SYMPLECTIC VORTICES WITH FIXED HOLONOMY AT INFINITY

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Abstract. Let Σ be a fixed surface with cylindrical end punctures, G a Lie group and let X denote a monotone symplectic manifold with a Hamiltonian G-action. Using the symplectic vortex equations for X on Σ and restricting the connections to those with prescribed holonomy at infinity, we (partially) define gauged Gromov Witten invariants, which are intended to be a Gromov-Witten theory for the quotient X/G.

Contents

Preliminary version.

1. Introduction

In this paper we study the moduli space of symplectic vortices on a fixed punctured curve for compact connected structure group with fixed holonomies at infinity. The moduli space of symplectic vortices on a closed curve was introduced by Cieliebak, Gaio, and Salamon [6] and Mundet [18], see also Cieliebak, Gaio, Mundet, and Salamon [11], [5]. The moduli spaces with fixed holonomies generalize the moduli spaces of flat bundles on a surface studied by Mehta-Seshadri [16], corresponding in the holomorphic language to bundles with parabolic structure at marked points. Similar moduli spaces are studied by Mundet and Tian [13] in the case $G = S^1$. Cieliebak, Gaio, and Salamon [6] also study invariants associated to marked surfaces, but with trivial holonomy around the

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markings. The motivation for studying the moduli spaces of symplectic vortices with fixed holonomy is the hope that the invariants defined by integration might yield a generalization of small quantum cohomology to the case of Hamiltonian G-manifolds, similar to the way that orbifold quantum cohomology generalizes quantum cohomology to the case of finite group actions [4].

To describe the results more explicitly let G be a compact group, X a compact, connected Hamiltonian G-manifold with moment map $\Phi : X \to \mathfrak{g}^*$ equipped with a compatible, G-invariant almost complex structure, Σ a compact complex surface equipped with a volume form Vol_Σ. The space $\mathcal{A}(\Sigma, X)$ of pairs (A, u) consisting of a connection A on the trivial G-bundle and a pseudoholomorphic section u of the associated bundle with fiber X has a natural action of the group $\mathcal{G}(\Sigma)$ of gauge transformations. The symplectic quotient

$$M(\Sigma, X) = \mathcal{A}(\Sigma, X) /\!\!/ \mathcal{G}(\Sigma)$$

is the moduli space of solutions to the *symplectic vortex equations*

$$\overline{\partial}_A u = 0, \qquad F_A + \mathrm{Vol}_\Sigma \, u^*\Phi = 0. \tag{1}$$

In the case that Σ is a non-compact punctured surface, the space of solutions depends crucially on the behavior of the volume form near the punctures relative to the cylindrical end metric. The moduli spaces in this paper are defined using a volume form that decays faster than the volume form defined by the metric; this leads to a moduli space in which the value of the section at infinity takes values in the fixed point set of the holonomy of the limiting connection. We show how to achieve transversality, for irreducible vortices, obtain a compactification via stable vortices and a pseudocycle structure whenever X is a monotone manifold and the underlying connections are irreducible. This leads to a definition of gauged Gromov-Witten invariants, for generic fixed holonomies at infinity. We describe situations in which gauged Gromov-Witten invariants are enumerative, and relating them to counts of parabolic bundles together with holomorphic maps with prescribed behavior at the markings.

There are several directions that would be interesting to investigate further. First, the underlying curve-with-volume-form might be allowed to vary. In the case of circle actions the study of variation of the curve is a project of Mundet and Tian [13]. In a separate paper [12], we study the dependence on the volume form on the projective line. Second, the moduli spaces should have a "derived" (also known as virtual or Kuranishi) structure in the case when X is not necessary monotone, which should allow the definition of the gauged Gromov-Witten invariants in general. Third, there should be analogs of these moduli spaces of vortices with varying holonomy at infinity, which fit into the framework of Hamiltonian loop group manifolds [17], or equivalently, Hamiltonian actions with group-valued moment map [1]. Finally, one would like to use the invariants to construct a "gauged version" of small equivariant quantum cohomology, which should be a deformation of the non-abelian equivariant cohomology of orbit-type strata of X in the sense of [2].

We elaborate briefly on the last point, and in particular, the conjectural ring defined using the vortex invariants with Givental's equivariant quantum cohomology. Roughly speaking, equivariant quantum cohomology of X should count pseudoholomorphic curves in $X_G = X \times_G EG$. Givental has introduced an analog of equivariant quantum cohomology, which is a module over $H(BG)$, and counts curves in X. Thus, in Givental's version, only the X-direction is quantized. Since $H(BG)$ is free, it does not admit any interesting deformation, so in some sense $H_G(X)$ cannot be quantized further. But this is taking quantization too literally: a curve in BG is a G-bundle, and thus one expects the ring structure on $QH_G(pt)$ to count bundles. There are a number of theories which do this, topological Yang-Mills in two dimension being the simplest, and various version of twisted K-theory being the most sophisticated to date. For the former, the corresponding Frobenius algebra is the convolution algebra of invariant distributions on G (which contains the center of the universal enveloping algebra, and hence $H(BG)$, as a sub-algebra.) The structure coefficients for the simplest version should be constructed as follows. For any μ in the n-fold product $\mathfrak{A}^n$ of the Weyl alcove let $M(\Sigma, G, X, \mu)$ denote the moduli space of symplectic vortices with holonomy given by the conjugacy classes. The finite energy condition provides the space $N(\Sigma, X, \mu)$ of symplectic vortices with framings at infinity with an evaluation map

$$\mathrm{ev} : N(\Sigma, X, \mu) \to X^\mu := X^{\mu_1} \times \ldots \times X^{\mu_n}.$$

Here X^{μ_j} is the fixed point set of a representative in the conjugacy class of the holonomy. The stabilizer group

$$G_\mu = G_{\mu_1} \times \ldots \times G_{\mu_n} \tag{2}$$

acts on $N(\Sigma, X, \mu)$ as well as X^μ. The quotient $M(\Sigma, X, \mu)$ inherits an *evaluation section* ev : $M(\Sigma, X, \mu) \to M(\Sigma, X, \mu, X^\mu)$ of the associated bundle $M(\Sigma, X, \mu, X^\mu) := N(\Sigma, X, \mu) \times_{G(\partial\Sigma)_\mu} X^\mu$ where $X^\mu = X^{\mu_1} \times \ldots \times X^{\mu_n}$. The evaluation section extends to proper compactifications

$$\mathrm{ev} : \overline{M}(\Sigma, X, \mu) \to \overline{M}(\Sigma, X, \mu, X^\mu).$$

Given a collection of equivariant cohomology classes $\alpha \in \prod H_{G_\mu}(X^{\mu_j})$ we can take its pullback and descend to obtain ordinary cohomology classes $\mathrm{ev}^* \alpha$ on $\overline{M}(\Sigma, X, \mu)$, one would like to define *gauged Gromov-Witten invariants* by

$$Z_{\Sigma,X}(\mu) \quad : \quad \prod_j H_{G_{\mu_j}}(X^{\mu_j}) \quad \to \quad \mathbb{R}, \qquad \alpha \quad \mapsto \quad \int_{M(\Sigma,X,\mu)} \mathrm{ev}^* \alpha \ \wedge \ \exp(t\omega_\mu) \tag{3}$$

where ω_μ is a symplectic form on $M(\Sigma, X, \mu)$, and t is a positive parameter. Let

$$X = \bigcup_L G \times_{N(L)} X_L$$

be the orbit-type decomposition of L. For each orbit-type subgroup $L \subset G$, let $\mathcal{H}_{N(L)}(\overline{X}_L)$ be the non-abelian equivariant cohomology of [2]; this is a module over the ring of invariant distributions on $N(L)$. Each space $M(\Sigma, X)$ and collection of classes $\alpha \in \mathcal{H}_{N(L)}(\overline{X}_L)$

is expected to give rise to a distribution $Z_{\Sigma,X} \in (\mathcal{D}(L)^{N(L)})^n$. Define

$$Q\mathcal{H}_G(X) := \bigoplus_L \mathcal{H}_{N(L)}(\overline{X}_L).$$

The gauged Gromov-Witten invariants should define a product on $Q\mathcal{H}_G$. It is not yet clear to us whether one should expect this product to be strictly associative.

Unfortunately the results in this paper are limited, we have only defined the invariants (3) in the case when the parameter t vanishes. In this case, by taking the three pointed sphere $\Sigma = S^2 \backslash \{1, 0, \infty\}$ with holonomies μ_1, μ_2, μ_3 arround the punctures, then $\mu_3^{-1} = \mu_2\mu_1$ and we obtain a product

$$H_{G_{\mu_1}}(X^{\mu_1}) \times H_{G_{\mu_2}}(X^{\mu_2}) \to H_{G_{\mu_1\mu_2}}(X^{\mu_1\mu_2}),$$

which can be thought as the Chen-Ruan quantum product equivalent for the quotient X/G. If the quotient X/G is a globaly presented orbifold, then by restricting vortices (A, u) with u constant, this product should agree with the orbifold product.

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2. Flat bundles with fixed holonomy at infinity

The papers by Gaio-Salamon [11], Mundet [18] etc. generally take the point of view that the moduli space of symplectic vortices generalizes the moduli space of pseudo-holomorphic curves. In this paper we will emphasize the other point of view, that the moduli space of vortices generalizes the moduli space of flat bundles over a surface. In this section, we describe the Mehta-Seshadri theory of moduli spaces of flat bundles on punctured surfaces with fixed holonomy around the punctures; this is the special case of moduli space of symplectic vortices on punctured surfaces where the symplectic manifold is a point.

Let Σ be a compact, oriented surface possibly with n boundary components, and G a compact, 1-connected Lie group. We denote by T a maximal torus and W the Weyl group. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}_+$ a choice of positive chamber, so that $\alpha_0 \in \mathfrak{t}^*$ is the highest root.

Let $\mathcal{A}(\Sigma, G)$ denote the affine space of connections on the trivial G-bundle over Σ. The choice of a non-degenerate invariant metric B on $\mathfrak{g}$ induces on $\mathcal{A}(\Sigma, G)$ the structure of a symplectic manifold, with symplectic form given by

$$(a_1, a_2) \mapsto \int_\Sigma B(a_1 \wedge a_2). \tag{4}$$

The group of gauge transformations $\mathcal{G}(\Sigma)$ fits into an exact sequence

$$1 \to \mathcal{G}_\partial(\Sigma) \to \mathcal{G}(\Sigma) \to \mathcal{G}(\partial\Sigma) \to 1$$

where $\mathcal{G}_\partial(\Sigma)$ is the group of gauge transformations that equal to the identity on the boundary. This group acts on $\mathcal{A}(\Sigma, G)$ with moment map given by the curvature

$$\mathcal{A}(\Sigma, G) \to \Omega^2(\Sigma, \mathfrak{g}), \quad A \mapsto F_A$$

and the symplectic quotient

$$M(\Sigma, \partial\Sigma; G) = \mathcal{A}_\flat(\Sigma, G)/\mathcal{G}_\partial(\Sigma)$$

is the moduli space of flat connections with framing on the boundary. If G is not 1-connected then there exist other topological types, and taking the union of the analogous construction for each topological type of bundle gives the moduli space of flat bundles with framing on the boundary. The group of gauge transformations $\mathcal{G}(\partial\Sigma)$ acts on $M(\Sigma, \partial\Sigma; G)$ by changing the framing on the boundary, with moment map given by restriction to the boundary

$$\Phi : M(\Sigma, \partial\Sigma; G) \to \mathcal{A}(\partial\Sigma, G).$$

In the case that $\partial\Sigma$ is connected, the $\mathcal{G}(\partial\Sigma)$-orbits on $\mathcal{A}(\partial\Sigma, G)$ are parametrized by the Weyl alcove

$$\mathcal{A} = \{\xi \in \mathfrak{t}_+, \quad \alpha_0(\xi) \leq 1\}.$$

The sequence of maps $\mathcal{A} \to \mathfrak{t} \to T \to G$ induce isomorphisms of quotient spaces

$$\mathfrak{A}^n \cong (T/W)^n \cong (G/\operatorname{Ad}(G))^n \cong \mathcal{A}(\partial\Sigma, G)/\mathcal{G}(\partial\Sigma);$$

the last map is induced by taking the holonomy around the boundary. For any $\mu \in \mathfrak{A}^n$, where n is the number of boundary components, we denote by $\mathcal{O}_\mu \subset \mathcal{A}(\partial\Sigma, G)$ the orbit labelled by $\mu \in \mathfrak{A}^n$ and denote the symplectic quotient

$$M(\Sigma, G, \mu) = \Phi^{-1}(\mathcal{O}_\mu)/\mathcal{G}(\partial\Sigma),$$

the *moduli space of flat bundles with fixed holonomies* $\exp(\mu)$ around the boundary. For generic μ, the space $M(\Sigma, G, \mu)$ is a compact orbifold.

The moduli spaces of flat bundles with fixed holonomy have a holomorphic description due to Mehta-Seshadri [16], see also [3], [19], as follows. Let $E \to \Sigma$ be a holomorphic principal $G_\mathbb{C}$-bundle. A *quasiparabolic structure* at a point $s \in \Sigma$ is a reduction of structure group of E_s to a parabolic subgroup $P \subset G$, that is, a point in the quotient E_s/P. A *parabolic structure* is a quasiparabolic structure together with conjugacy class $\mu \subset G$ of the type specified by the parabolic subgroup, that is, the Levi subgroup of the parabolic is isomorphic to the centralizer of any point in the conjugacy class. We suppose that the conjugacy classes are parametrized by $\mu_1, \ldots, \mu_b \in \mathfrak{A}$. One says that a parabolic bundle is *semistable* if a certain inequality is satisfied for each reduction of E to a maximal parabolic subgroup (at least, if none of the markings μ_j are contained in the opposite wall of the Weyl alcove, see [19]). On the set of semistable parabolic bundles one defines a *grade equivalence* relation, which equates parabolic bundles if their associated graded bundles are isomorphic. Let $M_\mathbb{C}(\Sigma, G, \mu)$ denote the moduli space of grade-equivalence classes of semistable parabolic bundles. Mehta-Seshadri prove that $M_\mathbb{C}(\Sigma, G, \mu)$ is a normal projective variety homeomorphic to $M(\Sigma, G, \mu)$.

In the case that each conjugacy class μ_j has finite order, one can describe $M_{\mathbb{C}}(\Sigma, G, \mu)$ in terms of equivariant bundles for a finite group action. Let $\tilde{\Sigma} \to \Sigma$ be a totally ramified $\mathbb{Z}_N$-covering, such that the order of each μ_j divides N; if necessary, we can take N infinite. Suppose we are given a $\mathbb{Z}_N$-equivariant holomorphic principal $G_{\mathbb{C}}$-bundle $\tilde{E} \to \tilde{\Sigma}$, with the following property: consider a local trivialization near s_j in which the generator of $\mathbb{Z}_N$ acts by g_j, and suppose that $\exp(\mu_j) = g_j$. At each puncture glue in the trivial bundle $D \times G_{\mathbb{C}}$ over the disk via the transition map

$$(z, g) \mapsto (z, \exp(N \ln(z) \mu_j / 2\pi i) g). \tag{5}$$

The group $\mathbb{Z}_N$ acts freely on the resulting completed bundle and the quotient is a holomorphic principal $G_{\mathbb{C}}$-bundle E over $\overline{\Sigma}$. The bundle E has a parabolic reduction at the fiber s_j, given as the image of $(0, P_j)$ in the local trivialization, where P_j is the parabolic corresponding to ξ_j; this is independent of the local trivializations used above. Mapping $\tilde{E}$ to E defines a correspondence between equivariant bundles on $\tilde{\Sigma}$ and parabolic bundles on Σ, see [16], [19].

3. Vortices with fixed holonomy at infinity

In this section we give a detailed analytical construction of the moduli spaces of vortices with fixed holonomy at infinity.

3.1. Weighted Sobolev spaces.

Let $\overline{\Sigma}$ be a compact, connected surface, and $z_1, \ldots, z_n \in \overline{\Sigma}$ distinct points. Let Σ denote the punctured surface $\overline{\Sigma} \backslash \{z_1, \ldots, z_n\}$. A *cylindrical end* at z_i is a holomorphic, proper embedding $\rho_i : (0, \infty) \times S^1 \to \Sigma$ such that $\lim_{r \to \infty} \rho_i(r, \theta) = z_i$ uniformly in $\theta \in S^1$. We will denote the coordinates on the i-th end by (r_i, θ_i).

Let E be a Euclidean rank r vector bundle over Σ, trivialized along the cylindrical ends. Weighted Sobolev spaces of sections of E are defined as follows. Fix a connection

$$D : \Omega^0(\Sigma, E) \to \Omega^1(\Sigma, E)$$

on E, trivial on the cylindrical ends. Let $r_\Sigma \in C^\infty(\Sigma)$ be a function equal to r_i and supported on each cylindrical end. Let $m > 1$, and let δ be a real number, and $\Omega^0_c(\Sigma, E)$ the space of compactly-supported sections. Define the *δ-weighted (p, m)-Sobolev norm* on $\Omega^0_c(\Sigma; E)$ by

$$\|\xi\|^p_{m,p,\delta} := \int_\Sigma e^{\delta r_\Sigma} \Big(\sum_{|\alpha| \le m} |D^\alpha(\xi^i)|^p \Big)$$

Let $W^p_{m,\delta}(\Sigma, E)$ be the completion of $\Omega^0_c(\Sigma; E)$ with respect to the weighted Sobolev norm $\| \cdot \|_{m,p,\delta}$. These spaces satisfy the following multiplication and embedding theorems, see e.g. [14, Lemma 7.2], [8, Proposition 3.23].

Proposition 3.1.1. (a) *If $m - 2/p \ge m' - 2/q$ and $\delta' < \delta$ then the identity on smooth sections induces a compact embedding $W^p_{m,\delta}(E) \to W^q_{m',\delta'}(E)$.*

(b) *If E, E' are vector bundles over Σ then tensor product on smooth sections induces an embedding*

$$W^p_{m,\delta}(E) \times W^p_{m',\delta'}(E') \to W^p_{m'',\delta''}(E \otimes E') \tag{6}$$

for $\delta'' \geq \delta + \delta'$ and $m + m' > m" + 2/p$.

