9.8. *If* $(\underline{\Sigma}_\bullet, A_\bullet, \underline{u}_\bullet) \to S, \varphi$ *is a universal deformation of constant type of* $(\underline{\Sigma}, A, \underline{u})$*, then* $(\#\underline{\Sigma}_\bullet, \#A_\bullet, \#\underline{u}_\bullet, \#\varphi)$ *is a partially smooth universal universal deformation of* $(\underline{\Sigma}, A, \underline{u})$*.*

The proof is similar to that of Theorem 4.3.7, and will be omitted. Let $\mathrm{aut}(\underline{\Sigma}, A, \underline{u})^\perp$ be a complement of $\mathrm{aut}(\underline{\Sigma}, A, \underline{u})$ in $\ker(\tilde{D}_{\underline{\Sigma},A,\underline{u}})$. The base of the universal deformation $\#S$ can be taken to be a neighborhood of 0 in $\mathrm{aut}(\underline{\Sigma})^\perp \times \mathbb{C}^m$.

5.10. **Charts.** Given a deformation $(\underline{\Sigma}_\bullet, A_\bullet, \underline{u}_\bullet), \varphi$ of a parametrized regular marked stable vortex $(\underline{\Sigma}, A, \underline{u}, \underline{z})$, define a map

$$\#_{\underline{\Sigma},A,\underline{u},\underline{z}} : \#S \to \overline{M}_n(P, X), \quad s \mapsto [(\underline{\Sigma}_s, A_s, \underline{u}_s, \underline{z}_s)]. \tag{76}$$

There is a similar construction for framed vortices. We wish to prove the following generalization of Theorem 4.4.1:

Theorem 5.10.1. *Let* $(\underline{\Sigma}, A, \underline{u})$ *be a parametrized regular stable map. Two fibers of the partially smooth universal deformation* (76) *are isomorphic if and only if they are related by an automorphism of* $(\underline{\Sigma}, A, \underline{u})$*. Furthermore,* (76) *is a partially smooth universal deformation of any of its fibers.*

The proof is complicated because we lack the analyticity arguments used in the corresponding statement e.g. for stable curves 2.1.5. Our proof depends on differentiability properties of the evaluation maps, using a non-standard gluing profile. Given a parametrized regular framed vortex $(\underline{\Sigma}, A, \underline{u}, \underline{z}, \underline{\phi})$ let

$$(\underline{\Sigma}_\bullet, A_\bullet, \underline{u}_\bullet, \underline{z}_\bullet, \underline{\phi}_\bullet) \to S$$

be the family given by the gluing construction. Consider the map

$$\mathrm{ev} : S \to X^n, \quad s \mapsto (\phi_j(\underline{u}_s(z_{s,j})))_{j=1}^n.$$

Proposition 5.10.2. *The map* ev *is* C^1 *for the exponential gluing profile.*

Proof. First we take the derivative with respect to (A, u), and then with respect to the gluing parameter. The first part is similar to that of McDuff-Salamon [32, p. 372], made more complicated by the presence of a connection, but on the other hand simpler because we are fixing a right inverse. Consider a family of stable vortices $(A_s, u_s) = (A + a_s, \exp_u(\xi_s))$ of the same combinatorial type. As before, for simplicity we assume that there is a single sphere bubble. Let $(A_s(\delta), u_s(\delta))$ denote the pre-glued family. Let $\tilde{D}_s = \tilde{D}_{\#_\delta(A_s,u_s)}$, $\tilde{D} = \tilde{D}_{A(\delta),\underline{u}(\delta)}$ and Q its right inverse as above. From (53) one obtains that for s sufficiently small $\|\tilde{D}_s - \tilde{D}\| \le C\|a_s, \xi_s\|$ for some constant C independent of s. Differentiating the equation

$$\mathcal{F}((\underline{\Sigma}, A, \underline{u})(\delta, a_s, \xi_s) = 0.$$

with respect to s gives

$$\tilde{D}_s(\partial_s A_s(\delta) + \partial_s a_s, E_1 \partial_s u_s + E_2 \nabla_s \xi_s) = 0$$

where

$$\frac{d}{ds}\exp_u(\xi) = E_1(u,\xi)\frac{d}{ds}u + E_2(u,\xi)\nabla_s\xi.$$

Hence

$$\tilde{D}_s(\partial_s a_s, E_2\nabla_s\xi_s) = \tilde{D}_s(\partial_s A_s(\delta), E_1\partial_s u_s).$$

Now by definition a_s, ξ_s is in the image of Q, that is,

$$(a_s,\xi_s) = Q(\phi_s,\psi_s,\eta_s)$$

for some ϕ_s, ψ_s, η_s. Since this is independent of s we have

$$\begin{aligned}\tilde{D}_s(\partial_s a_s, E_2\nabla_s\xi_s) &= (\tilde{D}_s - \tilde{D})(\partial_s a_s, E_2\nabla_s\xi_s) + \tilde{D}(\frac{d}{ds}a_s, \nabla_s\xi_s)\\ &= \tilde{D}_s(\partial_s A_s(\delta), E_1\partial_s u_s).\end{aligned}$$

Since $\tilde{D}$ is bounded on the image of Q uniformly in δ by the previous section, we obtain

$$\|\partial_s a_s, \nabla_s\xi_s\|_\delta \le cf(\delta,\eta)\|\partial_s A_s, \partial_s u_s\|_\delta$$

for some constant c, where $f(\delta,\eta)$ is the error term of (63), which approaches zero as $\delta \to 0$ and $\kappa \to \infty$. (In fact, using elliptic regularity one can replace the norm on the right hand side with the L^2-norm, as in [32], but we will not need this.) It follows that the restriction $\mathrm{ev}(\delta)$ of ev to a particular set of gluing parameters δ approaches $\mathrm{ev}(0)$ in the C^1-topology as $\delta \to 0$.

Next we take the derivative with respect to the gluing parameter. Let (A,u) be a symplectic vortex, (A^δ, u^δ) denote the pre-glued pair (we omit the parameter κ controlling the diameter of the gluing region from the notation) and consider the equation

$$\mathcal{F}(A^\delta + \zeta^\delta, \exp_{u^\delta}(\xi^\delta)) = 0.$$

Differentiating with respect to δ gives

$$\tilde{D}_{(\underline{\Sigma},A,\underline{u})(\delta)}\left(\frac{d}{d\delta}A^\delta + \frac{d}{d\delta}a^\delta, D\exp_{\underline{u}^\delta,\xi^\delta}(\frac{d}{d\delta}\underline{u}^\delta, \nabla_\delta\xi^\delta)\right) = 0.$$

Define

$$(\hat{a}^\delta, \hat{\xi}^\delta) = T_\delta\nabla_\delta\eta_\delta - (\nabla_\delta a^\delta, \nabla_\delta\xi^\delta)$$

where

$$\eta^\delta = (\tilde{D}_\delta T_\delta)^{-1}\tilde{D}_\delta(a^\delta,\xi^\delta).$$

Then

$$\begin{aligned}(77)\quad &\tilde{D}_{\#_\delta(A,\underline{u})}\left(I \times D_2\exp_{\underline{u}^\delta,\xi^\delta}\right)T_\delta\nabla_\delta\eta^\delta\\ &\qquad = -\tilde{D}_{\#_\delta(A,\underline{u})}(\frac{d}{d\delta}A^\delta, D_1\exp_{\underline{u}^\delta,\xi^\delta}(\frac{d}{d\delta}\underline{u}^\delta) + \tilde{D}_{\#_\delta(A,\underline{u})}(\hat{a}^\delta,\hat{\xi}^\delta).\end{aligned}$$

The same arguments as in the proof of Theorem 4.4.2 show that

$$\|\tilde{D}_{(\underline{\Sigma},A,\underline{u})(\delta,a,\underline{\xi})}(\frac{d}{d\delta}A^\delta, D_1\exp_{\underline{u}^\delta,\underline{\xi}^\delta}(\frac{d}{d\delta}u^\delta)\|_\delta \le C(\kappa\sqrt{\varphi})^{2/3}\log(\varphi)^2$$

which is exponentially small in the gluing parameter δ. Now

$$\tilde{D}_{(\underline{\Sigma},A,\underline{u})(\delta,a,\underline{\xi})}\left(I \times D_2\exp_{u^\delta,\xi^\delta}\right)T_\delta$$

is uniformly bounded from below for δ sufficiently small, since T_δ is an approximate inverse to $\tilde{D}_{A^\delta,u^\delta}$, the difference $\tilde{D}_{\underline{\Sigma},A,\underline{u}} - \tilde{D}_{\underline{\Sigma}^\delta,A^\delta,u^\delta}$ is small, and the exponential maps are approximately the identity. It follows that $\nabla_\delta \eta_\delta$ and hence $\nabla_\delta\zeta_\delta$, $\nabla_\delta\xi_\delta$ are exponentially small. Hence $\frac{d}{d\delta}\operatorname{ev}\circ\#$ approaches the map obtained by replacing the gluing map with the pregluing map, $\frac{d}{d\delta}\operatorname{ev}\circ\#^{\mathrm{pre}} = 0$, which shows the claim.