(c) *The identity induces a compact embedding $W^p_{m,\delta}(E) \to C^0(E)$ if $2/p < m$ and $\delta > 0$.*

(d) *A metric on E and integration over Σ induces a perfect pairing*

$$W^p_{m,\delta}(E) \times W^q_{m,-\delta}(E) \to \mathbb{R}$$

where $1/p + 1/q = 1$.

Applying these constructions to the vector bundle $\Lambda^k T^*\Sigma \otimes E$ we define the weighted Sobolev space

$$\Omega^k(\Sigma, E)_{m,p,\delta} = W_{m,p,\delta}(\Lambda^k T^*\Sigma \otimes E)$$

of k-forms with values in E. If $F : \Omega^{k+l}(\Sigma, E)_{m,2,\delta} \to \Omega^k(\Sigma, E)_{m,2,\delta}$ is a differentiable operator, the formal *weighted adjoint* in the space $W_{m,2,\delta}$ is

$$F^* = e^{-\delta r_\Sigma} F^\star e^{\delta r_\Sigma} \tag{7}$$

where $F^\star$ is the usual adjoint. This follows from the identity

$$\int_\Sigma e^{\delta r_\Sigma} \langle F\phi, \psi \rangle = \int_\Sigma e^{\delta r_\Sigma} \langle \phi, e^{-\delta r_\Sigma} F^\star e^{\delta r_\Sigma} \psi \rangle.$$

A similar analysis gives weighted adjoints for $p > 2$. In this paper we will always use formal weighted adjoints whenever we are using differential operators on weighted Sobolev spaces.

3.2. **Connections.** For similar constructions of moduli spaces see Donaldson [8] and Daskalopoulos-Wentworth [7]. Let P be a G-principal bundle over Σ, equipped with a trivialization over the cylindrical ends. Equip $\mathfrak{g}$ with a G-invariant metric $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, and consider the associated bundle

$$P(\mathfrak{g}) := P \times_G \mathfrak{g}.$$

Let A be an irreducible connection on P. Denote by $\rho_k^* A$ the restriction of A at the k-th cylindrical end. We write

$$\rho_k^* A = \xi_k d\theta, \quad \xi_k \in \mathrm{Map}((0,\infty) \times S^1, \mathfrak{g}).$$

We say that A is *asymptotically constant* if ξ_k is constant on $(R, \infty) \times S^1$ for some $R > 0$. Define the *asymptotic holonomy* $\mu_k \in G$ by

$$\mu_k = \lim_{r \to \infty} \exp(\xi_k(r, \theta)).$$

Consider the space of connections

$$\mathcal{A}(\mu)_{m,p,\delta} := A + \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta}.$$

Using Sobolev multiplication (6), for $\delta > 0$ the curvature defines a smooth map

$$F : \mathcal{A}(\mu)_{m,p,\delta} \to \Omega^2(\Sigma, P(\mathfrak{g}))_{m,p,\delta}, \qquad A \mapsto F_A.$$

Let $\mathcal{G} := \Omega^0(\Sigma, P(G))$ be the group of gauge transformations, where $P(G) = P \times_G G$. We say that $g \in \mathcal{G}$ is *asymptotically constant* if for each k, there exists an r_0 such that $\rho_k^* g(r,\theta)$ is constant for $r > r_0$. We say that the limit $\lim_{r\to\infty} \rho_k^* g(r,\theta)$ is the *asymptotic value* of g. Let $\mathcal{G}(\mu)$ denote the subgroup of $\mathcal{G}$ consisting of asymptotically constant gauge transformations whose asymptotic value on the k-th end centralizes $\exp(\mu_k)$.

Using a faithful representation $G \to \operatorname{Aut}(V)$ we define $\mathcal{G}_{\partial,m+1,p,\delta} = \Omega^0(\Sigma, P(G))_{m+1,p,\delta}$ the group of gauge transformations asymptotically equal to the identity. For $\delta > 0$ and $m > 1$, if $p = 2$ (if $p > 2$ take $m > 2/p$) the set $\mathcal{G}_{m+1,\delta}$ is a Banach Lie group. Charts are constructed using the exponential map $\Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta} \to \Omega^0(\Sigma, P(G))_{m+1,p,\delta}$, and multiplication is well defined and smooth by Sobolev multiplication (6). Similarly, we have a group $\mathcal{G}(\mu)_{m+1,p,\delta}$ whose elements are of the form $g\exp(\xi)$ with $g \in \mathcal{G}(\mu)$ and $\xi \in \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta}$. In order to construct charts for this group, suppose that φ_k is a cutoff function equal to 1 at infinity on each cylindrical end, and supported on that end. Define charts for $\mathcal{G}(\mu)_{m+1,p,\delta}$ by

$$\Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta} \oplus \mathfrak{g}^n \to \mathcal{G}(\mu)_{m+1,p,\delta}, \quad (\xi, \psi) \mapsto g\exp(\xi + \sum_k \varphi_k \psi_k).$$

Then $\mathcal{G}(\mu)_{m+1,p,\delta}$ is a Banach Lie group which fits into an exact sequence

$$1 \to \mathcal{G}_{\partial,m+1,p,\delta} \to \mathcal{G}(\mu)_{m+1,p,\delta} \to G_\mu \to 1$$

where the third map takes asymptotic values on the cylindrical ends. The formula

$$(a_1, a_2) \mapsto \int_\Sigma B(a_1 \wedge a_2). \tag{8}$$

defines a symplectic two-form on $\mathcal{A}(\mu)_{m,p,\delta}$. The group $\mathcal{G}(\mu)_{m+1,p,\delta}$ acts smoothly on $\mathcal{A}(\mu)_{m,p,\delta}$, with moment map given by the curvature. Taking the quotient

$$M(P,\mu) = \mathcal{A}_\flat(\mu)_{m,p,\delta}/\mathcal{G}(\mu)_{m+1,p,\delta}$$

gives another construction of the moduli spaces of flat bundles with fixed holonomy, which were constructed in e.g. Meinrenken-Woodward [17] using Sobolev spaces on surfaces with boundary.

3.3. **Gauged sections.** Let X be a compact, connected Hamiltonian G-manifold with moment map $\Phi : X \to \mathfrak{g}^*$. For each principal G-bundle P on Σ consider its associated bundle

$$\pi : P(X) = (P \times X)/G \to \Sigma.$$

Each connection A on P determines a symplectic connection on $TP(X)$:

$$TP(X) = TP(X)^{\mathrm{hor}} \oplus TP(X)^{\mathrm{vert}} \cong \pi^* T\Sigma \oplus TP(X)^{\mathrm{vert}}.$$

The basic 2-form on the product $P \times X$

$$p_2^* \omega_X + \mathrm{d}(p_1^* A, p_2^* \Phi)$$

descends to a closed 2-form on the associated fiber bundle

$$P(X) = (P \times X)/(pg, x) \sim (p, gx)$$

denoted $\omega_{P(X),A}$ or just by $\omega_{P(X)}$ assuming that the dependency on the connection is understood. The cohomology class $[\omega_{P(X),A}] \in H^2(P(X), \mathbb{R})$ is independent of the choice of A. The moment map $\Phi : X \to \mathfrak{g}^*$ induces a map

$$P(\Phi) : P(X) \to P(\mathfrak{g}^*).$$

Let $\Gamma(\Sigma, P(X))_\infty$ denote the space of asymptotically constant sections of $P(X)$, that is, sections independent of r, θ for r sufficiently large, for each cylindrical end. The tangent space to $\Gamma(\Sigma, P(X))_\infty$ is the space $\Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_\infty$ of asymptotically constant sections of the vertical tangent bundle $\Omega^0(\Sigma; T^{\mathrm{vert}}P(X))$. Let $\Gamma(\Sigma, P(X))_{m,p,\delta}$ denote the space of sections of $P(X)$ of the form $\exp_u(\xi)$ where $u \in \Gamma(\Sigma, P(X))_\infty$ and $\xi \in \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta}$.

Proposition 3.3.1. *For $m > 2/p$ the space $\Gamma(\Sigma, P(X))_{m,p,\delta}$ can be given the structure of a Banach manifold, with the action of the Banach Lie group $\mathcal{G}_{m+1,p,\delta}$.*

Proof. Charts are defined by $\xi \mapsto \exp_u(\iota(\xi))$, where ξ lies in

$$T_u\Gamma(\Sigma, P(X))_{m,p,\delta} := \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta} \oplus \bigoplus_{i=1}^{n} u(z_i)^*T^{\mathrm{vert}}P(X), \tag{9}$$

where $z_i \in \overline{\Sigma}$ is the point at infinity on the i-th cylindrical end, and the embedding ι is defined by

$$\iota(\xi_0, \xi_1, \ldots, \xi_n) = \xi_0 + \sum_{i=1}^{n} \varphi_i \xi_i$$

where φ_i is function equal to 1 in a neighborhood of infinity on the i-th cylindrical end, with support on the cylindrical end. The claim on the action of the gauge group is left to the reader. □

Any section u in the space $\Gamma(\Sigma, P(X))_{m,p,\delta}$ has a limit at infinity on the cylindrical ends and thus extends to a continuous section $\overline{\Sigma} \to \overline{P}(X)$. Moreover, the image of u is contained in a compact set.

Assumption 3.3.2. $\mathrm{Vol}_\Sigma \in \Omega^2(\Sigma)$ is a two-form on Σ with exponential decay on the cylindrical ends, that is, it is given by an expression of the form

$$\mathrm{Vol}_\Sigma(r, \theta) = C_\Sigma e^{-\kappa r} \mathrm{d}r \mathrm{d}\theta \tag{10}$$

on the cylindrical ends, for some $\kappa > 0$ and locally constant function C_Σ, possibly zero. For the most part, we assume that C_Σ is identically equal to 1, to simplify the notation.[†]

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The formula

$$(\xi_1, \xi_2) \mapsto \int_\Sigma u^*\omega_{P(X)}(\xi_1, \xi_2)\,\mathrm{Vol}_\Sigma \tag{11}$$

defines a symplectic form on $\Gamma(\Sigma, P(X))_\infty$, which extends naturally to the completion $\Gamma(P(X))_{m,p,\delta}$ for $m, \delta \geq 0$. The action of $\mathcal{G}_{m+1,p,\delta}$ on $\Gamma(\Sigma, P(X))_{m,p,\delta}$ has generating vector fields given by

$$\xi_{\Gamma(\Sigma,P(X))}(s) = (\xi(s))_X(u(s)).$$

The action is Hamiltonian with moment map given by the pull-back of the moment map times the volume form

$$\Psi : \Gamma(P(X))_{m,p,\delta} \to \Omega^2(\Sigma; P(\mathfrak{g}))_{m,p,\delta}, \quad u \mapsto \mathrm{Vol}_\Sigma\, u^* P(\Phi).$$

Combining the forms of equations (8) and (11) we obtain a 2-form on $\mathcal{A}(\mu)_{m,p,\delta} \times \Gamma(P(X))_{m,p,\delta}$ by

$$\omega_\mu((a_1, \xi_1)(a_2, \xi_2)) = \int_\Sigma B(a_1 \wedge a_2) + u^*\omega_{P(X)}(\xi_1, \xi_2)\,\mathrm{Vol}_\Sigma\,. \tag{12}$$

Non-degeneracy of the inner product and symplectic form on X imply that this form is also symplectic. The action of $\mathcal{G}(\mu)_{m+1,p,\delta}$ on $\mathcal{A}(\mu)_{m,p,\delta} \times \Gamma(P(X))_{m,p,\delta}$ is Hamiltonian respect of the form ω_μ and with moment map

$$\mathcal{A}(\mu)_{m,p,\delta} \times \Gamma(P(X))_{m,p,\delta} \to \Omega^2(\Sigma; P(\mathfrak{g}))_{m-1,\delta}\ ,\ (A, u) \mapsto F_A + \mathrm{Vol}_\Sigma\, u^*P(\Phi). \tag{13}$$

We take the *energy* of a pair (A, u) to be

$$E(A, u) = \frac{1}{2} \int_\Sigma \left(C_\Sigma e^{\kappa r_\Sigma} |\mathrm{d}_A u|^2 + C_\Sigma^2 e^{2\kappa r_\Sigma} |F_A|^2 + |u^*P(\Phi)|^2 \right) \mathrm{Vol}_\Sigma\,.$$

Note that the energy is always non-negative. On the cylinders, the energy is given by the integral of the energy density function

$$\chi(r, \theta) = \frac{1}{2}(|\mathrm{d}_A u|^2 + C_\Sigma^{-1} e^{\kappa r} |F_A|^2 + C_\Sigma e^{-\kappa r} |u^*P(\Phi)|^2) \tag{14}$$

with respect to the standard metric on the cylindrical ends. Note in particular that the term $u^*P(\Phi)$ is not required to go to zero at infinity.

The *action* of a pair (A, u) is by

$$D(A, u) := \int_\Sigma u^*(\omega_{P(X),A}) = ([\Sigma], u^*[\omega_{P(X),A}])$$

where the last expression denotes the pairing of $H_2(\Sigma, \partial\Sigma; \mathbb{Z})$ with $H^2(\Sigma, \partial\Sigma)$. It is somewhat awkward to phrase this as a pairing of a *degree* of u and we will avoid doing so.

3.4. **Almost complex structures.** We now introduce a suitable class of almost complex structures on $P(X)$. Let j denote a complex structure on Σ, we say that an almost complex structure I on $P(X)$ is $\omega_{P(X)}, j$-*compatible* if

(a) the projection $\pi : P(X) \to \Sigma$ is (I, j)-holomorphic,
(b) $I|_{\pi^{-1}(z)}$ is $\omega_{P(X)}|_{\pi^{-1}(z)}$-compatible on each fiber $\pi^{-1}(z)$,
(c) I is asymptotically constant on each cylindrical end.

Let $\mathcal{J}(P(X))$ be the space of $\omega_{P(X)}, j$-compatible almost complex structures. We have a canonical map

$$\mathcal{A}(\mu) \times \mathcal{J}(X)^G \to \mathcal{J}(P(X)), \quad (A, J_X) \mapsto J_A \tag{15}$$

given by defining J_A on the horizontal space by j and on the vertical space by J_X. This map extends to suitable Sobolev completions as follows. Let $\mathcal{J}(P(X))_\infty$ denote the space of $\omega_{P(X)}, j$-compatible almost complex structures locally of Sobolev class m, p, and asymptotically constant in a neighborhood of infinity. Given $J_0 \in \mathcal{J}(P(X))_\infty$, the space of compatible almost complex structures near J_0 is a smooth manifold with tangent space $T_{J_0}\mathcal{J}(P(X))$ the set of $\Delta J \in \mathrm{End}(TX)$ such that

$$(\Delta J)J_0 + J_0(\Delta J) = 0, \quad \pi \circ (\Delta J) = 0, \qquad \omega((\Delta J)\cdot, J_0\cdot) = -\omega(J_0\cdot, (\Delta J)\cdot).$$

Choose a local diffeomorphism

$$\exp_{J_0} : B_\epsilon(0, T_{J_0}\mathcal{J}(P(X))) \to \mathcal{J}(P(X)).$$

Let $\mathcal{J}(P(X), J_0)_{m,p,\delta}$ be the subspace of $\mathcal{J}(P(X))_{m,p,\mathrm{loc}}$ such that $J = \exp_{J_0}(\Delta I)$ for some section ΔI of Sobolev class (m, p, δ) near infinity. We have a continuous map from $\mathcal{J}(P(X), J_0)_{m,p,\delta}$ to $\mathcal{J}(P(X))_{m,p,\mathrm{loc}}$.

Lemma 3.4.1. *The map $(A + a, J) \mapsto J_{A+a}$ of* (15) *extends naturally to a map*

$$\mathcal{A}(\mu)_{m,p,\delta} \times \mathcal{J}(X)^G \to \mathcal{J}(P(X), J_A)_{m,p,\delta}.$$

† †review

Proof. To see that the image has the claimed regularity, note that

$$T_{[p,x]}P(X) = (T_pP \oplus T_xX)/\mathfrak{g}.$$

The splitting induced by a connection $A \in \Omega^1(P, \mathfrak{g})^G$ is

$$T_{[p,x]}P(X) \to T_{\pi(p)}\Sigma \oplus T_xX, \quad [w, v] \mapsto (D\pi(w - A(w)), A(w)_X(x) + v).$$

Hence J_A is given by

$$J_A[w, v] = [D\pi^{-1}j(D\pi)w - A(D\pi^{-1}j(D\pi)w), J(A(w)_X(x) + v)]. \tag{16}$$

The connection enters as a multiplication operator, and the claimed regularity follows from the Sobolev multiplication theorem. Given a base connection A_0 with limiting holonomy μ, and an element $A \in \mathcal{A}(\mu)_{m,p,\delta}$ sufficiently close to A_0, we have $J_A \in \mathcal{J}(P(X), J_{A_0})_{m,p,\delta}$, again using the formula (16). □

Given $J_X \in \mathcal{J}(X)^G$, for each section $u \in \Omega^0(\Sigma; P(X))_{m,p,\delta}$ define the operator

$$\bar{\partial}_{J_A}(u) := \frac{1}{2}(\mathrm{d}u + J_A(u) \circ \mathrm{d}u \circ j) \in \Omega^{0,1}(\Sigma; u^*T^{\mathrm{vert}}P(X))_{m-1,\delta}.$$

By [5, Proposition 3.1] we have an *energy-action relation*

$$E(A, u) = D(A, u) + \int_\Sigma |\overline{\partial}_{J_A} u|^2 e^{\kappa r_\Sigma} \,\mathrm{Vol}_\Sigma + \frac{1}{2} \int |F_A / \mathrm{Vol}_\Sigma + u^*\Phi|^2 \,\mathrm{Vol}_\Sigma, \tag{17}$$

where by abuse of notation $F_A/\operatorname{Vol}_\Sigma$ is the $\mathfrak{g}$-valued function φ such that $\varphi \operatorname{Vol}_\Sigma = F_A$. We say $u : \Sigma \to P(X)$ is (J_A, j)-*holomorphic* (or just *holomorphic*) if $\bar{\partial}_{J_A}(u) = 0$. The symplectic vortex equations (1) imply that the pair (A, u) minimizes the energy among pairs (A', u') with u' homotopic to u.

3.5. **Moduli space of vortices.** The moduli space of (finite energy) symplectic vortices can be defined by a symplectic quotient construction as follows. Define

$$\mathcal{A}(P, X, \mu)_{m,p,\delta} := \{(A, u) \in \mathcal{A}(\mu)_{m,p,\delta} \times \Gamma(P(X))_{m,p,\delta}, \quad \bar{\partial}_{J_A}(u) = 0, \quad E(A, u) < \infty\}.$$

In Section 4, we show that if a certain linearized operator is surjective then $\mathcal{A}(P, X, \mu)_{m,p,\delta}$ is a Banach manifold. The group of gauge transformations $\mathcal{G}(\mu)_{m+1,p,\delta}$ acts smooth on $\mathcal{A}(P, X, \mu)_{m,p,\delta}$. That is, u is (J_A, j)-holomorphic if and only if $g \cdot u$ is $(J_{g \cdot A}, j)$-holomorphic. Furthermore $\mathcal{A}(P, X, \mu)_{m,p,\delta}$ is a fiber bundle over the affine space of connections with fixed holonomy $\mathcal{A}(\mu)_{m,p,\delta}$ with fiber at A given by the set of J_A-pseudoholomorphic sections. The gauge group $\mathcal{G}(\mu)_{m+1,p,\delta}$ acts on $\mathcal{A}(P, X, \mu)_{m,p,\delta}$, with moment map given by the restriction of the product (13). The symplectic quotient

$$M(P, X, \mu)_{m,p,\delta} := \mathcal{A}(P, X, \mu)_{m,p,\delta} /\!\!/ \mathcal{G}(\mu)_{m+1,p,\delta} \tag{18}$$

is the *moduli space of vortices with fixed holonomy.* More generally, denote by

$$\mathscr{E}_{m,p,\delta} \to \mathcal{A}(\mu)_{m,p,\delta} \times \Gamma(\Sigma, P(X))_{m,p,\delta} \tag{19}$$

the vector bundle whose fibre at u is

$$\mathscr{E}_{m,p,\delta,u} := \Omega^2(\Sigma, P(\mathfrak{g}))_{m-1,\delta} \times \Omega^{0,1}(\Sigma; u^* T^{\mathrm{vert}} P(X))_{m-1,\delta}.$$

The moment map and holomorphic condition define a section of $\mathscr{E}_{m,p,\delta}$

$$\mathcal{F} : \quad (A, u) \mapsto (F(A) + \operatorname{Vol}_\Sigma u^* P(\Phi), \bar{\partial}_{J_A}(u)). \tag{20}$$

Then we have

$$M(P, X, \mu)_{m,p,\delta} = \mathcal{F}^{-1}(0) / \mathcal{G}(\mu)_{m+1,p,\delta}.$$

For future reference, we introduce

$$N(P, X, \mu)_{m,p,\delta} := \mathcal{F}^{-1}(0) / \mathcal{G}_{\partial,m+1,p,\delta},$$

the moduli space of vortices framed at infinity. Thus $N(P, X, \mu)_{m,p,\delta}$ fits into a fibration

$$G_\mu \to N(P, X, \mu)_{m,p,\delta} \to M(P, X, \mu)_{m,p,\delta}$$

where G_μ is the product of stabilizers of the holonomies in (2).

3.6. **Correspondence with parabolic vortices.** Suppose that X is a $G_{\mathbb{C}}$-variety. In this section we describe the holomorphic objects corresponding to vortices. First suppose that Σ is closed. A *holomorphic gauged map* from Σ to X is a pair (E, u) consisting of a holomorphic principal $G_{\mathbb{C}}$-bundle $E_{\mathbb{C}} \to \Sigma$ together with a holomorphic section $u : \Sigma \to E_{\mathbb{C}}(X)$. In the language of stacks, this is nothing but a holomorphic map from Σ to the quotient stack $X/G_{\mathbb{C}}$. The correspondence between connections on a given principal G-bundle $E \to \Sigma$ and holomorphic structures on its complexification extends to a correspondence between vortices and holomorphic gauged maps with underlying bundle E.

Suppose that Σ has cylindrical ends. Let $\overline{\Sigma}$ denote the associated closed surface, obtained by adding points at infinity $s_1, \ldots, s_n$.

Definition 3.6.1. A *parabolic holomorphic map* from Σ to X consists of a parabolic bundle $\overline{E} \to \overline{\Sigma}$ (see Section 2) together with a holomorphic section $u : \overline{\Sigma} \to \overline{E}(X)$.

There is no condition on the section at the marked points, as there is in the case of parabolic Higgs bundles. The latter corresponds, up to a twisting, with the case that X is a vector space, which is non-compact. Given a connection on a principal G-bundle $E \to \Sigma$ with fixed holonomies $\mu_1, \ldots, \mu_n$ around the ends, one obtains a parabolic $G_{\mathbb{C}}$ bundle $E_{\mathbb{C}} \to \Sigma$ by assigning to each fiber $(E_{\mathbb{C}})_{s_i}$ at infinity the parabolic reduction determined by μ_i, see e.g. [19], by gluing in trivial bundles using the twistings (5). By removal of singularities, a holomorphic section $u : \Sigma \to E_{\mathbb{C}}(X)$ of finite energy extends automatically to a section $\overline{u} : \overline{\Sigma} \to \overline{E}_{\mathbb{C}}(X)$. The value of $\overline{u}$ at the points $s_1, \ldots, s_n$ is described as follows. Let $B \subset G_{\mathbb{C}}$ denote the Borel subgroup whose Lie algebra contains the positive root spaces. Let P_{μ_j} be the parabolic determined by μ_j,

$$P_{\mu_j} = \{ g \in G_{\mathbb{C}} | \lim_{t \to \infty} \mathrm{Ad}(\exp(t\mu_j)) g \in B \}$$

whose Lie algebra is that of B plus the sum of negative root spaces vanishing on μ_j. The map $x \mapsto \exp(t\mu_j)x$ restricts on each P_{μ_j}-orbit on X to a retraction of $P_{\mu_j}x$ to $P_{\mu_j}x \cap X^{\mu_j}$; in particular, each P_{μ_j} orbit contains an element of X^{μ_j}. We write X^{μ_j} in terms of components.

$$X^{\mu_j} = \bigcup_k X_k^{\mu_j}, \qquad X = \bigcup_k P_{\mu_j} X_k^{\mu_j}$$

where $X_k^{\mu_j}$ are the connected components of X^{μ_j}. In the trivializations at the punctures, the section $\overline{u}$ is given by

$$\exp(N \ln(z) \mu_j / 2\pi i) u(z), z \neq 0.$$

Thus $u(0)$ lies in $X_k^{\mu_j}$ if and only if $\overline{u}(0)$ takes values in $P_{\mu_j} X_k^{\mu_j}$, that is, the limit of $\overline{u}(0)$ under the flow defined by μ_j is $u(0)$. This shows

Theorem 3.6.2. *There exists a one-to-one correspondence between pseudoholomorphic gauged maps (E, A, u) with holonomies μ_j and limits along the j-th cylindrical end in $X_k^{\mu_j}$ and parabolic holomorphic maps $(\overline{E}_{\mathbb{C}}, \overline{u})$ with $\overline{u}(s_j)$ in $P_{\mu_j} X_k^{\mu_j}$.*

We say that a parabolic holomorphic map is *semistable* if it is equivalent to a pseudoholomorphic gauged map satisfying the vortex equation (1). Note this depends on the choice of volume form Vol_Σ. In particular, if we choose Vol_Σ identically zero, then the stable parabolic holomorphic maps are those whose underlying parabolic bundles are parabolic semistable, by the Mehta-Seshadri theorem [16]. It would be interesting to investigate the Hilbert-Mumford criterion for stability more generally in this context; Mundet (unpublished) has described an answer in the circle case.

4. Fredholm theory and transversality

4.1. **Exponential decay on the cylinders.** First we show that any finite energy vortex has exponential decay along the cylindrical ends. This is a somewhat standard matter, but we need the details in order to specify our Sobolev weights. For the moment we will consider only vortices of Sobolev class $W_{m,p,loc}$. We will show that vortices of finite energy lie in a weighted Sobolev space for a properly chosen weight. The next lemma follows directly from the vortex equations (1) and compactness of X.

Lemma 4.1.1 (Exponential decay for curvature)**.** *If (A,u) is a vortex of class $W_{m,p,loc}, p > 1, m \geq 1$, then there is a constant C such that on the cylindrical ends the curvature decays exponentially in the form $|F_A| \leq Ce^{-\kappa r}$, for all r.*

Now we describe the exponential decay behaviour of the covariant derivative $\mathrm{d}_A u$ on the cylindrical ends. Similar results but with different hypotheses have been carefully studied in Ziltener [22, Theorem 1] by using the methods of [11].

Theorem 4.1.2 (Exponential Decay for sections)**.** *For any finite energy vortex (A,u) of class $W_{m,p,loc}, p > 1, m \geq 1$ there are positive constants k, r_0, C such that*

$$|d_A u(r,\theta)| < Ce^{-kr}$$

for $r > r_0$. In particular, the limit $\lim_{r\to\infty} u(r,\theta)$ is covariant constant and it lies in the fixed point set of the holonomy of A.

Proof. Since we are working on a cylinder, after gauge transformation we may assume that the connection A is in temporal gauge, thus $A = \alpha d\theta$. By Lemma 4.1.1 $|d_A\alpha(r,\theta)| \leq Ce^{-\kappa r}$, and thus by integration we can write $\alpha = \alpha_0 + a$ where α_0 is r independent and a has exponential decay $|a| \leq Ce^{-\kappa r}$ for some constant C. Consider $A_0 = \alpha_0 \mathrm{d}\theta$ as the base connection. The vortex equations for (A,u) are given by

$$\partial_r \alpha + e^{-\kappa r}\Phi(u) = 0, \quad \partial_r u + J(\partial_\theta u + \alpha_X(u)) = 0$$

and thus the norm of the derivative $\frac{1}{2}|d_A(u)|$ equals $|\partial_\theta u + \alpha_X(u)| = |\partial_r u|$. Using the connection A_0 one can trivialize the bundle P over the cylinder. Respect to this trivialization the section u yields a pseudo-holomorphic map $\tilde{u} : S^1 \times \mathbb{R} \to X$ that is periodic up to holonomy, i.e

$$\tilde{u}(\theta + 2\pi, r) = g\tilde{u}(\theta, r),$$

for some g in the holonomy group of A. The standard exponential decay result for pseudo-holomorphic maps also holds for $\tilde{u}$ see Proposition 4.1.4 below. Thus for positive constants r_0, C, k', the exponential decay estimate

$$|\mathrm{d}_{A_0} u| = |\mathrm{d}\tilde{u}| \leq Ce^{-k'r}$$

holds for all $r > r_0$. The derivatives $\mathrm{d}_A u$ and $\mathrm{d}_{A_0} u$ differ by a term containing a which also decays exponentially as $e^{-\kappa r}$. Therefore we have that

$$|\mathrm{d}_A u| \leq Ce^{-kr},$$

for some $C > 0$, all $r > r_0$ and $k = \min\{k', \kappa\}$. This shows that up to gauge transformation the exponential decay holds, but since $\mathrm{d}_{g \cdot A} g \cdot u = g^* \mathrm{d}_A u$, and the metric is gauge invariant, we obtain the desired estimate. □

For future reference, we now state a similar result for *flat* vortices on cylinders.

Proposition 4.1.3. *Let (A, u) denote a finite energy vortex on the cylinder $\mathbb{R} \times S^1$ of class $W_{m,p,loc}, p > 1, m \geq 1$, such that $F_A = 0$. Then, there are positive constants k, r_0, C such that*

$$|d_A u(r, \theta)| < Ce^{-k|r|}$$

for $|r| > r_0$. In particular, the limits $\lim_{r \to \pm\infty} u(r, \theta)$ lie in the fixed point set of the holonomy subgroup of A.

Proof. Up to gauge transformation we can write $A = A_0 + a\mathrm{d}\theta, a \in \mathfrak{g}$ where the base connection A_0 is flat and constant in r. Since A is flat, a is just an element $a \in \mathfrak{g}$. The proof of the statement is now identical to that of Theorem 4.1.2 using the finite energy condition of the pair (A, u). □

Proposition 4.1.4. *There are constants r_0, C, k' so that for each lift $\tilde{u} : S^1 \times \mathbb{R} \to X$ associated to u as above one has $|du(r, \theta)| \leq Ce^{-k'r}$ for all $r > r_0$.*

The proof is almost the same as the proof of the usual exponential decay as in [15, Lemma 4.7.3] but one needs to verify that there is an isoperimetric inequality and a well defined local action for paths that are closed up to the action of an element $g \in G$, as we now explain. A path $\gamma : [0, 2\pi] \to X$ is twisted by $g \in G$ if $\gamma(2\pi) = g\gamma(0)$. Let $\epsilon > 0$ and assume $\gamma : [0, 1] \to X$ is a path so that its length satisfies $\ell(\gamma) < \epsilon$. If ϵ is small enough, the path is contained in a contractible ball $U \subset X$ whose radius is smaller than ϵ_0, the injectivity radius. Moreover, if there is a point $x_0 \in X^g \cap U$, define the *extension* of γ by

$$u_\gamma : [0, 2\pi] \times [0, 1] \to X$$
$$u_\gamma(\theta, r) = \exp_{x_0}(r\xi(\theta))$$

where the vector field ξ is given by the equation $\gamma(\theta) = \exp_{\gamma(0)}(\xi(\theta))$. Then, define the local action of γ as

$$a(\gamma) := -\int u_\gamma^* \omega.$$

The action can only be defined if such point x_0 exists, but it does not depend on its choice, if γ is short enough. Suppose v_γ is the extension associated to a different point, then the connected sum $u := u_\gamma \# v_\gamma$ is contained in a contractible neighborhood, and thus the integral of u vanishes.

We have the following *isoperimetric inequality* for twisted paths.

Lemma 4.1.5. *Let ϵ_0 be the injectivity radius and γ be a path in X twisted by g. Suppose further that there is a fixed point $x_0 \in X^g$ within ϵ_0 of $\gamma(0)$. Then, for every constant $C > 1/4\pi$ there is an $\epsilon > 0$ smaller than ϵ_0 so that if $\ell(\gamma) < \epsilon$ then $|a(\gamma)| \leq C(\ell(\gamma))^2$.*

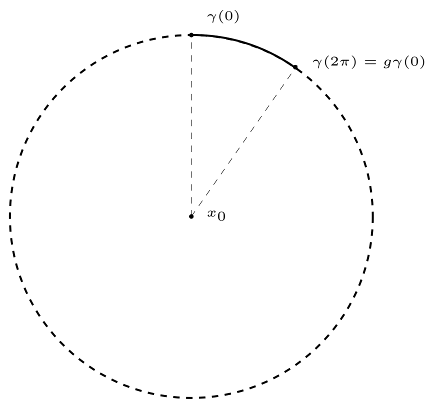

FIGURE 1. Twisted path and its extension. If g has finite order n, the path γ can be replaced by a loop.

Proof. It is not hard to see that this inequality holds for *some* constant C. Take ϵ smaller than half of the injectivity radius of X and let u_γ be its extension as defined above. We have $|\partial_r u_\gamma| = |\xi(\theta)| \leq \ell(\gamma)$ and $|\partial_\theta u_\gamma| \leq C|\dot{\xi}(\theta)| \leq C|\dot{\gamma}(\theta)|$ for some constant $C > 0$. Integrating, we get

$$|a(\gamma)| = |\int_0^{2\pi} \int_0^1 \omega(\partial_r u_\gamma, \partial_r u_\gamma)| \leq C'(\ell(\gamma))^2.$$

The proof of the general statement is done over a Darboux chart. It follows from the equivalent result on symplectic vector spaces. The arguments are a verbatim copy of the proof of Theorem 4.4.1 in [15]. □

Recall the mean value inequality for pseudoholomorphic maps:

Lemma 4.1.6 (Lemma 4.3.1 [15])**.** *There is a constant $\varepsilon_0 > 0$ such that if $r > 0$ and $u : B_r(0) \to X$ is a pseudo-holomorphic map on the ball of radius r in $\mathbb{C}$, then*

$$\int_{B_r(0)} |du|^2 \leq \varepsilon \quad \Rightarrow |du(0)|^2 \leq \frac{8}{\pi r^2} \int_{B_r(0)} |du|^2.$$

Proof of Proposition 4.1.4. For $r_0 > 0$ define the function $f(\tau)$ to be the energy of the map $\tilde{u}$ on the cylinder $[r_0, \infty) \times S^1$

$$f(\tau) := E(\tilde{u}; [\tau, \infty) \times S^1) = \frac{1}{2} \int_\tau^\infty \int_{S^1} |d\tilde{u}(r, \theta)|^2 \mathrm{d}r\mathrm{d}\theta. \tag{21}$$

Then

$$\lim_{\tau \to \infty} f(\tau) = 0.$$

Thus, by improving the constant r_0 if necessary, we can assume that the energy of u is smaller than the constant ε in the Mean Value Inequality, so we have for some $C > 0$

$$|d\tilde{u}(r, \theta)|^2 \leq CE(\tilde{u}; [r_0, \infty) \times S^1).$$

Let $\gamma_r(\theta) := \tilde{u}(r,\theta)$. We can apply lemma Lemma 4.1.5 to γ_r. To see this, note that the value of $\tilde{u}$, as r goes to infinity, approaches a point x_0 in the fixed point set X^g. Thus for r_0 big enough, $\gamma_r(0)$ is in an ϵ_0 neighborhood of a fixed point set x_0. By the mean value estimate above with a fixed radius from the point (τ,θ), $|\dot{\gamma}_\tau| \leq c_1 E(\tilde{u};[r_0,\infty)\times S^1)$, for a constant c_1 and thus $\ell(u)^2 < c_1 E(\tilde{u};[r_0,\infty)\times S^1)$. Therefore, by improving r_0 one can assume that γ_r has length shorter than the ϵ constant of Lemma 4.1.5. Thus for for all $r > r_0, |a(u_\gamma)| \leq c\ell(\gamma_r)^2$. The energy $E(\tilde{u},[r,\infty)\times S^1)$ agrees with the action $a(\gamma_r)$. To see this, consider the extension u_{γ_r} of the loop γ_r and use it to cap $\tilde{u}$. Since this resulting map has finite energy, we can extend this map to the a very small sphere that lies in a contractible chart. Integrating the symplectic form over this sphere one obtains that $E(\tilde{u},[r,\infty)\times S^1) - a(\gamma_r) = 0$. Putting all these estimates together we get

$$\begin{aligned} f(r) &= a(u_r) \leq c\ell(\gamma_r)^2 \\ &\leq 2\pi c \int_{S^1} |\dot{\gamma}_r(\theta)|^2 \mathrm{d}\theta = 2\pi c \int_{S^1} |\partial_r \tilde{u}(r,\theta)|^2 \mathrm{d}\theta \\ &= -k' f'(r) \end{aligned}$$

for a positive constant k' which is independent of $\tilde{u}$. The differential inequality $f'(\tau) \leq -k' f(\tau)$ yields the estimate

$$|\mathrm{d}\tilde{u}(\tau,\theta)| \leq f(\tau) \leq Ce^{-k'\tau}$$

for some $C > 0$. □

We can now state the following proposition.