The derivative of ev with respect to δ vanishes in the limit $\delta \to 0$, while the derivative with respect to ζ,ξ approaches the derivative of the evaluation map on the boundary stratum with respect to ζ,ξ as $\delta \to 0$. It follows that $D\operatorname{ev} : TU \to TX$ has a continuous limit as $\delta \to 0$, given by projecting (ζ,ξ,δ) onto (ζ,ξ) and then differentiating the boundary evaluation map. □

We now extend the rigidification construction of (42) to the case of gauged pseudoholomorphic maps. Let

$$(\underline{\Sigma}_\bullet, A_\bullet, \underline{u}_\bullet) \to S$$

be a family of gauged pseudoholomorphic maps. Given a codimension two submanifold $Y \subset X$ transverse to u, define the family of marked surfaces

$$\underline{\Sigma}_\bullet^{Y,\underline{u}} \to S$$

by (42). We rigidify the variation in the connections as follows. (Here one can imagine various possibilities.) Given a path $\gamma : [0,1] \to \underline{\Sigma}$ in the principal component and a trivialization of P over γ, we also obtain a family of parallel transport maps

$$\operatorname{Map}_{\gamma,\bullet} \to S$$

where

$$\operatorname{Map}_{\gamma,s} \in \operatorname{Map}_G(P_{\gamma(0)}, P_{\gamma(1)}) \cong G.$$

In particular, suppose that we fix a framing of $P_{\gamma(0)}$. Then parallel transport induces a family of framings

$$\phi_\bullet \to S$$

of $P_{\gamma(1)} \cong G$, giving rise to a map $S \to G$. We say that the family is versal resp. universal if it is submersive resp. a diffeomorphism onto the base of the universal deformation in a neighborhood of 0.

Let $\underline{Y} = (Y_1,\ldots,Y_k)$ be a collection of hypersurfaces transverse to $\underline{u}$. Fix l points $\underline{w} = (w_1,\ldots,w_l)$ on the principal component of $(A,\underline{u})$ as well as a base point $w_0 \in S$ and paths $\underline{\gamma} = (\gamma_1,\ldots,\gamma_l)$ from w_0 to $w_j, j = 1,\ldots,l$. From the family $(\underline{\Sigma}_\bullet, A_\bullet, \underline{u}_\bullet) \to S$ we obtain a family of marked surfaces $\underline{\Sigma}_\bullet^{\underline{Y},\underline{u}} \to S$ and a family of framings $\underline{\phi}_\bullet^{\underline{\gamma},A} \to S$.

Definition 5.10.3. We say that $\underline{Y},\underline{\gamma},A,\underline{u}$ are *compatible* if

(a) each Y_j intersects u_j transversally in a single point $z_j \in \underline{\Sigma}$;
(b) the map

$$\begin{aligned}\ker(\tilde{D}_{\underline{\Sigma},A,\underline{u}}) &\to \times_{j=1}^k T_{\underline{u}(z_j)}X/T_{\underline{u}(z_j)}Y_j \times TG^l \\ (\xi,a) &\mapsto \xi(z_j) \mod TY_j \oplus D_a\operatorname{Map}(\gamma,A)\end{aligned} \tag{78}$$

is an injection;

(c) the curve $\underline{\Sigma}$ marked with the additional points $z_1, \ldots, z_m$ is stable.

(d) if some automorphism of $(\underline{\Sigma}, \underline{u})$ maps z_i to z_j then Y_i is equal to Y_j.

Proposition 5.10.4. *Let $(\underline{\Sigma}, A, \underline{u})$ be a parametrized regular stable nodal vortex, and $(\underline{\Sigma}_\bullet, A_\bullet, \underline{u}_\bullet) \to S$ the partially smooth universal deformation constructed in (75).*

(a) *If $(\underline{Y}, \underline{\gamma})$ is compatible with $(A, \underline{u})$, then the rigidified family $(\underline{\Sigma}_\bullet^{\underline{Y},\underline{u}}, \phi_\bullet^{\underline{\gamma},A}) \to S$ of (42) is a partially smooth deformation of the marked-curve-with-framings $(\underline{\Sigma}^{\underline{Y},\underline{u}}, \phi^{\underline{\gamma},A})$.*

(b) *There exists a collection $(\underline{Y}, \underline{\gamma})$ compatible with $(\underline{u}, A)$.*

Proof. The proof of (a) is similar to Proposition 4.4.5 and will be omitted. (b) Suppose that the map (78) is not an injection. Let (a, ξ) maps to zero under (78). Suppose that ξ is non-zero. Let z_{n+1} be a point with $\xi(z_{n+1}) \neq 0$, and choose a codimension two submanifold Y_{n+1} transverse to u near $u(z_{n+1})$, and such that TY_{n+1} does not contain $\xi(z_{n+1})$. Adding Y_{n+1} to the list of dimension of the kernel decreases the dimension of the kernel of (78) by at least one. Repeating this process, we may assume that the only elements of the kernel of (78) have $\xi = 0$.

Suppose that ξ is zero, so that a is necessarily non-zero. Choose an additional marked point w_{m+1} and a path γ_{m+1} from the base point w_0 to w_{m+1} such that the derivative of the parallel transport over γ with respect to a over is non-zero. Appending γ_{m+1} to the list of path decreases the kernel of (78) by at least one. □

Theorem 5.10.5. *The universal deformations of Theorem 5.9.8 equip $\overline{M}_n^{\mathrm{reg}}(\Sigma, X, J, K)_\epsilon$ with the structure of a partially smooth topological orbifold, and (non-canonically) C^1-orbifold. Any choice of path $J_t \in \mathcal{J}(X)^G, t \in [0,1]$ induces a (non-compact) cobordism $\overline{M}^{\mathrm{reg}}(\Sigma, X, J_t)_\epsilon$ between $\overline{M}^{\mathrm{reg}}(\Sigma, X, J_0)_\epsilon$ and $\overline{M}^{\mathrm{reg}}(\Sigma, X, J_1)_\epsilon$ of* parametrized regular *ϵ-vortices, also equipped with a partially smooth, C^1 forgetful map and continuous evaluation maps.*

Proof. Charts are provided by the universal deformations of Proposition 5.10.4. Given two sets of hypersurfaces $\underline{Y}_1, \underline{Y}_2$, define $\underline{Y} = \underline{Y}_1 \cup \underline{Y}_2$ and $n = n_1 + n_2$ the total number. Similarly given two sets of paths on the principal component $\underline{\gamma}_1, \underline{\gamma}_2$, let $\underline{\gamma}$ be their union of total number $m = m_1 + m_2$. The family $\underline{\Sigma}_\bullet^{\underline{Y},\underline{u}} \to S$ admits a proper étale forgetful maps

$$(\underline{\Sigma}_\bullet^{\underline{Y},\underline{u}}, \underline{\phi}^{A,\underline{\gamma}_\bullet}) \to (\underline{\Sigma}_\bullet^{\underline{Y}_1,\underline{u}}, \underline{\phi}^{A,\underline{\gamma}_{1,\bullet}}) \times_S (\underline{\Sigma}_\bullet^{\underline{Y}_2,\underline{U}}, \underline{\phi}^{A,\underline{\gamma}_{2,\bullet}}).$$

The projections on each component are étale, by construction. Thus the fiber products have C^1-structures, that is, the orbifold charts are compatible. The results for the parametrized moduli space are similar and left to the reader. □

Remark 5.10.6. Using the differentiable structure defined above, the evaluation maps are differentiable but unfortunately the forgetful morphisms are not, unless one uses a different gluing profile for the moduli space of vortices with one less marking. More precisely, the forgetful morphism $\overline{M}_n^{\mathrm{reg}}(\Sigma, X)_\epsilon \to \overline{M}_n(\Sigma)$ is continuous and C^1 near any pair $(A, \underline{u}, z)$ whose domain is stable as an element of $\overline{M}_n(\Sigma)$, and a submersion near the boundary of $\overline{M}_n(\Sigma)$. For the standard smooth structure on $\overline{M}_n(\Sigma)$, the forgetful morphism $\overline{M}_n^{\mathrm{reg}}(\Sigma, X)_\epsilon \to \overline{M}_n(\Sigma)$ is smooth, but not submersive on a codimension two boundary stratum.