Proposition 4.1.7. *There is a constant $\delta_0 < \kappa$ such that for all vortices (A,u) of class $W_{m,p,loc}$ with finite energy, there exists a gauge transformation g of class $W_{m+1,p,loc}$ such that $(g\cdot A, g\cdot u)$ is of class $W_{m,p,\delta}$ for all weights $\delta < \delta_0$.*

Proof. Let δ be a positive weight. Assuming δ smaller than the constants κ, k of Lemma 4.1.1 and Theorem 4.1.2, the weighted Sobolev norm $\|(A,u)\|_{1,p,\delta}$ is finite. Assume that A is actually smooth, otherwise we can replace it by a nearby smooth connection. Let D be the linearization of the Cauchy-Riemann operator $\partial_r u + J(\partial_\theta u + \alpha_X(u))$ and D_0 the respective one by replacing A by A_0. For $k < m$, D_0 is a compact perturbation of a translation invariant elliptic operator $W_{k,p,\delta} \to W_{k-1,p,\delta}$ with smooth coefficients and since α and α_0 differ by a term that decays faster than δ, $D : W_{k,p,\delta} \to W^p_{k-1,\delta}$ is also elliptic . The regularity estimate for elliptic differential operators also holds in weighted Sobolev spaces [14, Equation 2.4], thus we have for some constant $C > 0$

$$\|u\|_{W^p_{k+1,\delta}} \leq C(\|Du\|_{W^p_{k-1,\delta}} + \|u\|_{W^p_{k,\delta}}),$$

and thus u is also in the $W_{2,p,\delta}$. The elliptic estimate for the curvature also holds. □

4.2. **Generic condition on the holonomies.** In this paper we only consider generic conditions on the holonomies. Suppose the case when there are only three cylindrical ends. Assume that a vortex (A, u) with prescribed holonomy (μ_1, μ_2, μ_3) is fixed by a one-parameter subgroup generated by ζ. Under this conditions the structure group reduces to G_ζ, then the vortex obtained by the induced projection onto this one-parameter subgroup is a-$U(1)$ vortex. It is not hard to see that the holonomies arround the punctures satisfy the relation

$$\mathrm{Hol}(\mu_1)\,\mathrm{Hol}(\mu_2) = \mathrm{Hol}(\mu_3)\exp(\int_\Sigma F_A).$$

Now, the curvature can be written as $F_A = \xi\,\mathrm{Vol}_\Sigma$ for some fixed vector ξ which is the image of a fixed point of X under the map (Φ, ζ). Therefore, the possible values of $\int_\Sigma F_A$ cannot be continuous, that is they are a discrete set. This puts a constraint on μ_3 which is a non-generic condition. By choosing the holonomies generically then this constraint cannot be satisfied.

We then make the following assumption.

Assumption 4.2.1. For this paper the holonomy for any vortex is assumed to be generic.

4.3. **Fredholm theory.** We introduce Hamiltonian perturbations of the vortex equations as follows. Let $\mathcal{D}(P(X)) \to \Sigma$ denote the bundle whose fibre at $s \in \Sigma$ is the space of compactly supported functions $C_c^\infty(P(X)_s)$. Let

$$\mathcal{H}(P(X)) = \mathrm{Map}_\mathcal{G}(\mathcal{A}(P,\mu), \Omega^1_c(\Sigma, \mathcal{D}(P(X)))$$

denote the space of 1-forms with compact support on Σ and values in $\mathcal{D}$, depending equivariantly on a connection $A \in \mathcal{A}(P,\mu)$, and on the cylindrical ends asymptotically equal to $H_i \mathrm{d}\theta_i$, for some functions $H_1, \dots, H_n \in C^\infty(X)$. Every $H_A \in \mathcal{H}(P(X))$ defines a 1-form σ_H on $P(X)_s$ given by

$$\sigma_{H_A}(x, v) = H_A(d\pi_x(v), x), v \in T_x(P(X))$$

Let X_{H_A} be the vector field on $P(X)$. By perturbing the symplectic connection $\omega_{P(X)}$ by $\omega_{H_A} = \omega_{P(X)} - d\sigma_{H_A}$ we get the equations

$$F_A + \mathrm{Vol}_\Sigma\, u^*\Phi = 0, \quad \bar\partial_{J_A} u - X_{H_A}(u)^{0,1} = 0, \tag{22}$$

which give a section of the bundle (19). Since $d\pi$ vanishes on vertical vectors, $\omega_{P(X),H_A}$ agrees with $\omega_{P(X)}$ on $TP(X)^{\mathrm{vert}}$. The solutions of (22), are holomorphic with respect to a new almost complex structure $J_{A,H_A} \in \mathcal{J}(P(X), \Omega_H)$, given by

$$J_{A,H_A} v = J_A v + J_A^{\mathrm{vert}} \circ X_{H_A \circ d\pi(v)} - X_{H_A \circ j \circ d\pi(v)}$$

Thus, the solutions of (22) are the solutions of the Cauchy-Riemann equation $\bar\partial_{J_{A,H_A}}(u) = 0$, which is a compact perturbation of $\bar\partial_{J_A}(u) = 0$. With respect to the connection induced by A, J_{A,H_A} has the matrix form

$$J_{A,H_A} = \begin{pmatrix} j & 0 \\ J \circ X_{H_A \circ d\pi(\cdot)} - X_{H_A \circ j \circ d\pi(\cdot)} & J \end{pmatrix} \tag{23}$$

therefore it is not compatible since the (j, J)-anti-linear term $J \circ X_{H_A} - X_{H_A} \circ j$ is not zero. Let $\mathcal{S}_H$ be the operator given by the perturbed vortex equations

$$\mathcal{A}(\Sigma,\mu)_{m,p,\delta} \times \Gamma(\Sigma, P(X))_{m,p,\delta} \to \Omega^2(\Sigma, P(\mathfrak{g}))_{m-1,\delta} \times \Omega^{0,1}(\Sigma; u^*T^{\mathrm{vert}}P(X))_{m-1,\delta} \tag{24}$$
$$(A,u) \mapsto (*F_A + u^*\Phi, \bar\partial_{J_{A,H_A}}(u)),$$

where (A, u) is a pair of class (m, δ). Recall that we are using a metric with finite volume form $\mathrm{Vol}_\Sigma = *1$, thus the star operator $*$ is well defined. Let $M(\Sigma, G, \mu, X, J, H)_{m,p,\delta}$ denote the moduli space of pairs satisfying (24) modulo gauge transformations $\mathcal{G}_{m+1,p,\delta}$. The linearization of the operator $\bar\partial_{J_{A,H_A}}(u)$, is

$$D^H : \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta} \to \Omega^{0,1}(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m-1,\delta}$$

constructed as follows. For any pair (A, u) of a connection and a section of the bundle $P(X)$ the map

$$\mathcal{F}_{(A,u,H)} : \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta} \to \Omega^{0,1}(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m-1,\delta}$$
$$\mathcal{F}_{(A,u,H)}(\xi) := \Psi_u(\xi)^{-1}\overline{\partial}_{J,H_A}(\exp_u(\xi))$$

where Ψ_u and $\exp_u$ are the parallel transport and exponential map, both along vertical geodesics on fibres using the vertical connection

$$\tilde\nabla_v := \nabla_v - \frac{1}{2} J_{A,H}(\nabla_v J_{A,H}),$$

obtained from the vertical Levi-Civita connection ∇. In this way $\tilde\nabla$ does preserves $J_{A,H}$ as well as the metric. Now, the operator D^H is given by $D^H := d\mathcal{F}_{(A,u,H)}(0)$. The operator D^H can be written as the sum of two terms

$$(a, \xi) \mapsto D_{u,J_{A,H_A}}(\xi) + D_2(a)$$

corresponding to the partial derivatives. The operator $D_{u,J_{A,H_A}}$ is the linearization of the Cauchy-Riemann operator with fixed almost complex structure J_{A,H_A}. The second term is the partial derivative respect to the variation of the operator $\mathcal{F}$ in the direction a. Using the expression (16) for the almost complex structure J_{A,H_A} we see that the term $D_2(a)$ is the $(0,1)$-component of the Lie derivative $L_u(a)$, given by the action of a along u. The linearized operator at (A, u) associated to the vortex equations is thus given by

$$D\mathcal{S}_{(A,u)} : \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta} \to \Omega^2(\Sigma, P(\mathfrak{g}))_{m-1,\delta} \times \Omega^{0,1}(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m-1,\delta}$$
$$(a,\xi) \mapsto (*d_A(a) + L_\xi u^*\Phi, D_{u,J_A,H_A}\xi + L_u^{0,1}(a)). \tag{25}$$

Where the first term is just the linearization of $*F_A + u^*\Phi = 0$. Using the equations for the slice (27) we obtain a linear slice for the $\mathcal{G}$ action on the space $\mathcal{A}(P, X, \mu)$. Let $\tilde D_{A,u}$ be the operator obtained from $D\mathcal{S}_{A,u}$ by adding the slice condition,

$$\tilde D_{A,u} : (a, \xi) \mapsto (*d_A a + L_\xi u^*(\Phi), D_{u,J_A,H_A}\xi + L_u^{0,1}(a), -d_A^* a + L_{J\xi}(u^*\Phi)). \tag{26}$$

Lemma 4.3.1. *$\tilde D_{A,u}$ is Fredholm.*

Proof. The terms involving the Lie derivatives are of lower order, and thus they can be ignored. We can assume that $D_A := D_{u,J_A,H_A}$ is the operator associated to the base connection A_0 since the difference $D_A - D_{A_0}$ is a compact operator. Again, $L_u^{0,1}(a)$ is compact thus $D_{u,J_A,H_A}\xi + L_u^{0,1}(a)$ is a lower order perturbation of the operator D_{A_0}. Thus, it is enough to show that the map

$$(a,\xi) \to (d_A \oplus d_A^* a,\ D_{A_0}(\xi))$$

is Fredholm. To see this, take the weight δ_0 smaller than any element in the spectrum of the limiting operators. If $\delta < \delta_0$ then $d_A \oplus d_A^*$ and D_{A_0} are both Fredholm, since the second factor in (9) is finite dimensional. Thus $\tilde{D}_{(A,u)}$ is Fredholm. □

4.4. **Transversality.** In order to apply the implicit function theorem, we need to show that we have a Banach space of Hamiltonian perturbations. One can either use Floer's C^∞_ϵ topology, or use the C^k topology with an appropriate weight function. For the latter let $\phi : \Sigma \to \mathbb{R}$ be a positive function approaching infinite on each cylindrical end, and let C^k_ϕ be the subspace of $f \in C^k_{\text{loc}}$ such that the norm

$$\|f\|^2_{\phi,k} = \sum_{j=0}^{k} \sup_{s\in\Sigma} \|\phi(s)(D_j f)(s)\|^2$$

is finite. Let $\Omega^1(\Sigma, C^k_\phi(P(X)))$ denote the space of one-forms taking values at s in $C^k(P(X)_s)$. Let $\mathcal{H}(P(X))_l$ denote the space of $\mathcal{G}(\mu)_{m+1,p,\delta}$-equivariant maps from $\mathcal{A}(P,\mu)_{m,p,\delta}$ to $\Omega^1(\Sigma, C^k(P(X)))$ that are class C^l. We define on $\mathcal{H}(P(X))_k$ the structure of a Banach manifold of class C^k. Given an element $A \in \mathcal{A}(P,\mu)_{m,p,\delta}$, let S_A be a slice for the action of $\mathcal{G}(\mu)_{m+1,p,\delta}$ at A. Local charts for $\mathcal{H}(P(X))_k$ are given by maps $S_A \to \Omega^1(\Sigma, C^k(P(X)))$ of class C^k.

Theorem 4.4.1. *There is a constant $\delta_0 > 0$ such that for all weights $0 < \delta < \delta_0$ the section S_H given by (22) is Fredholm and space of regular Hamiltonian perturbations is of second category in $\mathcal{H}(P(X))_k$. For any such perturbation the subset $M^*(\Sigma, G, \mu, X, J, H)_{m,p,\delta} \subset M(\Sigma, G, \mu, X, J, H)_{m,p,\delta}$ of irreducible elements has the structure of a smooth manifold, with tangent space isomorphic to the kernel of the operator (26).*

Proof. Consider the universal moduli space

$$\begin{aligned}\mathcal{A}^{\text{univ}}(P,X,\mu) =& \big\{(A,u,J,H) | \bar{\partial}_{J_A,H} u = 0, \quad F_A + \operatorname{Vol}_\Sigma u^*\Phi = 0\big\} \\ &\subset \mathcal{A}(\mu)_{m,p,\delta} \times \Gamma(\Sigma, P(X))_{m,p,\delta} \times \mathcal{J}(X)^G \times \mathcal{H}(P(X))_m.\end{aligned}$$

The idea of the proof is to show that for $\delta > 0$, the Hamiltonian term can be considered a compact perturbation of the linearized operator. Since the perturbations H vanish at infinity on the cylindrical ends, zero is not contained in the spectrum of the limiting operators on the cylindrical ends and so for sufficiently small Sobolev weights on the cylindrical ends the universal linearized operator

$$D^{\mathrm{u}} = D^{\mathrm{u}}_{A,u,J,H} : \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta} \times \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times T_H\mathcal{H}(P(X))_m \to \Omega^{0,1}(\Sigma, u^*T^{\mathrm{v}}P(X))_{m,p,\delta}$$
$$(a,\xi,h) \mapsto (D^H)_{(A,u)}(a,\xi) + \Delta_H D^u(h)$$

Here we denote by $h \mapsto \Delta_H D^u(h)$ the partial operator at H in the direction of h. We claim that it is onto. Because of the construction of $J_{A,H}$ as in Equation (23), this term $h \mapsto \Delta_H D^u(h)$ is (j,J)-anti-linear. Suppose that η is orthogonal to the image of $D^{\mathrm{u}}_{A,u,H}$. Denote by $(D^{\mathrm{u}})^*$ be the dual operator defined by integration and inner product

$$\int_\Sigma \langle D^{\mathrm{u}}\xi, \eta\rangle \operatorname{Vol}_\Sigma = \int_\Sigma \langle \xi, (D^{\mathrm{u}})^*\eta\rangle \operatorname{Vol}_\Sigma .$$

By integration by parts $D^*\eta = 0$ and also

$$\langle \eta, Y \circ D_u \circ j\rangle = 0, \qquad \text{for all } Y \in T_H\mathcal{H}(P(X)).$$

Suppose that there is a point z where $\eta_z \neq 0$. Let $x = u(z)$ and take a (j,J)-anti-linear map $Z : T\Sigma_z \to T_xP(X)^{\mathrm{v}}$, such that $\langle \eta_x, Z \circ j\rangle \neq 0$. Since $\mathcal{A}(P,\mu)$ is reducible-free, there exists an infinitesimal Hamiltonian perturbation Y such that $\langle \eta_z, (Y \circ Du \circ j)_z\rangle \neq 0$. Indeed, choose Y for A, extend it to a slice for the gauge group action, and then to the flow-out by equivariance. By multiplying Y by a bump function supported near $x = u(z)$ one can achieve that it lies in the space $T_I\mathcal{J}(J)$, and that $\langle \eta, (Y \cdot Du \cdot j)\rangle \neq 0$ which is a contradiction. Therefore $\eta = 0$ and D^{u} is onto.

Next we show that the action of $\mathcal{G}_{m+1,p,\delta}$ on $\mathcal{A}(P,X,\mu)_{m,p,\delta}$, resp. $\mathcal{A}^{\mathrm{univ}}(P,X,\mu)_{m,p,\delta}$ admits finite dimensional slices. The construction of slices for the gauge action is given by Gaio-Salamon [11], in the case without cylindrical ends. We first show that the action of $\mathcal{G}$ on the space $\mathcal{A}(\mu)_{m,p,\delta} \times \Gamma(\Sigma, P(X))_{m,p,\delta}$ admits slices and thus so does the action on $\mathcal{A}(P,X,\mu)_{m,p,\delta}$. The construction of the slice for $\mathcal{A}^{\mathrm{univ}}(P,X,\mu)_{m,p,\delta}$ is identical.

Recall that the generating vector fields for the gauge group action on $\mathcal{A}(\mu)_{m,p,\delta} \times \Gamma(\Sigma, P(X))_{m,p,\delta}$ are given by

$$\eta_{\mathcal{A}(P,X,\mu)} = (-d_A(\eta), (\eta_X)(u)) \text{ for } \eta \in \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta}.$$

We take as linear (local) slice the orthogonal complement to the orbit. A point (a,ξ) is L^2_δ-orthogonal if

$$\langle -d_A\eta, a\rangle_{L^2_\delta} + \langle \eta_X(u), \xi\rangle_{L^2_\delta} = 0, \text{ for all } \eta \in \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta}$$

or equivalently, using the formal weighted adjoints as in (7) we have

$$\langle \eta, -d_A^* a\rangle_{L^2_\delta} + \langle \eta, Z(\xi)\rangle_{L^2_\delta} = 0$$

where d_A^* is the weighted adjoint of d_A and

$$Z : \Omega^0(\Sigma, u^*TX)_{m,p,\delta} \to \Omega^0(\Sigma, P(\mathfrak{g}))_{m,p,\delta}$$

is the one corresponding to $\eta \mapsto \eta_X(u)$. For later use, we notice that using the definition of the moment map and the compatibility condition of the metric

$$\langle \xi, \eta_X(u) \rangle = \omega(J\xi, \eta_X(u)) = d(u^*\Phi\eta, J\xi)$$

so that the formal adjoint of $\eta \mapsto \eta_X(u)$ is the Lie derivative $L_{J\xi}(u^*\Phi)$ which in turn gives the weighted adjoint

$$Z(\xi) = e^{-\delta r} L_{Je^{\delta r}\xi}(u^*\Phi).$$

The desired linear slice is

$$K = \{(a, \xi) \in \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta}, \quad d_A^* a + Z(\xi) = 0\}. \tag{27}$$

So we have a splitting of the tangent space

$$T_{(A,u)}\mathcal{A}(P, X, \mu)_{m,p,\delta} = T_{(A,u)}(\mathcal{G}(\mu)_{m+1,p,\delta} \cdot (A, u)) \oplus K.$$

We claim that there exists a neighborhood U of the origin in

$$\Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}PX)_{m,p,\delta} \tag{28}$$

and a map $G : U \to \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta} \times K$, such that

$$(a, \xi) \mapsto (\mathrm{d}_A^* a + Z(\xi), G(a, \xi)) \tag{29}$$

is a diffeomorphism onto a neighborhood of the origin. The proof of the claim is a direct application of implicit function theorem applied to the map

$$I : \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta} \times \Omega^1(\Sigma, P(\mathfrak{g}))_{m,p,\delta} \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X))_{m,p,\delta} \to \Omega^0(\Sigma, P(\mathfrak{g}))_{m+1,p,\delta}$$

$$I(\eta, (a, \xi)) = d_A^*(e^\eta \cdot (A + a) - A) + Z(\log_u(e^\eta \cdot \exp_u \xi)). \tag{30}$$

that satisfies the following result.

Lemma 4.4.2. *The map I is smooth with and its partial derivative respect the first term is*

$$D_1 I : \eta \mapsto d_A^* d_A \eta + Z(\eta_X(u)).$$

Moreover, $D_1 I$ is invertible.

Proof of Lemma 4.4.2. The derivative of I can be computed by expanding each term in Equation (30) to obtain

$$I(\eta, (a, \xi)) = \mathrm{d}_A^* \mathrm{d}_A \eta + \mathrm{d}_A^* a + Z(\xi) + Z(\eta_X(u)) + \text{ quadratic terms.}$$

and thus its partial derivative respect to η at the origin is $d_A^* d_A \eta + Z(\eta_X(u))$. It rests to see that

$$D_1 I : \eta \mapsto \mathrm{d}_A^* \mathrm{d}_A \eta + Z(\eta_X(u))$$

is invertible. The second term $Z(\eta_X(u))$ has order zero and the operator $\mathrm{d}_A^* \mathrm{d}_A$ is elliptic, the result will follow from elliptic regularity and the fact that DI has trivial kernel, which in turn follows from the identity

$$\langle \eta, \mathrm{d}_A^* \mathrm{d}_A \eta + Z(\eta_X(u)) \rangle = \|\mathrm{d}_A \eta\|^2 + \|\eta_X(u)\|^2.$$

□

We now complete the proof of the Theorem. By applying the implicit function theorem to the map I we obtain the desired neighborhood U and map G of (28),(29). A similar construction shows that the universal space $\mathcal{A}^{\text{univ}}(P, X, \mu)$ is Banach and admits slices for the $\mathcal{G}$ action. If every connection is irreducible, then the stabilizer of any point is the center and so the quotient

$$\mathcal{M}^{\text{univ}}(P, X, \mu) = \mathcal{A}^{\text{univ}}(P, X, \mu)/\mathcal{G}(\mu)$$

is a smooth Banach manifold. The Sard-Smale theorem applied to the projection

$$\text{proj} : \mathcal{M}^{\text{univ}}(\mu) \to \mathcal{H}(P(X))_{\epsilon,\delta}.$$

shows that the space $\mathcal{H}(P(X))_k^{\text{reg},J}$ is of second category in $\mathcal{H}(P(X))_k$. □

Remark 4.4.3. For the rest of the paper, we will assume that all vortices are irreducible. The reducible vortex cases are treated as in Donaldson-Kronheimer [9, 4.2.2], using the stratification of vortices by fixing the centralizers of the connections. An analogous version of Theorem 4.4.1 is valid for each stratum.

4.5. **Regularity.**

Proposition 4.5.1. *Let δ_0 be the constant of Theorem 4.4.1, and consider $m \geq 1$ and $0 < \delta < \delta_0$. The moduli space $M(P, X, \mu, J, H)_{m,p,\delta}$ is independent of the choice of (m, δ), and in fact consists of equivalence classes of smooth solutions.*

Proof. Let $(A, u) \in M(P, X, \mu, J, H)_{m,p,\delta}$. There is a gauge transformation $g \in \mathcal{G}_{m+1,p,\delta}$ so that $g(A)$ is of class $(m+1, \delta)$. To see this note that for some $\epsilon > 0$, any A_0 such that $\|A - A_0\|_{m,p,\delta} < \epsilon$ is Coulomb gauge respect to A. That is, there is a gauge transformation $\mathcal{G}_{m+1,p,\delta}$ such that

$$d_A^*(g^{-1}(A_0) - A) = 0$$

By symmetry $g \cdot A$ is in Coulomb gauge relative to $g^{-1}A_0$. The difference $a = g(A) - A_0$ satisfies the condition $d_{A_0}^*(a) = 0$. Since A_0 is near A, we can assume A_0 is smooth. Thus by vortex equation for A_0 now give that a satisfies the equation

$$d_{A_0}^*(a) = 0, F_{A_0} + d_{A_0}a + [a, a] + u^*P(\Phi) = 0$$

The term $F_{A_0} + [a, a] + u^*P(\Phi)$ is of class (m, p, δ), so are $d_{A_0}^*(a)$ and $d_{A_0}(a)$. By elliptic regularity, a is of class $(m+1, p, \delta)$ and thus $g(A)$ is. We want to show that the same is true for $g \cdot u$. Suppose that we have a vortex (A, u) such that A is of class $(m+1, \delta)$ and u is of class (m, δ). u satisfies the Cauchy Riemann equation $\overline{\partial}_{J_A}(u) = 0$, since A is of class $m+1$ equation (16) shows that J_A is also of class $m+1$. The bootstrapping theorem for holomorphic maps [15, B.4.1] implies that u is of class $m+1$. Following this procedure we get that up to gauge transformation the pair (A, u) is indeed smooth. □

Remark 4.5.2. For a generic Hamiltonian perturbation $H \in C^\infty(X)_{\epsilon,\delta}$, the space $N(\Sigma, P, \mu)^*$ of simple J_A-holomorphic maps $u : \Sigma \to X$ is smooth. However, it is not possible to choose H to be invariant under the stabilizer of G_A, hence the moduli space $M(\Sigma, P, \mu)^*$ can not be made transversal.

5. Compactification

5.1. **Vortices with bounded first derivative.** Let $M(\Sigma, P, X, \mu)_{m,p,\delta}$ denote the moduli space of perturbed pseudoholomorphic sections.

Theorem 5.1.1. *Let (A_α, u_α) be a sequence of vortices with bounded energy $E(A_\alpha, u_\alpha)$ in $M(\Sigma, P, X, \mu)_{m,p,\delta}$. If du_α is bounded in C^0 on compact sets, then there exists a smooth vortex (A_∞, u_∞) such that after gauge transformation and passing to a subsequence (A_α, u_α) converges to (A_∞, u_∞) uniformly in all derivatives on compact sets.*

Proof. The vortex equation and bound on Φ give a pointwise bound on the curvature, and since Vol_Σ has exponential decay on the cylindrical ends, this implies an L^2 bound for the curvature cf. Lemma 4.1.2. Uhlenbeck compactness implies that there is a subsequence (still denoted) A_α and a sequence of gauge transformations $g_\alpha \in \mathcal{G}_{2,p,loc}$ such that $g_\alpha \cdot A_\alpha$ converges weakly to a connection A_∞ in the local Sobolev topology $W_{1,p,\mathrm{loc}}$ and strongly in the C^0-topology (see Wehrheim [21, Theorem A']). By hypothesis the sequence $g_\alpha \cdot u_\alpha$ is bounded in $W_{1,p,loc}$ the theorems Alaoglu and Rellich, after passing to a subsequence, $g_\alpha \cdot u_\alpha$ converges weakly in $W_{1,p,loc}$ and strongly in C^0 to a section u_∞. Since F_{A_α} converges to $F_{\mathcal{A}_\infty}$ and $\overline{\partial}_{J_{A_\alpha}} u_\alpha$ converges to $\overline{\partial}_{J_{A_\infty}} u_\infty$ both weakly in $W_{0,p,\delta}$, the pair (A_∞, u_∞) is a weak solution to the vortex equations in $W_{1,p,loc}$. By Proposition 4.1.7, after gauge transformations, we can assume that (A_α, u_α) converges to (A_∞, u_∞) in $W_{1,p,\delta}$, and by Theorem **??** we can assume that (A_∞, u_∞) is actually smooth. To show convergence in all derivatives on compact sets, we use the bootstrapping method of the proof of 4.5.1, as follows. By Coulomb gauge, there is a sequence of gauge transformations $g_\alpha \in \mathcal{G}_{2,p,loc}$ such that $d^*_{A_\infty}(g_\alpha A_\alpha - A_\infty) = 0$. The sequence $g_\alpha A_\alpha$ has also L^2-bounded curvature and thus a subsequence, still denoted by $g_\alpha A_\alpha$, converges to A_∞ weakly in $W_{1,p,loc}$ and strongly in C^0. g_α is uniformly bounded in $W_{2,p,loc}$ and converges to an element $g \in \mathcal{G}_{2,p,loc}$ strongly in $W_{1,p,loc}$ and weakly in $W_{2,p,loc}$. By Proposition 4.1.7 a subsequence $g_\alpha \cdot u_\infty$ converges to $g \cdot u_\infty$ strongly in C^0 and weakly in $W_{1,p,\delta}$, and $g \cdot A_\infty = A_\infty$. Using equation (16), one sees that the convergence of A_α gives the convergence of almost complex structures J_{A_α} in $W_{1,p,loc}$. In particular, for every small open set U of compact support, this sequence is bounded uniformly, that is there is a constant c_0 such that, $\|J_{A_\alpha}\| \leq c_0$, for all α. Now, by [5, Lemma 3.3] we have a constant c, depending on c_0 and U so that, the sequence u_α of J_{A_α}-holomorphic sections is also bounded in $W_{2,p,loc}$, and by Proposition 4.1.7 we can assume it is also bounded in $W_{2,p,\delta}$. The sequences $a_\alpha := A_\alpha - A_\infty, u_\alpha$ are bounded in $W_{1,p,\delta}$, and they satisfy

$$d^*_{A_\infty}(a_\alpha) = 0, F_{A_\infty} + d_{A_\infty} a_\alpha + [a_\alpha, a_\alpha] + \mathrm{Vol}_\Sigma(u_\alpha)^* P(\Phi) = 0.$$

Therefore, $d_{A_\infty}(a_\alpha)$ is bounded in $W_{1,p,\delta}$ as well as $d^*_{A_\infty}(a_\alpha)$, elliptic regularity shows that a_α is bounded in $W_{2,p,\delta}$. By passing to a subsequence if necessary, we can now assume that (A_α, u_α) converges in $W_{2,p,\delta}$. Continuing this process we get the convergence on compact sets in all derivatives up to gauge transformation. □

The case when the sequence $\mathrm{d}u_\alpha$ is not bounded on compact sets in the C^0 norm yields the existence of bubbles. More formally, we introduce the space of stable vortices which will be the proper compactification for the moduli space of vortices.

5.2. **Stable Vortices.** A *stable vortex* is defined as follows. Let Σ be a connected, oriented surface with n cylindrical ends, Γ be a tree with vertices

$$\mathrm{vert}(\Gamma) = \{0\} \cup \mathrm{vert}(\Gamma)_C \cup \mathrm{vert}(\Gamma)_\Sigma.$$

The vertex 0 is the root vertex. The vertices $\mathrm{vert}(\Gamma)_C$ resp $\mathrm{vert}(\Gamma)_\Sigma$ are *cylindrical* resp. *spherical*.The set of cylindrical vertices decomposes into subsets

$$\mathrm{vert}(\Gamma)_C = \bigcup_{i=1}^{n} \mathrm{vert}(\Gamma)_{C_i}$$

and each set $\mathrm{vert}(\Gamma)_{C_i}$ (possibly empty) is equipped with an ordering.

Definition 5.2.1. For any $(J, H) \in \mathcal{J}$ a tuple $(\underline{A}, \underline{u}) := ((A, u_0), (A_i, u_i), (v_j), \mathbf{z})$ is called a *stable vortex* of combinatorial type Γ if

- (root) (A_0, u_0) is is a (J, H)-perturbed vortex on the surface Σ.
- (broken cylinders) For each $i \in \mathrm{vert}(\Gamma)_C$, (A_i, u_i) is a flat vortex on the cylinder associated to the trivial principal bundle, so that $u_i : S^1 \times \mathbb{R} \to G \times X$ is holomorphic.
- (fibre bubbles) The set of vertices $\mathrm{vert}(\Gamma)_\Sigma$ is the disjoint union of two sets $\mathrm{vert}(\Gamma)_{\Sigma,0}$ and $\mathrm{vert}(\Gamma)_{\Sigma,\infty}$. Then, for each j we have a holomorphic spheres v_j on the fibres, either of the form $\mathbb{P}^1 \to P(X)_z$ if $j \in \mathrm{vert}(\Gamma)_{\Sigma,0}$ (fiber bubble on Σ) or $\mathbb{P}^1 \to X$ if $j \in \mathrm{vert}(\Gamma)_{\Sigma,\infty}$ (fibre bubble on $S^1 \times \mathbb{R}$).
- $\mathbf{z}$ is the tuple of attaching points where their entries are given as follows.
  - for each edge $i\Gamma j$ connecting the vertices $i, j \in \mathrm{vert}(\Gamma)_\Sigma$ there are two entries $z_{ij}, z_{ji} \in \mathbb{P}^1$.
  - for each edge $0\Gamma j$ connecting 0 and $j \in \mathrm{vert}(\Gamma)_\Sigma$, entries $z_{0,j} \in \Sigma$, $z_{j,0} \in \mathbb{P}^1$.
  - for each edge $j\Gamma i$ connecting $j \in \mathrm{vert}(\Gamma)_\Sigma$ and $i \in \mathrm{vert}(\Gamma)_C$, entries $z_{ij} \in S^1 \times \mathbb{R}$, $z_{ji} \in \mathbb{P}^1$.
  - for each edge $0\Gamma i$ connecting 0 and $i \in \mathrm{vert}(\Gamma)_C$, an entry $z_{0,i} \in \overline{\Sigma}$ which is a point at infinity on a cylindrical end of Σ. $z_{i,0}$ will be always understood as the south pole of $\mathbb{P}^1 = \overline{S^1 \times \mathbb{R}}$ after completing the cylinder to $\mathbb{P}^1$.
  - If $i, j \in \mathrm{vert}(\Gamma)_C$ are joined by an edge, then z_{ij}, z_{ji} are the south and north pole of $\overline{S^1 \times \mathbb{R}}$ respectively.

Definition 5.2.2. We say two stable vortices $(\underline{A}_\alpha, \underline{u}_\alpha), \alpha = 0, 1$ are *equivalent* if there exists

- a gauge transformation over the principal component
- automorphisms of the domains of the bubbles preserving the special points

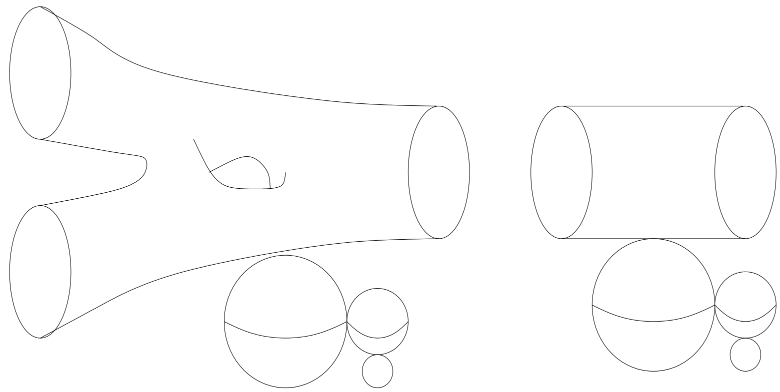

FIGURE 2. Stable vortex

such that the action of the gauge transformation on the principal component and on the bundles over the fibers by evaluation at the special points, together with the automorphisms of domains, transforms $(\underline{A}_0, \underline{u}_0)$ to $(\underline{A}_1, \underline{u}_1)$.