If $(\underline{\Sigma}, A, \underline{u})$ is a reducible polystable nodal vortex, then its restriction to the principal component (Σ_0, A, u_0) is a reducible vortex. It follows that the results of Section 5.4 extend:

Corollary 5.10.7. *There exists a discrete set of values $Z \subset (0, \infty)$ such that if $\epsilon \notin Z$, then every polystable nodal vortex is stable.*

Proposition 5.10.8. *If X satisfies the assumptions in Theorem 5.5.5, then there exists a discrete subset $Z \subset (0, \infty)$ such that if $\epsilon \notin Z$ then for a generic Hamiltonian perturbation K as in Theorem 5.5.5, every stable nodal vortex is regular.*

Proof. This is a repetition of the argument for Theorem 5.5.5, but using Sard-Smale to ensure in addition transversality of the evaluation maps at the nodes. □

5.11. **Boundary structure.** The boundary structure of $\overline{M}_n(\Sigma, X)_\epsilon$ is similar to that for $\overline{M}_n(\Sigma)$. For each subset $I \subset \{1, \ldots, n\}$ (possibly empty) and splitting $d_1 + d_2 = d$ there is a subset

$$\iota_I : D_I(d_1, d_2) \to \overline{M}_n(\Sigma)$$

where the markings for $i \in I$ have bubbled off onto an (unparametrized) sphere bubble of degree d_1, and a homeomorphism

$$\varphi_I : D_I(d_1, d_2) \cong \overline{M}_{0,|I|+1}(X, d) \times_X \overline{M}_{n-|I|+1}(X, \Sigma)_\epsilon. \tag{79}$$

The restriction of ι_I resp. φ_I to the regular locus is an immersion resp. isomorphism of orbifolds. Furthermore, the restriction of the forgetful morphism to the boundary gives maps

$$f : D_I(d_1, d_2) \to D_I$$

which are fibrations away from the collapsing locus. In particular, suppose that we have chosen a Hamiltonian perturbation so that every element in $\overline{M}(\Sigma, X)_\epsilon$ is regular. Let

$$\delta_I(d_1, d_2) \in H^2(\overline{M}(\Sigma, X)_\epsilon)$$

denote the dual class to $D_I(d_1, d_2)$. Then we have an equality

$$f^* \delta_I = \sum_{d_1 + d_2 = d} \delta_I(d_1, d_2). \tag{80}$$

Remark 5.11.1. If $\overline{M}_n(\Sigma)$ is equipped with a differentiable structure using the same gluing profile as $\overline{M}_n(\Sigma, X)$, then f is differentiable away from the collapsing locus and a submersion. Using a softer gluing profile on $\overline{M}_n(\Sigma)$ as $\overline{M}_n(\Sigma, X)$, then f is differentiable globally but not a submersion near the boundary.

5.12. **Vortex invariants.** We suppose in this section that there is an almost complex structure and Hamiltonian perturbation such that every polytstable nodal ϵ-vortex is stable and regular. As we explained in Theorem 5.5.5, we can guarantee that these conditions are satisfied at least for some convex varieties satisfying a genericity assumption.

The action of G^n on $\overline{M}^{\mathrm{fr}}(\Sigma, X)_\epsilon$ is locally free by definition, hence $\overline{M}^{\mathrm{fr}}(\Sigma, X)_\epsilon \to \overline{M}(\Sigma, X)_\epsilon$ has the structure of a orbifold principal G-bundle.

Assume first that the action of G^n is free. Let $\psi : \overline{M}_n^{\mathrm{fr}}(\Sigma, X)_d \to EG^n$ be a classifying map. Define *framed evaluation maps*

$$\mathrm{ev}^{\mathrm{fr}} : \overline{M}_n^{\mathrm{fr}}(\Sigma, X)_\epsilon \to X^n, \quad (A, \underline{u}, \underline{\phi}, z) \mapsto \underline{\phi} \circ \underline{u}(\underline{z}).$$

Combining the evaluation maps with the classifying map gives rise to an equivariant evaluation map

$$\mathrm{ev} : \overline{M}_n(\Sigma, X)_\epsilon \to X_G^n, \quad [A, \underline{u}, z] \mapsto [\psi(A, \underline{u}, z), \mathrm{ev}^{\mathrm{fr}}(A, \underline{u}, z)].$$

We denote by

$$f : \overline{M}_n(\Sigma, X)_\epsilon \to \overline{M}_n(\Sigma)$$

the forgetful morphism obtained by forgetting the data $(A, \underline{u})$ and collapsing any unstable component. In particular, we obtain a pull-back map in cohomology

$$f^* : H(\overline{M}_n(\Sigma)) \to H(\overline{M}_n(\Sigma, X)_\epsilon).$$

Pulling back and integrating defines the *vortex invariants*

$$\langle \alpha; \beta \rangle_{d,\epsilon} = \int_{\overline{M}_n(\Sigma, X)_\epsilon} \mathrm{ev}^* \alpha \wedge f^* \beta. \tag{81}$$

Summing over degrees gives a map

$$\tau^n : QH_G(X)^n \otimes H(\overline{M}_n(\Sigma)) \to \Lambda[[q]], \qquad \sum_{d \in H_2^G(X)} q^{([\omega_G], d)} \langle \alpha; \beta \rangle_{d,\epsilon}.$$

The compactness result of Theorem 5.6.10 implies that the sum is well-defined.

More generally, if the action of G^n is only locally free, then passing to the classifying spaces gives a topological principal G^n-bundle

$$B\overline{M}_n^{\mathrm{fr}}(P, X)_\epsilon \to B\overline{M}_n(P, X)_\epsilon.$$

Since $\overline{M}_n(P, X)_\epsilon$ is an orbifold, $B\overline{M}_n(P, X)_\epsilon$ has the homotopy type of a CW complex and the bundle above admits a classifying map

$$\psi : B\overline{M}_n^{\mathrm{fr}}(\Sigma, X)_\epsilon \to EG^n.$$

On the other hand, we have a map from the classifying space onto the underlying space

$$B\overline{M}_n^{\mathrm{fr}}(\Sigma, X)_\epsilon \to \overline{M}_n^{\mathrm{fr}}(\Sigma, X)$$

and pulling back the framed evaluation maps gives maps

$$B\overline{M}_n^{\mathrm{fr}}(\Sigma, X)_\epsilon \to X^n.$$

The combined map

$$B\overline{M}_n^{\mathrm{fr}}(\Sigma, X)_\epsilon \to EG^n \times X^n$$

is equivariant for the action of G^n and so descends to a map

$$B\overline{M}_n(\Sigma, X)_\epsilon \to X_G^n.$$

Pull-back in rational cohomology and descent under the rational equivalence (17) induces a map

$$\mathrm{ev}^* : H_G(X, \mathbb{Q})^n \to H(\overline{M}_n(\Sigma, X)_\epsilon, \mathbb{Q}).$$

The equation (81) then defines vortex invariants in this case as well.

Theorem 5.12.1. *With X, J as above, suppose that ϵ is such that every element of $\overline{M}_n(\Sigma, X)_\epsilon$ is strictly stable. Then the invariants τ^n define a cohomological trace on the cohomological field theory of Theorem 4.6.4, invariant of the Hamiltonian perturbation.*

Proof. By (80) and (79). □

Unfortunately, one cannot say that these are symplectic invariants of X, one cannot in general interpolate between two such convex almost complex structures. For this one needs a theory of equivariant Kuranishi fundamental classes.

5.13. **Relaxed vortices.** We describe a method of defining the invariants by weakening the vortex equations, as opposed to Hamiltonian perturbation, in the case that the target is convex. The construction is essentially that of a global Kuranishi structure. In general, the construction of the Kuranishi structure and invariants is somewhat involved; however, in this case it is substantially easier because the cokernels arise from the linearized operator on the principal component.

For any point $v \in \Sigma$, section $u : \Sigma \to P(X)$, and connection $A \in \mathcal{A}(P)$, construct a map

$$\eta_v = \eta_{v,u,A} : \Lambda^{0,1}(T^{\mathrm{vert}}_{u(v)} P(X)) \to \Omega^{0,1}(u^* T^{\mathrm{vert}} P(X))$$

as follows. First, parallel transport extends $\eta(v)$ to a local section η in a neighborhood U of v. Second, multiply by a cutoff function $\rho : U \to \mathbb{R}$. For any collection of distinct points

$$\underline{v} = (v_1, \ldots, v_l) \in \Sigma^l$$

and disjoint neighborhoods

$$\underline{U} = (U_1, \ldots, U_l) \subset \Sigma^l$$

the forms $\eta_{v_i}(\zeta_i), i = 1, \ldots, l$ are linearly independent. We denote by

$$\mathcal{R}(P, X) \to \mathcal{A}(P) \times \Gamma(P(X))$$

the vector bundle with fiber

$$\mathcal{R}(P, X)_{A,u} = \bigoplus_{j=1}^{l} \Lambda^{0,1}(T^{\mathrm{vert}}_{u(v_j)} P(X)).$$

Definition 5.13.1. A datum $(A, u, \underline{\zeta}) \in \mathcal{R}(P, X)$ is a *relaxed ϵ-vortex* iff

$$F_A + \mathrm{Vol}_\Sigma u^*\Phi = 0, \quad \overline{\partial}_A u + \sum_{j=1}^{l} \eta_{v_i}(\zeta_i) = 0. \tag{82}$$

The intersection of the zero section of $\mathcal{R}(P, X)$ with the space of relaxed vortices may be identified with the space of ordinary vortices. More explicitly, let $\pi^*\mathcal{R}(P, X)$ denote the pull-back bundle over $\mathcal{R}(P, X)$ by $\pi : \mathcal{R}(P, X) \to \mathcal{A}(P) \times \Gamma(P(X))$. The identity map defines a section of $\pi^*\mathcal{R}(P, X)$, so that $\mathcal{A}(P, X)$ is the zero section.

More informally, the space of relaxed vortices may be identified with the space of pairs (A, u) that satisfy the vortex equations up to the finite dimensional subspace generated by the forms $\eta_{v_i}(\cdot)$, that is, satisfy a relaxed version of the symplectic vortex equations. The motivation for this particular version arises from a similar situation in the study of the Higgs moduli spaces, in which one allows poles in the Higgs field to construct the derived structure on the moduli stack.

The group $\mathcal{G}(P)$ of gauge transformations on $\mathcal{A}(P) \times \Gamma(P(X))$ lifts to an action on $\mathcal{R}(P, X)$ by push-forward of tangent vectors, and preserves the relaxed vortex equation in (5.13.1). A relaxed vortex is *stable* if it has finite automorphism group, that is, its stabilizer under the gauge group is finite.

Let $M(P, X, \underline{v})_\epsilon$ denote the coarse moduli space of isomorphism classes of relaxed vortices, and $M^{\mathrm{reg}}(P, X, \underline{v})_\epsilon$ the subset of stable, regular relaxed vortices, that is, those for which the linearized operator is surjective and the automorphism group is finite. The bundle $\pi^*\mathcal{R}(P, X)$ over $\mathcal{A}(P) \times \Gamma(P(X))$ descends to an orbifold *obstruction bundle*

$$R(P, X, \underline{v})_\epsilon \to M(P, X, \underline{v})_\epsilon$$

equipped with a canonical section given by the identity map, so that the zero set is the coarse moduli space $M(P, X)_\epsilon$ of ordinary vortices.

Marked and framed versions are defined as before. If $\underline{\Sigma}$ is a rooted nodal curve, then a *relaxed nodal vortex* on $\underline{\Sigma}$ is a pair of a connection A on the principal component Σ_0, and a section $\underline{u} : \underline{\Sigma} \to P(X)$, so that $\underline{u}$ is pseudoholomorphic on the bubble components and (A, u_0) is a relaxed vortex on Σ_0. A relaxed vortex is *polystable* if each bubble on which the section is constant has at least three special points, and *stable* if it has finite automorphism group. Let $\overline{M}_n(P, X, \underline{v})_\epsilon$ denote the moduli space of isomorphism classes of relaxed polystable vortices with n markings $\underline{z}$ (disjoint from the nodes but possibly equal to the points $\underline{v}$), $\overline{M}_n^{\mathrm{reg}}(P, X, \underline{v})_\epsilon$ the locus of regular, stable relaxed vortices. We denote by $\overline{M}_n(\Sigma, X, \underline{v})$ the union over topological types, and

$$\overline{R}_n(\Sigma, X, \underline{v}) \to \overline{M}_n^{\mathrm{reg}}(\Sigma, X, \underline{v})$$

the obstruction bundle.

Theorem 5.13.2. *$\overline{M}_n^{\mathrm{reg}}(\Sigma, X, \underline{v})$ has the structure of a partially smooth topological orbifold in a neighborhood of $\overline{M}_n^{\mathrm{reg}}(\Sigma, X)$, and (non-canonically) the structure of C^1-orbifold.*

We will not give a proof of this theorem; it relies on a redo of the gluing construction to the setting of 5.13.1. The additional terms in the error estimate, estimate for the right inverse, and quadratic estimate of Section 5.9 arising from the last term in 5.13.1 vanish in the limit $\overline{\partial}_A u \to 0$, and do not affect the construction. As before, we have (at least rationally) maps

$$\mathrm{ev} : \overline{M}_n^{\mathrm{reg}}(\Sigma, X, \underline{v})_\epsilon \to X_G^n$$

and

$$f : \overline{M}_n(\Sigma, X, \underline{v})_\epsilon \to \overline{M}_n(\Sigma).$$

Theorem 5.13.3. *Suppose that $J \in \mathcal{J}(X)^G$ is convex and $d \in H_2^G(X)$. There exists a set $\underline{v}$ in a neighborhood of $\overline{M}_n(\Sigma, X, d)_\epsilon$ in $\overline{M}_n(\Sigma, X, \underline{v}, d)_\epsilon$, every relaxed vortex is regular.*

Proof. It suffices to choose $\underline{v}$ so that if η lies in the cokernel of some linearized operator $D_{A,\underline{u}}$ for $(A, \underline{u}) \in \overline{M}_n(P, X)_\epsilon$ then η is non-vanishing at at least one of the points $p_j, j = 1, \ldots, l$. Then every element of $\overline{M}_n(P, X)_\epsilon$ is regular as a relaxed vortex, and since regularity is an open condition the claim follows. □

Let $\mathrm{Eul}(\overline{R}_n(\Sigma, X, \underline{v})_\epsilon) \in H_c(U, \mathbb{Q})$ denote a compactly supported Euler class of the orbifold bundle $\overline{R}_n(\Sigma, X, \underline{v})_\epsilon$, constructed by pulling back a Thom class supported near the zero set in $\overline{R}_n(\Sigma, X, \underline{v})_\epsilon$, under the canonical section. If every relaxed vortex is stable and regular in a neighborhood of the usual vortices, we may define *vortex invariants*

$$\langle \alpha; \beta \rangle_{d,\epsilon} = \int_{\overline{M}_n(\Sigma, X, \underline{v})_\epsilon} \mathrm{Eul}(\overline{R}_n(\Sigma, X, \underline{v})_\epsilon) \wedge \mathrm{ev}^* \alpha \wedge f^* \beta. \tag{83}$$

It is left to the reader to check that these invariants are independent of the choice of $\underline{v}$, using properties of the Euler class, and agree with those defined before, if every element of $\overline{M}_n(\Sigma, X)_\epsilon$ is regular and stable. Under these assumptions, the invariants define a cohomological trace on the cohomological field theory of Theorem 4.6.4, independent of the choice of $\underline{v}$, but possibly depending on the choice of convex $J \in \mathcal{J}(X)^G$.

6. Polarized vortices and wall-crossing formulae

In this section we study the variation of the vortex invariants with respect to the vortex parameter ϵ. Perhaps the following finite analogy is helpful. Suppose that $M_j, j = 0, 1$ are finite dimensional Hamiltonian G-manifolds with symplectic forms ω_j and moment maps $\Phi_j : M_j \to \mathfrak{g}^*$. Let M_0^ϵ denote M_0 equipped with the re-scaled symplectic form $\epsilon\omega_0$, and consider the family of symplectic quotients $(M_0^\epsilon \times M_1) /\!\!/ G$ by the diagonal action of G. Given a cohomology class $\alpha \in H_G(M_0 \times M_1)$, we wish to study the variation of the integral

$$\int_{(M_0^\epsilon \times M_1) /\!\!/ G} \kappa_G(\alpha)$$

where

$$\kappa_G : H_G(M_0 \times M_1) \to H((M_0^\epsilon \times M_1) /\!\!/ G)$$

is the Kirwan map. This problem can be treated as a variation of symplectic quotient, as in Section 3.3. The resulting formula computes the difference

$$(84) \quad \int_{(M_0^{\epsilon_1} \times M_1)/\!\!/ G} \kappa_G(\alpha) - \int_{(M_0^{\epsilon_0} \times M_1)/\!\!/ G} \kappa_G(\alpha)$$
$$= \sum_{\epsilon \in [\epsilon_0, \epsilon_1]} \sum_\zeta \operatorname{Res}_{(\xi,\zeta)} \sum_{F \subset M_0^\zeta \times M_1^\zeta} \int_F \iota_F^* \beta \wedge \operatorname{Eul}(N_F)^{-1}$$

Our goal is to generalize this to the infinite dimensional situation, where $M_0 \times M_1$ is replaced by $\mathcal{A}(P, X)$. Of course, this is not a product, but the idea is the same.