Let $M(\Sigma, P, X, \mu, \Gamma)$ denote the moduli space of (perturbed) stable vortices of combinatorial type Γ. It will be helpful to have a notation for the set of nodes of the domain of $\underline{A}, \underline{u}$. We denote the union of nodes by $V(\underline{A}, \underline{u})$, which decomposes into two subsets

$$V(\underline{A}, \underline{u}) = V_0(\underline{A}, \underline{u}) \cup V_1(\underline{A}, \underline{u})$$

where V_0 denotes the set of nodes connecting ends of cylinder bubbles and V_1 denotes the set of nodes connecting sphere bubbles to the interior of cylinder bubbles, other sphere bubbles, or the principal component. For each $w \in V_1(\underline{A}, \underline{u})$ we denote by $T_w^{\mathrm{vert}} P(X)$ the corresponding linearized fiber of P, and for each $w \in V_0(\underline{A}, \underline{u})$ we denote by $T_w X^\mu$ the corresponding tangent fiber of X^μ. We denote by $\tilde{D}_{\underline{A},\underline{u}}$ the operator from

$$\Omega^1(\Sigma, P(\mathfrak{g})) \oplus \Omega^0(\Sigma, u_0^* T^{\mathrm{vert}} P(X)) \oplus \bigoplus_\alpha (\Omega^1(S^1 \times \mathbb{R}, \mathfrak{g}) \oplus \Omega^0(S^1 \times \mathbb{R}, u_\alpha^* X)) \oplus \bigoplus_\beta \Omega^0(\mathbb{P}^1, v_\beta^* TX)$$

to

$$(\Omega^0 \oplus \Omega^2)(\Sigma, P(\mathfrak{g})) \oplus \Omega^{0,1}(\Sigma, u_0^* T^{\mathrm{vert}} P(X)) \oplus \bigoplus_\alpha ((\Omega^0 \oplus \Omega^2)(S^1 \times \mathbb{R}, \mathfrak{g}) \oplus \Omega^{0,1}(S^1 \times \mathbb{R}, u_\alpha^* X)) \oplus \bigoplus_\beta \Omega^{0,1}(\mathbb{P}^1, v_\beta^* TX) \oplus \bigoplus_{w \in V_0(\underline{A},\underline{u})} T_w X^\mu \oplus \bigoplus_{w \in V_1(\underline{A},\underline{u})} T_w^{\mathrm{vert}} P(X)$$

obtained by combining the various linearized operators on the principal component and bubbles, and taking the difference of the infinitesimal sections at the nodes. The kernel of the operator $\tilde{D}_{\underline{A},\underline{u}}$ contains a finite dimensional subspace $\mathrm{aut}(\underline{A}, \underline{u})$ generated by the infinitesimal automorphisms of the sphere and cylinder bubbles.

We say that a stable vortex is *simple* if it is irreducible and all components somewhere injective. Denote by $M^*(\Sigma, P, X, \mu, \Gamma)$ the subset of $M(\Sigma, P, X, \mu, \Gamma)$ consisting of simple stable vortices. By [15, Lemma 2.1], any sphere bubble that is not somewhere injective is a multiple cover. We need a related argument for cylinder bubbles. We write the connection in temporal gauge $A = \mu d\theta$, so that the pseudoholomorphic section gives a map

$$u : \Sigma \to X, \quad \partial_r u(z) + J(u(z))(\partial_\theta u(z) + \mu_X(u(z))) = 0. \tag{31}$$

Lemma 5.2.3. *Either u is almost everywhere injective, or u is a multiple cover, that is, there exists a finite covering $\phi : \Sigma \to \Sigma$, a connection A' on $G \times \Sigma$, a map $u' : \Sigma \to X$, and a (possibly ramified) covering $\phi : \Sigma \to \Sigma$ such that u' is $J_{A'}$-holomorphic and $(A, u) = \phi^*(A', u')$.*

Proof. The proof follows the case that A is trivial in [15, Lemma 2.1]. Locally we can use the trivialization of $\Sigma \times X$ given by the connection A, and then u in a neighborhood of any point z is given by a pseudoholomorphic map. Hence the critical points of u have isolated singularities. Let $C \subset \Sigma$ denote the set of critical points of u and consider the equivalence relation

$$\Gamma \subset (\Sigma \backslash C) \times (\Sigma \backslash C), \quad \Gamma = \{(z_0, z_1), \quad z_{i,\nu} \to z_i, \quad u(z_{0,\nu}) = u(z_{1,\nu}).$$

As in the case that A is trivial, the projection π of Γ on the first factor is a finite covering, with order independent of $z \in \Sigma \backslash C$. The claim that π is finite requires some explanation, since Σ is not compact. However, u decays exponentially on the cylindrical ends, and this implies that the set of points w such that $|du(w)| = |du(z)|$ is finite. The closure $\overline{\Gamma}$ is an equivalence relation on Γ, and one defines $\Sigma' = \Sigma/\overline{\Gamma}$. The argument that Σ' has a holomorphic structure induced by the structure j and Σ, and the construction of the maps u' is the same as before. We obtain a bundle $P' \to \Sigma'$ from $P = \Sigma \times G$ by identifying $(s, g) \sim (s', g)$ whenever $s \sim s'$. The connection A on P induces a connection A' on P', so that u' is $J_{A'}$-holomorphic. □

By Lemma 5.2.3 and [15, Lemma 2.5.1], any stable vortex that is not simple is either reducible or is a multiple cover of a simple vortex. Let (A^0_α, u^0_α) and (A^1_α, u^1_α) be two cylinder bubbles with the same asymptotic limits, and $A^0_\alpha = A^1_\alpha = \xi d\theta$. By gluing, we can form a section $u_\alpha : S^1 \times S^1 \to X$ pseudoholomorphic with respect to the glued connection A_α, with energy $E(A_\alpha, u_\alpha) = E(A^0_\alpha, u^0_\alpha) + E(A^1_\alpha, u^1_\alpha)$ and degree $\deg(u_\alpha) = \deg(u^0_\alpha) + \deg(u^1_\alpha)$.

Lemma 5.2.4. *The index of the corresponding linearized operators differs by*

$$\mathrm{Ind}(D_{A^0_\alpha, u^0_\alpha}) - \mathrm{Ind}(D_{A^1_\alpha, u^1_\alpha}) = 2(c_1(X), \deg(u^0_\alpha) - \deg(u^1_\alpha)).$$

Proof. By the gluing law for indices, we have

$$\mathrm{Ind}(D_{A^0_\alpha, u^0_\alpha}) + \mathrm{Ind}(D_{A^1_\alpha, u^1_\alpha}) - 2\dim(X^\xi) = 2(c_1(X), \deg(u_\alpha)).$$

$$\mathrm{Ind}(D_{A^0_\alpha, u^0_\alpha}) + \mathrm{Ind}(D_{A^0_\alpha, u^0_\alpha}) - 2\dim(X^\xi) = 2(c_1(X), \deg(u_\alpha)).$$

Subtracting gives the desired formula. □

We now prove transversality for the moduli space of vortices of a fixed combinatorial type. Let J be an G-equivariant almost complex structure on X, and A a connection on P. Let

$$M^{\mathrm{vert}}(P(X), J_A) := \{v : S^2 \to P(X) | \overline{\partial}_{J_A} v = 0, \pi \circ v = cte\}$$

denote the moduli space of vertical holomorphic spheres. Consider the space

$$\mathcal{JH} := \mathrm{Map}_{\mathcal{G}}(\mathcal{A}(P,\mu), \mathcal{J}^G(X) \times \Omega^1_c(\Sigma, \mathcal{D}(P(X)))$$

of gauge equivariant almost complex structures and perturbations depending on the connections A. One provides this space with a Banach structure by using the same spaces and perturbations of Theorem 4.4.1.

Recall that the vertices $\mathrm{vert}(\Gamma)$ of Γ decompose into the root $\{0\}$, cylindrical $\mathrm{vert}(\Gamma)_C$, and spherical $\mathrm{vert}(\Gamma)_\Sigma$ type. The moduli space $N(J, H; \Gamma)$ of framed vortices at infinity of combinatorial type Γ is equipped with an evaluation map at the attaching points

$$\mathrm{ev} : N(J, H; \Gamma) \to \mathbb{X}$$

taking values in the space

$$\mathbb{X} = \prod_{\text{edges of } \Gamma} (X \times X)$$

Let Δ_Γ be the diagonal defined by making equal the entries corresponding to the vertices joined by edges. The space $N(\Sigma, P, X, \mu; (J, H), \Gamma)$ of stable vortices of combinatorial type Γ is the inverse image of Δ_Γ via ev. We say that an element $(H, J) \in \mathcal{JH}$ is *regular* for the combinatorial type Γ if the following properties are satisfied.

- For every element $((A, u_0), (A_i, u_i), (v_j), \mathbf{z}) \in N(J, H; \Gamma)$ the vertical operator $D^{\mathrm{v}}_{J_A, v_j}$ is onto for the spherical components corresponding to $j \in \mathrm{vert}(\Gamma)_\Sigma$.
- The perturbation (J, H) is regular for the root component (A_0, u_0).
- The evaluation map ev is transversal to the relevant diagonal Δ_Γ.

Th

Theorem 5.2.5. *Let Γ be a fixed combinatorial type, for each regular element (J, H) the moduli space $N^*(\Sigma, P, X, \mu; (J, H), \Gamma)$ is a manifold with tangent space at $\underline{A}, \underline{u}$ isomorphic to the quotient of the kernel of the operator $\tilde{D}_{\underline{A}, \underline{u}}$ by $\mathrm{aut}(\underline{A}, \underline{u})$. Moreover, the evaluation map*

$$ev^\Gamma : N^*(\Sigma, P, X, \mu; (J, H), \Gamma) \to X^\mu$$

is smooth and the space $\mathcal{JH}^{reg}(\Gamma)$ of regular perturbations is of second category in $\mathcal{JH}$.

The proof is similar to the one of Theorem 4.4.1 and will be omitted. We denote the union over combinatorial types

$$\overline{N}(\Sigma, P, X, \mu)_{m,p,\delta} = \bigcup_\Gamma N(\Sigma, P, X, \mu, \Gamma)_{m,p,\delta}.$$

Similarly we define $\overline{M}(\Sigma, P, X, \mu)_{m,p,\delta}$.

5.3. **Topology on the moduli of stable vortices.**

Definition 5.3.1. Suppose that (A_α, u_α) is a sequence of vortices on Σ and $(A, \underline{u})$ is a stable vortex. We say that (A_α, u_α) *Gromov converges* to $(A, \underline{u})$ if there exist a sequence g_α of gauge transformations such that

(a) $g_\alpha A_\alpha$ converges uniformly to A on compact sets;
(b) $u_{0,\alpha}$ converges to u_∞ uniformly on compact subsets of the complement of the bubbling set $Z_0 \subset \Sigma_0$ of $\underline{u}$;
(c) for every bubbling component Σ_i of $\underline{\Sigma}$, there exists a sequence $\epsilon_{i,\alpha} \to \infty$ and maps $\phi_{i,\alpha} : B_{\epsilon_{i,\alpha}(w_i)} \to \Sigma_i$ such that $u_\alpha \circ \phi_{i,\alpha}$ converges uniformly on compact subsets of the complement of the bubbling set $Z_i \subset \Sigma_i$ to $u_{i,\infty}$.
(d) for any bubble point w_j on Σ_i, the energy lost

$$m(w_j) := \lim_{\epsilon \to 0} \lim_{\alpha \to \infty} E(u_\alpha \circ \phi_{i,\alpha}; B_\epsilon(z_j))$$

is equal to the sum of the energies on the components of $\underline{u}$ attached to w_j.
(e) for any bubble point w_j on Σ_i, $\phi_{\alpha,i}^{-1} \circ \phi_{\alpha,j}$ converges to w_j uniformly on compact sets in a neighborhood of w_i.
(f) if z_i is contained in Σ_j, then $z_i = \lim_{\alpha\to\infty} \phi_{j,\alpha}^{-1}(z_{i,\alpha})$.

To definine when a sequence $(A_\alpha, \underline{u}_\alpha)$ of stable vortices Gromov-converges to a stable vortex $(A, \underline{u})$ one needs to ask for the existence of contractions on the trees $\Gamma_\alpha \to \Gamma$ associated to each stable map and modify the above definition accordingly. The details are left to the reader.

A subset C of $\overline{N}(\Sigma, P, X, \mu)_{m,p,\delta}$ is *Gromov closed* if any convergent sequence in C has limit point in C, and *Gromov open* if its complement is closed. This induces a topology in $\overline{N}(\Sigma, P, X, \mu)_{m,p,\delta}$.

We want to show that these spaces are Hausdorff. For that we need to describe Gromov convergence of stable vortices in terms of an auxiliary function called the distance function. Our function is the same as in [15, p. 134] with an added term including the connection.

$$\rho((\underline{A}, \underline{u}), (\underline{A'}, \underline{u'})) = \inf_{f:T\to T'} \inf_{\phi_i} \rho_\epsilon((\underline{A}, \underline{u}), (\underline{A'}, \underline{u'}); f, \phi_i),$$

where the function

$$\begin{aligned}\rho((\underline{A},\underline{u}),(\underline{A}',\underline{u}');f,\phi) :=&\|A'-A\|_{L^2}\\ &+\sup_{ij}|E_i(u;B_\epsilon(z_{ij}))-E_{f(i)}(u';\phi_j(B_\epsilon(z_{ij})))|\\ &+\sup_{i\in\Gamma}\sup_{z\notin B_\epsilon(z_i)} d(u'_{f(i)}\phi_i,u_i)\\ &+\sup_{i\neq j,f(i)=f(j)}\sup_{z\notin B_\epsilon(z_ij)} d(\phi_j^{-1}\phi_i,z_{ij})\\ &+\sup_{f(i)\neq f(j)} d(\phi_j^{-1}(z_{f(i)f(j)}),z_{ij})\\ &+\sup_{i\in T,1\leq j\leq n} d(\phi_i^{-1}(z'_{f(i)j}),z_{ij})\end{aligned}$$

depends on the contraction $f:\Gamma\to\Gamma'$ such that map the nodes $i\to i'$ and ϕ_i is an automorphisms of the i-th component. We set $\rho_\epsilon=\infty$ if there are no contractions f.

Lemma 5.3.2. *For ϵ sufficiently small, $(\underline{A}_\alpha,\underline{u}_\alpha)$ Gromov converges to $\underline{A},\underline{u}$, if and only if $\rho((\underline{A}_\alpha,\underline{u}_\alpha),(\underline{A},\underline{u}))$ is zero.*

Proof. The forward direction is immediate from the definition of Gromov convergence. The backwards direction follows from [15, Remark 5.5.7] and by noticing that the term that contains the connection goes to zero, if and only if A_α converges to A after gauge transformation. □

Proposition 5.3.3. *The Gromov open sets form a topology for which any convergent sequence is Gromov convergent. Furthermore, any convergent sequence has a unique limit.*

Proof. By [15, Lemma 5.6.5] it suffices to show that the function ρ_ϵ satisfies the following properties:

(a) For all $(\underline{A},\underline{u})$ there is an $\epsilon_0>0$ such that for all $0<\epsilon<\epsilon_0$,
(b) $\rho_\epsilon((\underline{A},\underline{u}),(\underline{A}',\underline{u}'))=0$ only if $(\underline{A},\underline{u})=(\underline{A}',\underline{u}')$.
(c) $(\underline{A}_\alpha,\underline{u}_\alpha)$ converges to $(\underline{A},\underline{u})$ if and only if $\rho_\epsilon((\underline{A}_\alpha,\underline{u}_\alpha),(\underline{A},\underline{u}))$ converges to 0.
(d) Suppose that $(\underline{A}_\alpha,\underline{u}_\alpha)$ converges to $(\underline{A},\underline{u})$. Then $\limsup_\alpha\rho_\epsilon(\underline{\Sigma}_\alpha,\underline{u}_\alpha)\geq\rho_\epsilon(\underline{A},\underline{u})$.

(b) Suppose $\rho_\epsilon((\underline{A},\underline{u}),(\underline{A}',\underline{u}'))=0$. Then $\underline{A}=\underline{A}'$, and $\underline{u}|\Sigma=\underline{u}'|\Sigma$ and the bubble points all agree. These implies that $\underline{u}=\underline{u}'$, by unique continuation. (b) and (c) follow from Lemma 5.3.2. □

Before we show that adding stable vortices yields a compact space, we need to prove exponential decay for the twisted energy of the gauged sections.

5.4. **Exponential decay on annuli.**

5.4.1. *Twisted action.* Let X be a compact Hamiltonian G-manifold with symplectic form ω and moment map Φ. Recall that by the Cartan construction the equivariant symplectic form $\omega \in \Omega^2_G(X)$ descends to a closed two-form $\omega_A \in \Omega^2(P(X))$, by the formula

$$\pi^* \omega_A = \omega + \mathrm{d}(\Phi, A). \tag{32}$$

In general, ω_A is closed but not symplectic. Let $\omega_\Sigma \in \Omega^2(\Sigma)$ be a volume form on Σ, and for any $c > 0$ let $\omega_{A,c} = \omega_A + c\pi^*\omega_\Sigma$.