6.1. **The Chern-Simons line bundle.** We denote by $\mathcal{L}(P) \to \mathcal{A}(P)$ the Chern-Simons line bundle, whose curvature is the symplectic form on $\mathcal{A}(P)$. We review its construction, see e.g. [33]. We fix a base connection A_0. The trivial line bundle $\mathcal{A}(P) \times \mathbb{C}$ with connection 1-form

$$\alpha \in \Omega^1(\mathcal{A}(P) \times \mathbb{C}), \quad \alpha_A(a, \lambda) = \lambda + \frac{1}{2} \int_\Sigma B(a \wedge (A - A_0))$$

has curvature equal to $-2\pi i$ times the symplectic form corresponding to the inner product B. The central $U(1)$-extension $\widehat{\mathcal{G}(P)}$ of the gauge group $\mathcal{G}(P)$ defined by the cocycle

$$(g_1, g_2) \mapsto \exp\left(\pi i \int_\Sigma (g_1^{-1}\, \mathrm{d}g_1 \wedge \mathrm{d}g_2\, g_2^{-1})\right)$$

($g_1^{-1}\, \mathrm{d}g_1$, resp. $\mathrm{d}g_2\, g_2^{-1}$ are the pull-backs of the left, resp. right Maurer-Cartan forms on G) acts on $\mathcal{A}(P) \times \mathbb{C}$ by connection preserving automorphisms by the formula

$$(g, z) \cdot (A, w) = \left(k \cdot A,\ \exp\left(\pi i \int_\Sigma (g^{-1}\, \mathrm{d}k \wedge A)\right) z\, w\right).$$

One may use the Chern-Simons three-form to trivialize the central extension $\widehat{\mathcal{G}(P)}$. Hence the action of the group of gauge transformations $\mathcal{G}(P)$ lifts to an action on $\mathcal{A}(P) \times \mathbb{C}$. Let

$$\phi : \mathcal{A}(P) \times \mathbb{C} \to \mathbb{R}, \quad \tilde{A} \mapsto |\tilde{A}|$$

denote the norm function. Consider on $\mathcal{A}(P) \times \mathbb{C}$ the closed two-form given by $\mathrm{d}(\phi, \alpha)$. The action of $\mathcal{G}(P)$ on $\mathcal{A}(P) \times \mathbb{C}$ is Hamiltonian with moment map given by $\tilde{A} \mapsto \phi(\tilde{A}) F_A$.

6.2. **Polarized vortices.** Let $\mathcal{A}(P, X) \times \mathbb{C}$ denote the pull-back of $\mathcal{A}(P) \times \mathbb{C}$ to $\mathcal{A}(P, X)$, that is,

$$\mathcal{A}(P, X) \times \mathbb{C} = \{(\tilde{A}, u, z), \overline{\partial}_A u = 0\} \subset \mathcal{A}(P) \times \operatorname{Map}(\Sigma, P(X)) \times \mathbb{C}.$$

The action of $\mathcal{G}(P)$ on $\mathcal{A}(P, X) \times \mathbb{C}$ has moment map given by

$$\mathcal{A}(P, X) \times \mathbb{C} \to \Omega^2(\Sigma, P(\mathfrak{g})), \quad \tilde{A} \mapsto \phi(\tilde{A}) F_A + \operatorname{Vol}_\Sigma u^* P(\Phi).$$

Let $L(P,X)$ denote the symplectic quotient

$$L(P,X) = \{(\tilde{A},u) \mid \phi(\tilde{A})F_A + \mathrm{Vol}_\Sigma u^*\Phi = 0, \quad \overline{\partial}_A u = 0\}/\mathcal{G}(P).$$

We call a pair $(\tilde{A},u)$ in the zero level set a *polarized vortex*. We say that $(\tilde{A},u)$ is *stable* if the automorphism group $\mathrm{Aut}(\tilde{A},u)$ is finite. A *smooth family of polarized vortices* on Σ is a smooth family $(\tilde{A}_s,u_s), s \in S$ of vortices over a smooth parameter space S. Deformations of polarized vortices are defined similarly.

One defines *regular vortices* by the analog of (53), and $L^{\mathrm{reg}}(P,X) \subset L(P,X)$ the locus of stable regular vortices. Deformations of polarized vortices are defined as in Proposition (5.3.1). Existence and properties of universal deformations of regular stable polarized vortices with smooth domain are similar to those of stable vortices, and will be omitted. The following is the analog of Theorem 5.5.1:

Theorem 6.2.1. *Universal deformations of regular stable polarized vortices provide $L^{\mathrm{reg}}(P,X)$ with the structure of a smooth, finite dimensional manifold with dimension*

$$\dim L^*(P,X) = 2((1-g)\dim(X) + (c_1(TX),d) - \dim(G) + 1).$$

Next we give a criterion for every polarized vortex to be stable, if the symplectic form on X is "sufficiently irrational".

Proposition 6.2.2. *If X is generic as in Definition 5.5.3, then every polarized vortex with smooth domain is stable.*

Proof. A pair $(\tilde{A},u)$ is fixed by a one-parameter subgroup $\xi \in \mathfrak{g}(P)$ if and only if (A,u) is ξ-fixed and ξ acts trivially on the fiber, that is, $(\xi,F_A) = 0$. Now if A is ξ-fixed then the structure group reduces to the centralizer G_ξ of ξ, and we may assume that ξ is a constant gauge transformation. Hence $(u^*P(\Phi),\xi)$ also vanishes, which implies that u maps to a component F of the fixed point set X^ξ such that $(\Phi(F),\xi)$ vanishes. □

We can partially compactify $L(P,X)$ by passing to *polystable nodal polarized vortices*; the definition is the same as that for polystable vortices but on the principal component we have a lift $\tilde{A}$ of A to $\mathcal{A}(P) \times \mathbb{C}$. Two nodal polarized vortices are equivalent if there exists an automorphism of the domain and a gauge transformation mapping one to the other. A *marked* polarized vortex is a marked vortex $(\underline{\Sigma},A,\underline{z},\underline{u})$, equipped with a lift of A to an element $\tilde{A} \in \mathcal{A}(P) \times \mathbb{C}$. Similarly, we can defined *framed polystable nodal polarized vortices*, as a framed polystable nodal vortex together with a lift of the connection.

We say that a polystable nodal polarized vortex $(\tilde{A},\underline{u})$ is *stable* if its automorphism group $\mathrm{Aut}(\tilde{A},\underline{u})$ is finite. By definition we have an injection

$$\mathrm{Aut}(\tilde{A},\underline{u}) \to \mathrm{Aut}(A,\underline{u})$$

given by projecting $\mathcal{G}(P) \times U(1)$ onto $\mathcal{G}(P)$. Thus, if $(\tilde{A},\underline{u})$ is unstable, then so is its projection $(A,\underline{u})$. It is a crucial point for this paper that *the converse does not hold*:

there exist stable polarized vortices whose underlying vortices are only polystable. This is what makes it possible to obtain wall-crossing formulas.

Let $\overline{L}(P,X)$ denote the coarse moduli space of isomorphism classes of polystable polarized vortices. The definition of the Gromov topology on $\overline{L}(P,X)$ is similar to that of Definition 5.6.5, and will be omitted.

Let $\overline{L}_n(P,X)$ resp. $\overline{L}_n^{\rm fr}(P,X)$ denote the moduli space of polystable marked, resp. framed vortices. This space is certainly *not* compact, because the value of the norm ϕ can run off to 0 or ∞. The correct analog of compactness is the properness statement in the theorem below:

Theorem 6.2.3. *$\overline{L}_n(P,X,J)$ is a Hausdorff topological space. The map ϕ extends to a proper map $\overline{L}_n(P,X,J) \to (0,\infty)$. The subset $\overline{L}_n^{\rm reg}(P,X,J)$ of stable regular polarized vortices admits the structure of a partially smooth orbifold. Any one-parameter family $J_t \in \mathcal{J}(X)^G, t \in [0,1]$ induces a (possibly non-compact) oriented partially smooth orbifold cobordism $\overline{L}_n^{\rm reg}(P,X,J_t)$ between $\overline{L}_n^{\rm reg}(P,X,J_0)$ and $\overline{L}_n^{\rm reg}(P,X,J_1)$.*

The proof is similar to that of Theorem 5.10.5 and will be omitted. Evaluation at the marked points defines equivariant evaluation maps

$$\mathrm{ev}^{\rm fr} : \overline{L}_n^{\rm fr}(P,X) \to X^n.$$

If $\overline{L}n^{\rm fr,reg}(P,X) \to \overline{L}_n^{\rm reg}(P,X)$ is a principal G-bundle then combining this with a classifying map gives an evaluation map

$$\mathrm{ev} : \overline{L}_n^{\rm reg}(P,X) \to X_G^n$$

by the same argument as in Section 5.12. More generally, if $\overline{L}_n^{\rm fr}(P,X) \to \overline{L}_n(P,X)$ is an orbifold principal G-bundle then we obtain a classifying map on the classifying space $B\overline{L}_n(P,X)$ of the orbifold $\overline{L}_n(P,X)$, and hence a map

$$\mathrm{ev} : B\overline{L}_n^{\rm reg}(P,X) \to X_G^n$$

and a pull-back map in cohomology

$$\mathrm{ev}^* : H_G(X)^n \to H(B\overline{L}_n^{\rm reg}(P,X),\mathbb{Q}) \cong H(\overline{L}_n^{\rm reg}(P,X),\mathbb{Q}).$$

6.3. **Extended vortices.** Consider the action of $U(1)$ on $\overline{L}_n(\Sigma,X)$. As in the finite dimensional case (15), the projection of any polarized vortex to $\overline{M}_n(\Sigma,X)$ must be fixed by a one parameter subgroup $U(1)_\zeta$ generated by a vector ζ in the Lie algebra of the gauge group. The image of the restriction of $\underline{u}$ to the principal component Σ_0 lies in the fixed point set $P(X)^\zeta$, and the structure group of P admits a canonical reduction to a subgroup $G_\zeta \subset G$, defined as the stabilizer of the evaluation of ζ at any point in Σ_0. The bubbles of a fixed point need not take values in $P(X)^\zeta$, since the action of $U(1)$ can be cancelled by the action of reparametrizations.

We give a description of the fixed point sets as moduli spaces of *extended vortices.* Let $P_\zeta \to S$ be a principal G_ζ-bundle.

Definition 6.3.1. An n-marked *extended vortex* is a datum $(\underline{\Sigma}, \underline{z}, A, \underline{u})$ where $\underline{\Sigma}$ is a nodal curve of with principal component $\Sigma_0 = \Sigma$ and genus zero bubble components $\Sigma_1, \ldots, \Sigma_m$, (A, u_0) is a symplectic vortex with values in the Hamiltonian G_ζ-manifold X_ζ, and for each component Σ_i of $\underline{\Sigma}$, $u_i : \Sigma_i \to X$ is a pseudoholomorphic map invariant under the $U(1)$-action generated by ζ. The data should satisfy matching conditions at the nodes.

Let $\overline{M}_n(P_\zeta, X^\zeta, X)$ denote the coarse moduli space of isomorphism classes of extended vortices with values in X^ζ and bubbles in X. $\overline{M}(P_\zeta, X^\zeta, X)$ can be identified with a fiber product

$$\overline{M}_n(P_\zeta, X^\zeta, X) = \bigcup_{r, I_1 \cup \ldots \cup I_r \subset \{1,\ldots,n\}} \left(M_r^{\mathrm{fr}}(P_\zeta, X^\zeta) \times_{(X^\zeta)^r} \prod_{j=1}^r \overline{M}_{|I_j|+1}(X)^{U(1)} \right) / G^r$$

by identifying each extended vortex with a vortex in X^ζ and a collection of bubble trees in X.

We say that an extended vortex is *regular* if each component is regular, and the evaluation maps are transverse to the diagonal on each stratum, which implies that the above fiber product is smooth on the regular locus in each stratum. Hence the moduli space of regular extended polystable vortices $\overline{M}_n^{\mathrm{reg}}(P_\zeta, X^\zeta, X)$ has a canonical structure of a partially smooth topological orbifold.

If every extended polystable vortex vortex is stable and regular, we define an *extended vortex invariant*

$$\langle \alpha; \beta \rangle_{X^\zeta, X, G_\zeta, d_\zeta, \epsilon} = \int_{\overline{M}_n(P^\zeta, X, X^\zeta, d_\zeta)} \mathrm{ev}^* \alpha \wedge f^* \beta \in \mathbb{Q}.$$

More generally, let V be a representation of G_ζ such that the action of ζ is fixed-point free. Associated to any $A \in \mathcal{A}(P^\zeta)$ we have a Fredholm operator

$$\mathrm{d}_A(V) : \Omega^0(\Sigma, P^\zeta(V)) \to \Omega^1(\Sigma, P^\zeta(V)).$$

Let $\mathrm{Ind}(V) \in K^0_{\mathcal{G}(P^\zeta)}(\mathcal{A}(P^\zeta), \mathbb{Z})$ denote the associated class in $U(1)$-equivariant K-theory. If $\mathcal{A}(P^\zeta)_d \subset \mathcal{A}(P^\zeta)$ denotes the subset on which the kernel of $\mathrm{d}_A(V)$ has dimension d then the restriction of $\mathrm{Ind}(V)$ to $\mathcal{A}(P)_d$ is represented by the G_ζ-equivariant virtual vector bundle

$$\mathrm{Ind}(V)_d \to \mathcal{A}(P)_d, \quad \mathrm{Ind}(V)_{d,A} = -\ker(\mathrm{d}_A(V)) + \mathrm{coker}(\mathrm{d}_A(V)).$$

$\mathrm{Ind}(V)$ descends to a $U(1)$-equivariant class still denoted $\mathrm{Ind}(V)$ on $\overline{M}_n(P^\zeta, X^\zeta, X)$, at least rationally if $\overline{M}_n(P^\zeta, X^\zeta, X)$ has singularities caused by finite stabilizers in the gauge group. Associated to V we have the *twisted extended vortex invariants*

$$\langle \alpha; \beta \rangle_{X^\zeta, X, G_\zeta, V, d_\zeta, \epsilon} = \int_{\overline{M}_n(P^\zeta, X, X^\zeta, d_\zeta)} \mathrm{ev}^* \alpha \wedge f^* \beta \wedge \mathrm{Eul}_{U(1)}(\mathrm{Ind}(V))^{-1} \in \mathbb{Q}[\zeta, \zeta^{-1}].$$

Finally we have version twisted by the Euler classes of the operator $D_{A,u}$ with values in the normal bundle to the stratum. Namely, let

$$T \to \overline{M}_n(P^\zeta, X^\zeta, X), \quad T_{\underline{\Sigma},A,\underline{u}} = \ker(D_{A,\underline{u}})$$

denote the orbifold vector bundle whose fibers are the kernels of the twisted linearized Cauchy-Riemann operator $D_{A,\underline{u}}$ of (53), that is, the tangent bundle to $\overline{M}_n(P, X)$. Let

$$T^\zeta \to \overline{M}_n(P^\zeta, X^\zeta, X), \quad T^\zeta_{\underline{\Sigma},A,\underline{u}} = \ker(D_{A,\underline{u}})^{U(1)}$$

denote the fixed point bundle of the $U(1)$-action, whose fiber over $\underline{\Sigma}, \underline{u}, A$ is the invariant part of the kernel. Define

$$\langle \alpha; \beta \rangle_{X^\zeta,X,G_\zeta,V,T/T^\zeta,d_\zeta,\epsilon} = \int_{\overline{M}_n(P^\zeta,X,X^\zeta,d_\zeta)} \mathrm{ev}^* \alpha \wedge f^*\beta \wedge \mathrm{Eul}_{U(1)}(\mathrm{Ind}(V) \oplus T/T^\zeta)^{-1}.$$

Let $W_\zeta \subset W$ denote the Weyl group of G_ζ, or equivalently, the stabilizer of ζ under the adjoint action of W.

Lemma 6.3.2. *Inclusion of $\mathfrak{g}_\zeta$ in $\mathfrak{g}$ induces a map $\iota_\zeta : \overline{M}_n(P_\zeta, X^\zeta, X)_\epsilon \to \overline{M}_n(P_\zeta \times G)/G_\zeta, X)^{U(1)}$.*

(a) *ι_ζ is surjective onto the fixed point set of $U(1)$, and has fiber isomorphic to W/W_ζ.*
(b) *The pull-back under ι_ζ of the normal bundle to $\overline{M}_n^{\mathrm{reg}}(P, X)_\epsilon^{U(1)}$ is isomorphic in K-theory to minus the index bundle associated to the representation $(\mathfrak{g}/\mathfrak{g}_\zeta)^{\mathbb{C}}$, plus T/T^ζ.*

Proof. Local surjectivity is already proved. The fiber is the set of G^ζ-vortices inducing the same G-vortex; assuming the structure group does not reduce further, the set of G-gauge transformations which preserve the set of G^ζ-vortices is the space of gauge transformations with values in the normalizer $N(G^\zeta)$, and the result follows from the isomorphism $N(G^\zeta)/G^\zeta \cong W/W_\zeta$. The second assertion follows by deforming the linearized operator to $\mathrm{d}_A \oplus \mathrm{d}_A^* \oplus D_u$, at which point the contributions from $\mathfrak{g}$ and $\mathfrak{g}^\zeta$ split as claimed. □

6.4. **Polarized vortex invariants for convex varieties.** Suppose that every polarized vortex is regular. The map induced from the norm function

$$\phi : \overline{L}_n(\Sigma, X) \to \mathbb{R}$$

is the moment map for the $U(1)$-action on the open stratum. For parameters $\epsilon_1 < \epsilon_2$ we denote by $\overline{L}_n(\Sigma, X)_{[\epsilon_1,\epsilon_2]}$ the cut space

$$\begin{aligned} \overline{L}_n(\Sigma, X)_{[\epsilon_1,\epsilon_2]} &= \phi^{-1}(\epsilon_1)/U(1) \cup \phi^{-1}(\epsilon_1, \epsilon_2) \cup \phi^{-1}(\epsilon_2)/U(1) \\ &= \overline{M}_n(\Sigma, X)_{\epsilon_1} \cup \phi^{-1}(\epsilon_1, \epsilon_2) \cup \overline{M}_n(\Sigma, X)_{\epsilon_2}. \end{aligned}$$

If these quotients are free then $\overline{L}_n(\Sigma, X)_{[\epsilon_1,\epsilon_2]}$ has the structure of a C^1-manifold with $U(1)$-action, still equipped with evaluation and forgetful maps. (At least if the bundle of framed polarized vortices has a free action by changing of framings; otherwise, the evaluation maps exist only rationally as in (17)).