Lemma 5.4.1. *Let Σ, X be compact. For any $c_1 > 0$, there exists a $c_2 > 0$ such that if $\sup |A|_{C_1} < c_1$ and $c > c_2$ then $\omega_{A,c}$ is symplectic.*

Proof. It suffices to add on a sufficiently large multiple of $\pi^*\omega_\Sigma$ so that $\omega_{A,c}$ is positive on the horizontal subspace. Since the norm of ω_A on the horizontal subspace depends linearly on the C^1-norm of the connection and the moment map, the claim follows. □

The almost complex structure J_A determined by $J \in \mathcal{J}(X)^G$ and the connection is automatically compatible with $\omega_{A,c}$. We denote by $g_{A,c}$ the metric determined by $J_A, \omega_{A,c}$ on $P(X)$. From now on, c is fixed as in the lemma above. For a loop $\gamma : S^1 \to P(X)$ whose length is smaller than the half the g_A-injectivity radius of $P(X)$, we may define the *twisted energy* as the energy with respect to the metric g_A and the *twisted symplectic action* by

$$D_A(\gamma) = \int_D u^*\omega_A$$

where $u : D \to P(X)$ is any smooth map whose image is contained in a geodesic ball and whose boundary values are the given loop, that is, $u|\partial D = \gamma$. For any J_A-pseudoholomorphic map we have the energy action relation $E_A(u) = D_A(\partial u)$.

Lemma 5.4.2 (Uniform isoperimetric inequality)**.** *There exist constants $c_1, c_2, c_3 > 0$ such that if* $\sup |A|_{C_2} < c_1$ *then*

(a) *the g_A-injectivity radius of $P(X)$ is greater than c_2,*
(b) *if $l_A(\gamma) < c_2/2$ then $D_A(\gamma) \leq c l_A(\gamma)^2$.*

See [15, p.86]; the injectivity radius depends continuously on the C^2-norm of the metric.

5.4.2. *Bootstrapping.* In this section we consider solutions to the vortex equations

$$F_A + f \operatorname{Vol} u^*\Phi = 0, \quad \overline{\partial}_A u = 0$$

where f is an arbitrary function. We call a solution an *f-vortex*. We wish to show that if A and f are sufficiently small and A is in Coulomb gauge with respect to a smooth connection A_0, then all derivatives of A, u are controlled by e.g. the C^0 norm of A and the C^1 norm of u.

Lemma 5.4.3 (Bootstrapping)**.** *Let X be a compact Hamiltonian G-manifold equipped with an invariant almost complex structure J. Let $U \subset \mathbb{R}^2$ be an open set and $K \subset U$ a compact subset. For all integers $k > 0, p > 2$, there exists constants c_1, c_2, c_3 such that for all f-vortices (A, u) on U, if*

$$\sup |A| < c_1, \quad d^*_{A_0} A = 0 \quad \sup |d_A u| < c_2, \quad \|f\|_{k,p} < c_3, \quad \|A\|_{W^{k,p}(U)} \leq c_4$$

then there exists a constant $c = c(c_1, c_2, c_3, c_4, K, U, k, p)$ such that

$$\|A, u\|_{W^{k+1,p}(K)} \leq c\|A, u\|_{W^{k,p}(U)}.$$

In addition, for the case $k = 0$ we have

$$\|A\|_{W^{1,p}(K)} \leq c\|A\|_{L^p(U)} + c\|u\|_{W^{1,p}(U)}.$$

Proof. Consider first the case $X = \mathbb{R}^{2n}$. The elliptic estimate for the pseudoholomorphic section proved in [6, Lemma 3.3] gives for $k > 0$,

$$\|u\|_{k+1,p} \leq c\|u\|_{k,p} + \|u\|_{1,\infty}.$$

We carry out a similar estimate for the connection A, for any $k \geq 0$. Let $\rho : \mathbb{R} \times S^1$ be a cutoff function with compact support, equal to 1 on K. The connection ρA satisfies

$$\begin{aligned} \mathrm{d}(\rho A) &= (\mathrm{d}\rho)A + \rho \mathrm{d}A \\ &= (\mathrm{d}\rho)A - \rho[A, A]/2 - \rho f \operatorname{Vol} u^*\Phi \end{aligned}$$

and

$$\mathrm{d}^*(\rho A) = *(\mathrm{d}\rho) \wedge *A + \rho * [A_0, *A].$$

It follows that for $\|A\|_{k,p} \leq 1$,

$$\|F_{\rho A} + f \operatorname{Vol}(\rho u)^*\Phi\|_{k,p} \leq c\|A\|_{k,p}$$

and

$$\|\mathrm{d}^*(\rho A)\|_{k,p} \leq c\|A\|_{k,p}.$$

By the elliptic estimate for the $\mathrm{d} + \mathrm{d}^*$,

$$\|\rho A\|_{k+1,p} \leq c(\|A\|_{k,p} + \|u\|_{k,p})$$

To prove the claim for arbitrary X, choose r sufficiently small so that on any ball $B_r(z)$, the image of u lies in a Darboux chart on X, and cover X by a finite number of Darboux charts and Σ by a finite number of balls so that the image of each ball lies in a Darboux chart. Summing the estimates for the restriction of u to each ball proves the claim. □

5.4.3. *Exponential decay.* The exponential decay estimate depends on the exponential decay of the energy, which is derived from the mean value inequality. We prove the following uniform version:

Lemma 5.4.4 (Uniform mean value inequality)**.** *Let Σ be a Riemann surface (possibly with boundary), X a compact symplectic manifold, $P \to \Sigma$ the trivial G-bundle, $P(X)$ the associated fiber bundle. Fix $c > 0$, and for any connection $A \in \Omega^1(\Sigma, P(\mathfrak{g}))$ let E_A*

denote the twisted energy of Section 5.4.1. For any $c_1 > 0$*, there exist constant* $\delta > 0$ *such that if* $\|A\|_{C_2} < c_1$ *then*

$$E_A(u|B_r(z)) \leq \delta \implies |d_A u(z)|^2 \leq (8/\pi r^2) E_A(u|B_r(z)).$$

.

Proof. The computation on [15, p.90] uses only an estimate on the second derivative of the almost complex structure (hence the C^2-norm of the connection) and the metric (determined by the choice of c above and the C^0-norm of the connection. □

Let $z \in \Sigma$ be a point, and fix a trivialization of P in a neighborhood of z. Let $A(\delta, \epsilon)$ denote a small annulus around z, conformally equivalent to the cylinder $[\log(\delta), \log(\epsilon)] \times S^1$.

Lemma 5.4.5 (Uniform exponential decay)**.** *Let* X *be a compact Hamiltonian* G*-manifold equipped with an invariant almost complex structure* J*. There exists constants* $\kappa, c_0, c_1, c_2, c_3, c_4$ *such that for all* $\delta, \epsilon > 0$*, all* $A \in \Omega^1(A(\delta,\epsilon), \mathfrak{g})$ *and all* $u : A(\delta, \epsilon) \to X$ *pseudoholomorphic with respect to* J_A*, if*

$$\epsilon/\delta > c_0, \quad \|A\|_{C_2} < c_1, \quad E(u; A(\delta, \epsilon)) < c_2$$

then the twisted energy

$$E_A(u;T) := E_A(u|A(e^T\delta, e^{-T}\epsilon))$$

satisfies an exponential decay estimate

$$E_A(u;T) < c_3 e^{-\kappa T} E_A(u;0).$$

Furthermore, the distance between $u(z)$ *and* $u(z')$ *for* $z, z' \in A(e^T\delta, e^{-T}\epsilon)$ *satisfies the exponential decay estimate*

$$\sup_{z,z'} \operatorname{dist}(u(z), u(z')) \leq c_4 e^{-\kappa T} E_A(u;0)^{1/2}$$

Finally, if in addition (A, u) *satisfies*

$$F_A + f \operatorname{Vol} u^*\Phi = 0 \quad d^*A = 0$$

then there exist constants c_5, c_6 *such that if* $\|f\|_{C_2} < c_5$ *and* $\|A\|_{C_0} < c_6$*, then the same conclusions hold.*

We do *not* claim that the constant κ can be chosen arbitrarily close to 1 as in [15].

Proof. We may assume that the energy and C^2 norm of A are sufficiently small so that the mean value theorem applies, so that the twisted action is well-defined. Since u is

J_A-holomorphic,

$$\begin{aligned} E_A(u;T) &\leq D_A(u;T) - D_A(u;-T) \\ &\leq c_7(l_A(T)^2 - l_A(-T)^2) \\ &\leq c_8 \int_{S^1} |\partial_t u(T,\theta)|_A^2 \mathrm{d}\theta + c_2 \int_{S^1} |\partial_t u(-T,\theta)|_A^2 \mathrm{d}\theta \\ &\leq -c_8 \frac{\mathrm{d}}{\mathrm{d}T} E_A(u;T) \end{aligned}$$

for some constants c_7, c_8 depending on the previous constants. Thus

$$\frac{d}{dt} E_A(u;T) \leq -(1/c_8) E_A(u;T)$$

which implies

$$E_A(u;T) \leq c_3 \exp(-\kappa T) E_A(u;0)$$

where $\kappa = 1/c_8$ which proves the third claim. Applying the mean value inequality completes the proof of the second claim. The final claim uses the bootstrapping estimate to obtain a C^2 bound on A from the C^0 bound on A and the bound on $\mathrm{d}_A u$. □

5.5. **Compactness theorem.**

Theorem 5.5.1. *For any $C > 0$ and , the space $\overline{M}(\Sigma, P, X, \mu)^C_{m,p,\delta}$ is compact.*

Proof. The proof requires a detailed analysis of the different type of bubbling. We will show that all possible degenerations are already in the space of stable vortices. We do the proof in several cases, ans each case in several steps.

Case A. *Spherical fibre bubbling on the principal component.* Suppose that (A_α, u_α) is a sequence of vortices such that $\mathrm{d}_{A_\alpha} u_\alpha$ is unbounded in C^0 on compact sets. That is, there exists a convergent sequence $z_\alpha \to s$ such that $\mathrm{d}_{A_\alpha} u_\alpha(z_\alpha) \to \infty$. For this case assume further that no such s is a point at infinity (i.e. $s \in \Sigma$. We call such point s a singular or bubbling point for the sequence (A_α, u_α). We have the following.

Step 1. (Hard rescaling)

Proposition 5.5.2. *If (A_α, u_α) is a sequence of vortices whose energy is bounded by C, then there is a finite set of bubbling points $Z \in \Sigma$, and a vortex (A_∞, u_∞) on Σ such that a subsequence still denoted (A_α, u_α), converges after gauge transformations to (A_∞, u_∞) on compact sets of $\Sigma \backslash Z$ in all derivatives.*

Proof. To show this, for a bubbling point s, $\lim_{z_\alpha \to s} |\mathrm{d}_{A_\alpha} u_\alpha(z_\alpha)| = \infty$. Let $\epsilon > 0$ small enough so that a neighborhood of s is consider as an open set in $\mathbb{C}$. By Hofer's lemma [15, 4.6.4] to the function $z \mapsto |d_{A_\alpha} u(z)|$ for $|z - s| < \epsilon$, the points z_α and the constants $\delta_\alpha := |d_{A_\alpha} u_\alpha(z_\alpha)|^{-1/2}$, there exist sequences $\zeta_\alpha \in \Sigma$, $\epsilon_\alpha > 0$ such that

$$\zeta_\alpha \to s; \quad \sup_{|z-\zeta_\alpha|<\epsilon_\alpha} |d_{A_\alpha} u_\alpha| \leq 2c_\alpha; \quad \epsilon_\alpha \to 0; \quad \epsilon_\alpha c_\alpha \to \infty,$$

where $c_\alpha := |d_{A_\alpha} u_\alpha(\zeta_\alpha)|$. Let ψ_α be the sequence defined on the ball $B_{\epsilon_\alpha c_\alpha}(0)$ given by $\psi_\alpha(z) := (\zeta_\alpha + z/c_\alpha)$. This sequence converges to s uniformly in compact sets. The rescaled sequence

$$v_\alpha(z) := u_\alpha(\psi_\alpha(z))$$

has uniformly bounded first derivative on compact sets, on arbitrarily large domains, since the sequence $\epsilon_\alpha c_\alpha \to \infty$. The maps v_α are pseudoholomorphic with respect to the almost complex structure determined by the re-scaled connections $c_\alpha^{-1} A_\alpha$, which have uniformly bounded curvature. Using Theorem 5.1.1 we get, after gauge transformations, that the sequence $(c_\alpha^{-1} A_\alpha, v_\alpha)$ converges uniformly to a pair (A_0, v) on compact sets, in all derivatives. Note that the limit A_0 is necessarily the trivial connection, since $c_\alpha \to \infty$, and that the map v necessarily lies on the fibre s,

$$v : \mathbb{C} \to P(X)_s$$

since standard removal of singularities for finite energy maps implies that v extends to a smooth J-holomorphic map $v : \mathbb{P}^1 \to P(X)_s$, where J is the given almost complex structure on X, and v is non constant.

We now show that there are only finitely many of points z where bubbling can occur. Let $\hbar$ denote the energy quantization constant for X, so that any non-constant pseudoholomorphic map $v : \mathbb{P}^1 \to X$ captures energy at least $E(v) > \hbar$. Since the energy is non-negative, sphere bubbles can develop at most at finitely many points.

Finally, on the complement $\Sigma - Z$, $d_\alpha u_\alpha$ is uniformly bounded in compact sets and (A_α, u_α) has bounded energy, then by Theorem 5.1.1 A_α converges to a connection A_∞ on compact subsets of Σ and u_α converges to a section u_∞ on compact subsets of $\Sigma - Z$ in all derivatives, so that (A_∞, u_∞) is a solution to the vortex equations on $\Sigma - Z$. Since u_∞ is a finite energy pseudoholomorphic map from Σ to $P(X)$, removal of singularities implies that it extends to all of Σ. (A_∞, u_∞) is the *principal component* of the limiting sequence. □

To justify why energy quantization also holds for vortices, we state the following.

Lemma 5.5.3. *There is a constant $\hbar(\Sigma)$ such that any non-trivial vortex (A, u) has energy $E(A, u) > \hbar(\Sigma)$. In particular, there is only a finite collection of spherical fibre bubbles.*

Proof of Lemma 5.5.3. Suppose otherwise. Then there exists a sequence of vortices (A_α, u_α) with non-zero energy $E(A_\alpha, u_\alpha)$ such that $E(A_\alpha, u_\alpha) \to 0$. In particular, this sequence must have no sphere bubbling, so by the above argument it converges to a solution (A_∞, u_∞) of energy zero. Hence the connection A_∞ is flat and the section u_∞ covariant constant. The mean value inequality implies that u_α is homotopic to u_∞. Since the action is a topological invariant, this implies that $E(A_\infty, u_\infty) = 0$, which is a contradiction. □

We now deal with the construction of the bubble tree.

Step 2. We first show how to capture the first bubble. For simplicity assume that the set of bubbling points Z for the sequence consists of a single element s. To ensure that we capture bubbles so that there is no bubbling in between, we need to choose the rescaling sequences more carefully and we need to show that the bubbles connect argument still holds for vortices. Fix a trivialization of P in a neighborhood of s.

Let $m(s) := \lim_{\epsilon \to 0} \lim_\alpha (u_\alpha; B_\epsilon(s))$ be the energy of the sequence being captured at s. By choosing a subsequence, this limit exists since the energy is uniformly bounded. Since bubbling occurs near s, we can restrict the sequence u_α to the ball $B_R(s)$ for $R > 0$. Moreover, let $\zeta'_\alpha \in B_R(s)$ be the point where the function $d_{A_\alpha} u_\alpha$ attains its supremum on $B_R(s)$. The section $\tilde{u}_\alpha := u_\alpha(z + \zeta'_\alpha)$ is such that attain its sup at $z = 0$ and thus the sequence $\tilde{u}_\alpha$ has a bubbling point at $z = 0$, since the sequence $\zeta'_\alpha \to s$. Choose the rescaling constants ϵ_α so that

$$E(\tilde{u}_\alpha; B_{\epsilon_\alpha}(0)) = m(s) - h/2, \tag{33}$$

where $h < \min\{\hbar/2, c_2/2\}$ where $\hbar$ is the energy quantization constant and c_2 is the constant of Lemma 5.4.5. For any $\epsilon > \epsilon_\alpha$ the energy of $\tilde{u}_\alpha$ on the annulus $B_\epsilon(0) \backslash B_{\epsilon_\alpha}(0)$ is at the most $\delta/2$ and then there is not enough energy to form another bubble in this annulus. Let $\psi_\alpha(z) := \zeta'_\alpha + \epsilon_\alpha z$ and let $(A'_\alpha(z), v_\alpha(z)) := (A_\alpha(\psi_\alpha), u_\alpha(\psi_\alpha))$ denote the rescaled sequence defined on the ball $B_{R/\epsilon_\alpha}(0)$.

Now, we can state the following.

Proposition 5.5.4. *Bubbles connect*

(i) *There is a finite energy pseudoholomorphic map $v : \mathbb{C} \to P(X)_s$ and a finite set of points $Z_1 \subset \mathbb{C}$ so that the sequence v'_α converges uniformly in all derivatives on $\mathbb{C} \backslash Z_1$ to v. By removal of singularities v extends to $\mathbb{P}^1$.*

(ii) *Let $m_j := \lim_{\epsilon \to 0} \lim_\alpha E(v_\alpha; B_\epsilon(s_j))$ is the energy being captured by the bubbling point $s_j \in Z_1$, then*

$$m_0 := E(v) + \sum_{s_j \in Z_1} m_j.$$

Therefore, there is no other possible bubbles forming at s, only at the points $s_j \in Z_1$.

(iii) *We have $u(z) = v(\infty)$ in $P(X)_z$.*

Proof. This sequence by construction has uniformly bounded energy. By Step 1, there exists a finite set $Z_1 \subset \mathbb{C}$, and a vortex (A'_0, v) such that a subsequence still denoted (A'_α, v_α) converges uniformly on compact sets in all derivatives to (A'_0, v). Since $\epsilon_\alpha \to 0$, A' is necessarily the trivial connection. Also note that the function $|dv(z)|$ has its maximum at 0, thus $0 \in Z_1$. v has finite energy, and since $\delta_\alpha \to 0$, it is defined on arbitrarily big subsets of $\mathbb{C}$. Removal of singularities shows that it extends to a J-holomorphic map $v : \mathbb{P}^1 \to P(X)_s$, with J exactly the almost complex structure on X, since the connection A' is trivial. Now, recall that by Step 1, a subsequence of (A_α, u_α) converges in all derivatives on compact sets of $\Sigma \backslash Z$ to the principal component vortex (A_∞, u_∞). The pair (A_0, v) is the *first bubble* that appears attached to the principal component.