Define *polarized gauged Gromov-Witten invariants* as follows. Given a collection $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_G(X)^n$ and an interval $[\epsilon_1, \epsilon_2]$ such that J is ϵ_1, ϵ_2 regular define

$$\langle \alpha; \beta \rangle_{J,d,[\epsilon_1,\epsilon_2]} = \int_{\overline{L}_n(P,X,d,J)_{[\epsilon_1,\epsilon_2]}} \operatorname{ev}^* \alpha \wedge f^* \beta \in \mathbb{Q}[\zeta].$$

Here ζ is the equivariant parameter for the $U(1)$-action and the subscript $[\epsilon_1, \epsilon_2]$ denotes the symplectic cut as in (12). Define *polarized correlators*

$$\begin{aligned} \tau^n_{[\epsilon_1,\epsilon_2]} &: QH_G(X)^n \otimes H(\overline{M}_n(\Sigma)) \to \Lambda[[q]][\zeta] \\ (\alpha, \beta) &\mapsto \sum_{d \in H_2^G(X)} q^{([\omega_G],d)} \langle \alpha; \beta \rangle_{d,[\epsilon_1,\epsilon_2]}. \end{aligned}$$

These invariants are independent of the choice of Hamiltonian perturbation $K \in \mathcal{H}(\Sigma, X)$, by an argument involving parametrized moduli spaces.

Theorem 6.4.1. *Suppose that (X, ω) is a compact Hamiltonian G-manifold, $J \in \mathcal{J}(X)^G$ a convex almost complex structure, and $[\epsilon_1, \epsilon_2]$ such that every element of $\overline{L}_n(\Sigma, X)_{[\epsilon_1,\epsilon_2]}$ is regular and stable. Then the invariants $\tau^n_{[\epsilon_1,\epsilon_2]}$ define a cohomological trace on the cohomological field theory of Theorem 4.6.4 with values in $\Lambda[[q]][\zeta]$.*

Proof. As for Theorem 5.12.1. □

In general, one cannot achieve transversality for the moduli spaces of polarized vortices using the strategy of invariant Hamiltonian perturbations described in Section 5.5, and one needs the derived structures constructed e.g. using relaxed vortices in Section 5.13.1. The construction is the same as before; let $\overline{L}_n(\Sigma, X, \underline{v})_\epsilon$ denote the moduli space of polystable nodal polarized relaxed vortices. Polarized vortex invariants are defined by inserting the Euler class of the obstruction bundle into the formula above, using a $U(1)$-equivariant Euler class $\operatorname{Eul}_{U(1)}(\overline{R}_n(P, X, \underline{v})_\epsilon)$:

$$\langle \alpha; \beta \rangle_{J,d,[\epsilon_1,\epsilon_2]} = \int_{\overline{L}_n(\Sigma,X,\underline{v},d)_{[\epsilon_1,\epsilon_2]}} \operatorname{Eul}(\overline{R}_n(\Sigma, X, \underline{v})_\epsilon) \wedge \operatorname{ev}^* \alpha \wedge f^* \beta \in \mathbb{Q}[\zeta]. \tag{85}$$

6.5. **Wall-crossing formulae.** Suppose that X, J are as above, and every polarized vortex is stable and regular. The wall-crossing arguments of Section 3.3 (see Remark 3.3.3) yield the wall-crossing formula

$$\langle \alpha; \beta \rangle_{d,\epsilon_2} - \langle \alpha; \beta \rangle_{d,\epsilon_1} = \operatorname{Res}_\zeta \left(\sum_F \int_F \iota_F^* (\operatorname{ev}^* \alpha \wedge f^* \beta) \wedge \operatorname{Eul}(N_F)^{-1} \right) \tag{86}$$

where F ranges over the fixed point components of $U(1)$ on $\overline{L}(P, X)$, with value of ϕ between ϵ_1 and ϵ_2. As described in Section 6.3, each fixed point component is the image of a component of extended vortices. We denote by

$$\rho_G^{G_\zeta} : H_\bullet^{G_\zeta}(X) \to H_\bullet^G(X).$$

We use the same notation for the dual map

$$\rho_G^{G_\zeta} : H_G^\bullet(X) \to H_G^\bullet(X).$$

We write $d_\zeta \mapsto d$ if $d \in H_2^G(X)$ is the image of $d_\zeta \in H_2^{G_\zeta}(X)$.

Theorem 6.5.1 (Wall-crossing for vortex invariants)**.** *Suppose that X, J are as above. Then*

$$\begin{aligned}(87)\quad & \langle \alpha; \beta \rangle_{G,d,\epsilon_2} - \langle \alpha; \beta \rangle_{G,d,\epsilon_1} \\ &= \sum_{\zeta \in \mathfrak{g}} \sum_{d_\zeta \mapsto d} \sum_{\epsilon \in (\epsilon_1, \epsilon_2)} \operatorname{Res}_{(\zeta,\xi)} \frac{\# W_\zeta}{\# W} \langle \rho_{G_\zeta}^G \alpha; \beta \rangle_{X^\zeta, X, G_\zeta, d_\zeta, (\mathfrak{g}/\mathfrak{g}_\zeta)_{\mathbb{C}}, T/T^\zeta, \epsilon}.\end{aligned}$$

Here (ζ, ξ) is the equivariant parameter for the action of ζ.

Proof. By (86) and Lemma 6.3.2. □

More generally, if the polarized vortex invariants are defined by (85), then equivariant localization applies to the integral since the classes are compactly supported and the same conclusion holds.

Lemma 6.5.2. *Suppose that P is not trivializable. There exists $\epsilon_0 > 0$ such that the moduli space $\overline{M}(P, X)_\epsilon$ is empty for $\epsilon > \epsilon_0$*

Proof. Since P is not trivializable, there exists a constant c such that any connection A on P has curvature F_A with $\|F_A\|_{L_2} > c$, see [2], [11]. Choose ϵ_0 so that $\epsilon_0^{-1} \sup \Phi$ is smaller than c; the vortex equation $F_A + \epsilon^{-1} \operatorname{Vol}_\Sigma u^* P(\Phi)$, is then impossible to satisfy for $\epsilon > \epsilon_0$. □

We say that a degree $d \in H_2^G(X)$ is *strictly equivariant* if it is not in the image of the map $H_2(X) \to H_2^G(X)$. For any vortex with strictly equivariant degree, the first Chern class and hence the bundle P are non-trivial. Hence

Corollary 6.5.3. *If d is strictly equivariant, then for any $\epsilon_0 > 0$,*

$$(88)\quad \langle \alpha; \beta \rangle_{G,d,\epsilon_0} = \sum_{\zeta \in \mathfrak{g}} \sum_{d_\zeta \mapsto d} \sum_{\epsilon_1 > \epsilon_0} \operatorname{Res}_{(\zeta,\xi)} \frac{\# W_\zeta}{\# W} \langle \rho_{G_\xi}^G \alpha; \beta \rangle_{X^\zeta, X, G_\zeta, d_\zeta, (\mathfrak{g}/\mathfrak{g}_\zeta)_{\mathbb{C}}, T/T^\zeta, \epsilon}.$$

Recursively, the same arguments can be applied to study the dependence on ϵ in the extended invariants $\langle \rho_{G_\xi}^G \alpha; \beta \rangle_{X^\zeta, X, G_\zeta, d_\zeta, (\mathfrak{g}/\mathfrak{g}_\zeta)_{\mathbb{C}}, T/T^\zeta, \epsilon}$. Namely, we say that a *polarized extended vortex* is an extended vortex, together with a lift to a polarized vortex on the principle component. Let $\overline{L}_n(P_\zeta, X^\zeta, X)$ denote the moduli space of polarized extended vortices,

$$\overline{L}_n(P_\zeta, X^\zeta, X) = \bigcup_{r, I_1 \cup \ldots \cup I_r \subset \{1,\ldots,n\}} \left(L_r^{\mathrm{fr}}(P_\zeta, X^\zeta) \times_{(X^\zeta)^r} \prod_{j=1}^r \overline{M}_{|I_j|+1}(X)^{U(1)} \right) / G^r.$$

The same wall-crossing formulas now apply to the reductions of $\overline{L}_n(P_\zeta, X^\zeta, X)$ as ϵ_1 varies. The resulting formula expresses the extended invariants at ϵ_1 as a sum over $\epsilon_2 > \epsilon_1$ of *doubly extended* invariants which are invariant under one-parameter subgroup generated by $\zeta_2 \in \mathfrak{g}_\zeta/\mathbb{R}\zeta$. The wall-crossing terms represent integrals over moduli spaces of G_{ζ_2}-vortices in $X^{\zeta_2} \subset X^\zeta$ with sphere bubbles in X^ζ and in X. The full description of the fixed point contributions is somewhat unwieldy:

$$\overline{M}_n(P^{\zeta_2}, X^{\zeta_2}, X^\zeta, X) = \bigcup M_r^{\mathrm{fr}}(P_{\zeta_2}, X^{\zeta_2}) \times_{(X^{\zeta_2})^r} \prod_{j=1}^r \Big(\overline{M}_{|I_j|+1}(X)^{U(1)_{\zeta_2}} \times_{(X^\zeta)^{s_j}} \prod_{k=1}^{s_j} \overline{M}_{|J_k|+1}(X)^{U(1)_{\zeta_2} \times U(1)_\zeta} \Big) / G^r \tag{89}$$

where the union is over double partitions $r, I_1 \cup \ldots \cup I_r = \{1, \ldots, n\}$ and $s_j, J_1 \cup \ldots \cup J_{s_j} = I_j, j = 1, \ldots, r$. The union is not disjoint; bubbles that are both ζ_1 and ζ_2 fixed lead to identifications of the various moduli spaces.

Continuing with the assumption that d does not lie in the image of $H_2(X)$, one eventually obtains a sum of contributions from moduli spaces of T-vortices, with bubbles fixed by T.

7. Abelianization

The *abelianization* or *quantum Martin conjecture* of Bertram, Ciocan-Fontanine and Kim [4] relates the Gromov-Witten invariants of a symplectic or equivalently geometric invariant theory quotient with the twisted Gromov-Witten invariants of the quotient by the maximal torus T. Consider the canonical map $H_2^T(X, \mathbb{Z}) \to H_2^G(X, \mathbb{Z})$. Any element $d_T \in H_2^G(X, \mathbb{Z})$ in the pre-image of $d_G \in H_2^G(X, \mathbb{Z})$ is called a *lift* of d_G. In particular, given a class $d_G \in H_2(X /\!\!/ G)$, we may embed $H_2(X /\!\!/ G) \to H_2^G(X)$ and say that $d_T \in H_2(X /\!\!/ T)$ is a lift of d_G if its image is a lift of the image of d_G. If so, we write $d_T \mapsto d_G$.

The quantum Martin conjecture of Bertram et al involves *twisted Gromov-Witten invariants*, given by integrating pull-back classes together with Euler classes of index bundles defined as follows. For any vector bundle $E \to X$, let

$$\mathrm{Ind}(E) = R f_{n+1,*}(\mathrm{ev}_{n+1}^* E)$$

denote the index class of E in the rational K-theory of $\overline{M}_{0,n}(X, d)$. Let

$$\mathrm{Eul}(\mathrm{Ind}(E)) \in H_{U(1)}(\overline{M}_{0,n}(X, d), \mathbb{Q})$$

denote its Euler class, defined formally using the $U(1)$-action by scalar multiplication on the fibers of $\mathrm{Ind}(E)$. Define the *E-twisted Gromov-Witten invariant*

$$\langle \alpha \rangle_{X,E,d} = \int_{[\overline{M}_{0,n}(X,d)]} \mathrm{ev}^* \alpha \wedge \mathrm{Eul}(\mathrm{Ind}(E)) \in \mathbb{Q}[\zeta]$$

where ζ is the equivariant parameter. For X a $G_{\mathbb{C}}$-variety, and V a T-representation let $V/\!\!/T := \Phi_T^{-1}(0) \times_T V$ denote the associated vector bundle over $X/\!\!/T$. In particular,

$$(\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}})/\!\!/T \to X/\!\!/T$$

is the sum of the line bundles associated to the roots. Let W denote the Weyl group. Motivated by a conjecture of Hori and Vafa relating Gromov-Witten invariants of the Grassmannian with those of products of projective spaces, Bertram-Ciocan-Fontanine-Kim [4] conjectured the following

Conjecture 7.0.4 (Quantum Martin formula). *For any classes $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_G(X)^n$ and $\beta \in H(\overline{M}_{0,n})$, the Gromov-Witten invariants for $X/\!\!/T$ and $X/\!\!/G$ are related by*

$$\langle \kappa_G(\alpha); \beta\rangle_{X/\!\!/G, d_G} = (\#W)^{-1} \sum_{d_T \to d_G} \langle \kappa_T(\rho_T^G \alpha); \beta\rangle_{X/\!\!/T, d_T, (\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}})/\!\!/T}.$$

As we explain elsewhere, we believe that this conjecture holds only after suitable "quantum Kirwan corrections" for arbitrary X. However, for vortex invariants we expect that the conjecture is true as stated. We prove the following special case:

Theorem 7.0.5 (Abelianization for vortices of strictly equivariant degree). *Suppose that X is a compact symplectic Hamiltonian G-manifold equipped with a convex G-invariant integrable almost complex structure J, satisfying the genericity condition in Definition 5.5.3. For any vortex parameter $\epsilon \in (0, \infty)$ such that the moduli spaces $\overline{M}(\Sigma, X, G)_\epsilon$ and $\overline{M}(\Sigma, X, T)_\epsilon$ have no strictly polystable elements, $d_G \in H_2^G(X)$ strictly equivariant, and $\alpha \in H_G(X)^n, \beta \in H(\overline{M}_n(\Sigma))$, we have*

$$\langle \alpha; \beta\rangle_{G,\epsilon,d_G} = (\#W)^{-1} \sum_{d_T \mapsto d_G} \langle \rho_T^G \alpha; \beta\rangle_{T,\epsilon,d_T,\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}}.$$

We do not claim that our twisting is the same as that of [4]; we are twisting by the Euler class of the index bundle of the connection on the principal component while [4] twist by an index bundle associated to the section. However, we expect that the two twistings are equivalent. The proof proceeds by proving a more general conjecture regarding extended vortex invariants. Namely, for any sequence $\underline{\zeta} = (\zeta, \zeta_2, \zeta_3, \ldots, \zeta_k)$ of Lie algebra vectors as in the previous section, let $\langle \alpha; \beta\rangle_{X^{\underline{\zeta}}, \mathfrak{g}/\mathfrak{g}_{\zeta_k}, T/T^{\zeta_k}, \epsilon, d_{G_{\zeta_k}}}$ denote the iterated extended vortex invariant of the previous section, obtained by recursively taking fixed points under the various $U(1)$-actions generated by the ζ_j.

Theorem 7.0.6 (Abelianization for extended vortices of strictly equivariant degree). *Suppose that X, J are as in the previous theorem. For any vortex parameter $\epsilon \in (0, \infty)$ such that the moduli spaces $\overline{M}(\Sigma, X^{\underline{\zeta}}, G_{\zeta_k}, d_{\zeta_k})_\epsilon$ and $\overline{M}(\Sigma, X^{\underline{\zeta}}, T, d_T)_\epsilon$ have no strictly polystable elements for all $d_T \mapsto d_{\zeta_k}$, $d_{\zeta_k} \in H_2^{G_{\zeta_k}}(X)$ strictly equivariant,* *and* $\alpha \in H_G(X)^n, \beta \in H(\overline{M}_n(\Sigma))$, *we have*

$$\langle \rho^G_{G_{\zeta_k}} \alpha; \beta \rangle_{X \underline{\zeta}, (\mathfrak{g}/\mathfrak{g}_{\zeta_k})_{\mathbb{C}}, T/T^{\zeta_k}, \epsilon, d_{G_{\zeta_k}}} = (\# W_\zeta)^{-1} \sum_{d_T \mapsto d_{G_{\zeta_k}}} \langle \rho^G_T \alpha; \beta \rangle_{X \underline{\zeta}, (\mathfrak{g}/\mathfrak{t})_{\mathbb{C}}, T/T^{\zeta_k}, \epsilon, d_T}. \tag{90}$$

Proof. The proof is by induction on $\dim(G_\zeta/T)$, by comparing the contributions to the wall-crossing formulae in the previous section. Clearly the theorem holds for $G_\zeta = T$. Next, suppose that the theorem holds for all groups $G_{\zeta'}$ with $\dim(G_{\zeta'}/T) < \dim(G_\zeta/T)$. The wall-crossing formulas of the previous section express the extended vortex invariants for G_ζ and T, as a sum over fixed point contributions, given by extended vortex invariants for smaller groups G_{ζ_1} fixed by successions of one-parameter subgroups, with Euler classes of the representation $(\mathfrak{g}_\zeta/\mathfrak{t})_{\mathbb{C}}$ added in the first case. It follows from the inductive hypothesis that the equality holds for the fixed point contributions, hence for extended G_ζ-vortex invariants. □

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