It rests to show the last two claims. These facts will follow as in the proof of [15, 4.7.1], using that the exponential decay for the energy on annuli near s is still valid for the sequence u_α of J_{A_α}-holomorphic curves.

After a sequence of gauge transformations, we may assume that the connections A_α are in Coulomb gauge with respect to the trivial connection in a neighborhood of z. Let $m(s)$ be the energy lost at the point s. Note that we do not need to distinguish the Yang-Mills-Higgs energy with the twisted energy $E_{A_\alpha}(u_\alpha; B_\epsilon(s))$ here, since the energy of the connection on $B_\epsilon(s)$ approaches zero. That is, we also have

$$m(s) = \lim_{\epsilon\to 0}\lim_{\alpha\to\infty} E_{A_\alpha}(u_\alpha; B_\epsilon(s)).$$

We first note that $\lim_{R\to\infty}\lim E_{A_\alpha}(u_\alpha; B_{R\epsilon_\alpha}) = m(s)$, since otherwise it would exist a subsequence still denoted by u_α and a constant $\rho > 0$ such that for $R \geq 1$,

$$\lim_{\alpha} E_{A_\alpha}(u_\alpha; B_{R\epsilon_\alpha}) \leq m(s) - \rho.$$

Thus for $R > 1$, the energy in the annuli satisfies $\lim_\alpha E_{A_\alpha}(u_\alpha; A(\epsilon_\alpha, R\epsilon_\alpha)) \leq c_2/2 - \rho$, which is a contradiction.

To finish the proof, note that $E_{A_\alpha}(u_\alpha; B_{\epsilon_\alpha}(s)) = E_{A_\alpha}(v_\alpha; B_1(s)) = m_0 - h/2 \geq m_0 - \hbar/2$, and both sequences u_α, v_α capture energy $m(s)$ at s. This shows that $Z_1 \subset B_1(s)$, that is all bubbling points of v_α are in the unit ball of s. Then, for all balls $B_\epsilon(s) \subset B_1(s)$ we have

$$\begin{aligned} m(s) &= \lim_{R\to\infty}\lim_{\alpha} E_{A_\alpha}(v_\alpha; B_R(s)) \\ &= \lim_{R\to\infty}\lim_{\alpha} E_{A_\alpha}(v_\alpha; B_R(s)\backslash B_\epsilon(s)) + \lim_{\alpha} E_{A_\alpha}(v_\alpha; B_\epsilon(s)) \\ &= \lim_{R\to\infty} E(v; B_R(s)\backslash B_\epsilon(s)) + \lim_{\alpha} E_{A_\alpha}(v_\alpha; B_\epsilon(s)) \\ &= E(v; \mathbb{C}\backslash B_\epsilon(s)) + \lim_{\rho\to 0}\lim_{\alpha} E_{A_\alpha}(v_\alpha; B_\epsilon(s)\backslash \cup_{j\in Z_1} B_\rho(s_j)) + \sum_{s_j\in Z_1} m(s_j) \\ &= E(v) + \sum_{s_j\in Z_1} m(s_j). \end{aligned}$$

To prove the last claim, note that by definition, there exists a sequence $\kappa_\alpha \to 0$ such that

$$\lim_{\alpha\to\infty} E_{A_\alpha}(u_\alpha; B_{\kappa_\alpha}(s)) \to m(s).$$

Map the annulus $A(\delta_\alpha, \kappa_\alpha)$ onto the annulus $[\log(\delta_\alpha), \log(\kappa_\alpha)] \times S^1$. By the uniform mean value inequality (5.4.4), on the subset $[\log(\delta_\alpha) + 1, \log(\kappa_\alpha) - 1] \times S^1$ the twisted derivatives $\mathrm{d}_{A_\alpha} u_\alpha$ are uniformly bounded. The exponential decay lemma 5.4.5 shows exponential decay of the energy on this region. Recall from [15, p. 103] that since there is not enough energy for bubbling, the energy on the outer region of the annulus must approach zero. Hence the energy density on the annulus is controlled by the energy on the inner region.

Since the connections A_α are already in Coulomb gauge with respect to the trivial connection, exponential decay of the distance. We have

$$u_\infty(z) = \lim_{\alpha\to\infty}(u_\alpha(\kappa_\alpha)), \quad v_\infty(\infty) = \lim_{\alpha\to\infty}(u_\alpha(\delta_\alpha))$$

since there is not enough energy on the annulus $A(\delta_\alpha, \kappa_\alpha)$ for further bubbling.

Hence

$$\operatorname{dist}(v_\infty(\infty), u_\infty(0)) \leq \lim_{\alpha\to\infty} \operatorname{dist}(u_\alpha(\kappa_\alpha), u_\alpha(\delta_\alpha)) = 0$$

by the exponential decay estimate in Lemma 5.4.5. □

Step 4. We can now inductively construct the spherical components of a *stable* holomorphic section on the fibre $P(X)_s$ by applying the previous step to all points $s_j \in Z_1$. All bubble maps are holomorphic with respect to the trivial connection, and thus one can just think of lines in X. This process stops since the energy m_0 is finite. The case when Z has more than one element is left to the reader.

Case B: *Cylindrical bubbles and spherical bubbles at infinity.* If $\sup_\alpha |\mathrm{d}_{A_\alpha}(u_\alpha(z_\alpha))|$ is unbounded for some sequence $z_\alpha \to z_i$ going to infinity on the i-th cylindrical end, there are two possibilities that can happen. First, on the cylindrical ends there is also translational symmetry, which means that other vortices can form at infinity on the cylindrical ends. Second, spherical bubbles on these cylindrical bubbles can also form. For any time $s \geq 0$, let $\tau_s : S^1 \times (0,\infty) \to S^1 \times (0,\infty)$ denote translation by s.

Step 1. Suppose that there is a sequence of points z_α on the cylindrical end such that the r-coordinate of z_α goes to infinity and $\mathrm{d}_{A_\alpha} u(z_\alpha)$ is bounded from above and below. Thus, there must be a sequence of numbers r_α,a vortex (B, v) on the trivial bundle over the cylinder $S^1 \times \mathbb{R}$, satisfying the flat limit of the vortex equations

$$F_B = 0, \quad \overline{\partial}_{J_B} v = 0, \tag{34}$$

and a finite set Z so that $\tau_{r_\alpha}(A_\alpha, u_\alpha)$ converges on compact sets on the complement $\Sigma\backslash Z$ to (B, v) in all derivatives. Note that after gauge transformation any connection on the cylinder is in temporal gauge, if flat it is then of the form $A = \mathrm{d} + a\mathrm{d}\theta$, for a constant element $a \in \mathfrak{g}$, and d the trivial connection. Then, the holonomy at infinity of the limit vortex agrees with the holonomy at infinity of the sequence.

To justify that Z is finite, we use a similar argument to that of Lemma 5.5.3 shows energy quantization for solutions to Equation (34). This also shows, that only finitely many cylindrical bubbles can occur.

Lemma 5.5.5. *Flat vortices on the cylinder with fixed holonomy at infinity satisfy an inequality $E(A, u) \in \{0\} \cup (\hbar, \infty)$ for some $\hbar > 0$.*

Proof. Any flat vortex on the cylinder which is in temporal gauge lifts to a vortex, still denoted by (A, u), on the strip $[0, 1] \times \mathbb{R}$ which is periodic up to the action of the holonomy around the puncture: $u(1, \cdot) = \mu u(0, \cdot)$. In particular, each u yields a holomorphic map $\tilde{u} : [0, 1] \times \mathbb{R} \to \overline{X} \times X$ onto the product $\overline{X} \times X$, with symplectic form $-\omega \times \omega$. Thus $\tilde{u}$ satisfies the Lagrangian boundary conditions: $\tilde{u}(0)$ is required to be in the diagonal and $\tilde{u}(1)$ lies in the graph of μ. Lemma [15, Lemma 4.1.4] now finishes the proof. □

Step 2. Construction of the bubble tree for bubbles at infinity. Similarly to what we have said in Case A, we need to ensure that we capture bubbles in an optimal way, so that bubbling in between bubbles is not possible. To capture the first bubble, we need to choose the rescaling constants correctly. Let $m_0 := \lim_T \lim_\alpha (A_\alpha, u_\alpha; (T;\infty) \times S^1)$ be the energy of the sequence that dissipates at infinity. Let h be a constant smaller than $\min\{\hbar, k\}$, where k is the constant in Theorem 4.1.2. Choose r_α so that the energy $E(u_\alpha; (r_\alpha, \infty) \times S^1) = m_0 - h/2$, by gauge transforming the pairs (A_α, u_α) if necessary and choosing a subsequence, we can assume that r_α is bigger than the constant r_0 of Theorem 4.1.2, so that the covariant derivative $\nabla_A u$ and the energy $E(u)$ have exponential decay on the cylindrical end. Let $(B_\alpha, v_\alpha) := \tau_{r_\alpha}(A_\alpha, u_\alpha)$ be the rescaled sequence. This sequence has bounded energy vortices. By using Step 1, there is a finite set $Z \subset S^1 \times \mathbb{R}$ and a flat vortex (B, v) on the cylinder such that (B_α, v_α) converges in $\mathbb{R} \times S^1 \backslash Z$ in compact sets with all derivatives. Moreover, the choice of rescaling ensures that no further bubbling for the original sequence can happen at infinity in the i-th cylindrical end. Exponential decay of u at infinity, as well as exponential decay for flat vortices as stated in Proposition 4.1.3 ensures the existence of limits which by construction connect: $u(\infty) = v(-\infty)$.

We now apply this process inductively, after capturing the first bubble, the limiting sequence is as follows. For $j = 1, \ldots, M$ there are sequences $\{r^j_\alpha\}$ of positive numbers such that $r^j_\alpha \to \infty$ as $\alpha \to \infty$, flat vortices (A_j, v_j) on the cylinder $\mathbb{R} \times S^1$ and finite sets of points $Z_j \subset \mathbb{R} \times S^1$ so that $\tau_{r_\alpha}(A_\alpha, u_\alpha)$ converges to (B_j, v_j) on $\mathbb{R} \times S^1 \backslash Z_j$ on compact sets and there is spherical fibre bubbling occurring on points $s \in Z_j$. Now apply the arguments used in Case 1 to build the bubble tree on the fibres $P(X)_s$, $s \in Z_j$. The cylindrical bubbles also have limits at infinity, since the exponential decay for finite energy flat vortices Proposition 4.1.3 and by construction they connect: $v_j(\infty) = v_{j+1}(-\infty)$. This finishes the proof of the theorem.

□

Later we will also need a gluing result for the operator $\tilde{D}_{\underline{A},\underline{u}}$:

Proposition 5.5.6. *Suppose that $\underline{A}, \underline{u}$ is the limit of a sequence (A_α, u_α). For sufficiently large α, the index of the operator $\tilde{D}_{\underline{A},\underline{u}}$ is equal to that of D_{A_α, u_α}.*

Proof. This is a fairly standard gluing argument for indices. In general, a four-term exact sequence relates the kernels and cokernels of the two operators, see [10, Lemma 3.1] or [20, Theorem 2.4.1]. The case that the cokernels vanish is somewhat easier, and the proof will be omitted. □

6. Gauged Gromov-Witten invariants

6.1. **Evaluation maps.** Recall that $N(\Sigma, P, X, \mu)$ denotes the moduli space of vortices with framings at the points $z_1, \ldots, z_n$. Define the *evaluation map*

$$\mathrm{ev} : N(\Sigma, P, X, \mu) \to X^\mu := X^{\mu_1} \times \cdots \times X^{\mu_n}$$

as follows. By elliptic regularity, any finite energy section u pseudoholomorphic with respect to some pair (J, H) has

$$\lim_{r\to\infty} D^k u(r, \theta) \to 0$$

along each cylindrical end. The same holds for any infinitesimal change ξ to u. Define

$$\mathrm{ev}_i([A, u]) := \lim_{r\to\infty} u(\rho_i(r, \theta)).$$

In the local charts (9), this map is given simply by

$$\mathrm{ev}_i([A, u]) = \exp_{u(z_i)}(\xi_i)$$

and is therefore smooth.

The stabilizer group $G_\mu := \prod_i G_{\mu_i}$ acts smoothly on X^μ as well as on $N(\Sigma, P, X, \mu)$. The evaluation map ev is equivariant, and thus it lifts to a map

$$\hat{\mathrm{ev}} : N(\Sigma, P, X, \mu)_{G_\mu} \to X^\mu_{G_\mu}$$

where for any G_μ-space Y, Y_{G_μ} denotes the quotient $Y \times_{G_\mu} EG_\mu$, EG_μ is the classifying space of G_μ equipped with a Banach manifold structure, so that $\hat{\mathrm{ev}}_\mu$ is smooth. The "bubbles connect" argument implies that the evaluation map extends continuously to the compactification, that is we have a continuous map

$$\hat{\mathrm{ev}} : \overline{N}(\Sigma, P, X, \mu)_{G_\mu} \to X^\mu_{G_\mu}.$$

The projection $\overline{N}(\Sigma, P, X, \mu)_{G_\mu} \to \overline{M}(\Sigma, P, X, \mu)$ is a bundle with fiber G_μ. Choosing a classifying map $\overline{M}(\Sigma, P, X, \mu) \to EG_\mu$ which is smooth on each stratum of combinatorial type Γ one obtains a map $\mathrm{ev} : \overline{M}(\Sigma, P, X, \mu) \to X^\mu_{G_\mu}$.

Consider a family of finite dimensional approximations $\{EG^N_\mu\}$ of EG_μ. For N big enough, the higher homotopy groups of EG^N_μ vanish, thus the bundle $\overline{N}(\Sigma, P, X, \mu)^N_{G_\mu} := \overline{N}(\Sigma, P, X, \mu) \times_{G_\mu} EG^N_\mu \to \overline{M}(\Sigma, P, X, \mu)$ admits a section

$$s_N : \overline{M}(\Sigma, P, X, \mu) \to \overline{N}(\Sigma, P, X, \mu)^N_{G_\mu}.$$

The natural inclusions $EG_\mu^N \to EG_N^{N+1}$ define a section s_{N+1} that fits in the commutative diagram

$$\begin{array}{ccccc} \overline{M}(\Sigma, P, X, \mu) & \xrightarrow{s_N} & \overline{N}(\Sigma, P, X, \mu)^N_{G_\mu} & \xrightarrow{\hat{\text{ev}}} & X^{\mu,N}_{G_\mu} \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \overline{M}(\Sigma, P, X, \mu) & \xrightarrow{s_{N+1}} & \overline{N}(\Sigma, P, X, \mu)^{N+1}_{G_\mu} & \xrightarrow{\hat{\text{ev}}} & X^{\mu,N+1}_{G_\mu} \end{array} \tag{35}$$

where $\hat{\text{ev}} : \overline{N}(\Sigma, P, X, \mu)^N_{G_\mu} \to X^{\mu,N}_{G_\mu} := X \times_{G_\mu} EG^N$ is the restriction of ev. Let $ev_N : \overline{M}(\Sigma, P, X, \mu) \to X^{\mu,N}_{G_\mu}$ denote the *evaluation map* defined by the composition $\hat{\text{ev}} \circ s_N$. Diagram (35) shows that the family of sections s_N are compatible with the inclusions and thus taking the direct limit we have

Proposition 6.1.1. *Taking the limit of the pseudoholomorphic section u along the cylindrical ends gives rise to a continuous evaluation map*

$$\text{ev} : \overline{M}(\Sigma, P, X, \mu) \to X^\mu_{G_\mu}. \tag{36}$$

Moreover, ev *is smooth when restricted to any stratum $M(\Sigma, P, X, \mu; \Gamma)$.*

Theorem 6.1.2. *If X is a monotone symplectic manifold, then* ev *is a pseudo-cycle for any $(J, H) \in \mathcal{JH}^{reg}$.*

Proof. The countable intersection over all the possible combinatorial types given by the trees Γ is the set $\mathcal{JH}^{\text{reg}}$, which is still a second category subset of $\mathcal{JH}$. This proves the first part of the theorem.

Recall that in Section 4 in the case that the connections are assumed to be irreducible, we constructed perturbations so that the space of somewhere injective J-holomorphic spheres in X, as well as somewhere injective flat vortices on each cylindrical end, had moduli spaces of the expected dimension in Theorem 5.2.5. Note that in the flat case, the only contribution to $E(A, u)$ is from u. Furthermore, any flat vortex on the cylinder not somewhere injective is multiply covered. It follows from transversality, Proposition 5.5.6, and Lemma 5.2.4 that the images of the multiply covered components are contained in smooth manifolds of codimension at least two lower than the expected dimension of $N(P, G, X, \mu)$. Therefore ev is a pseudocycle. □

As a corollary of the previous theorem we have

Corollary 6.1.3. *The evaluation map* ev : $\overline{M}(\Sigma, P, X, \mu) \to X^\mu_{G_\mu}$ *is a pseudo-cycle for regular choices (J, H).*

6.2. **Definition of the invariants.** In the previous section we proved that the restriction of the evaluation section ev to the open stratum $M(\Sigma, P, X, \mu)$ (assuming no reducibles) is a pseudocycle for a generic choice (J, H). The space $X^\mu_{G_\mu}$ is not finite dimensional; thus what we mean is that all finite dimensional approximations ev_N of the evaluation section are pseudo-cycles. Classes in homology are uniquely represented as pseudocycles up to cobordism and that the pseudo-cycle product is the same as the intersection product of homology classes.

Definition 6.2.1. For For an equivariant class $\alpha \in \prod_{i=1}^n H_{G_{\mu_i}}^\bullet(X^{\mu_i})$ we degine the *gauged Gromov-Witten* invariant

$$Z_{G,X,\mu}(\alpha) := \operatorname{ev} \cdot Y \tag{37}$$

6.3. **Enumerative meaning for the convex, flat case.** The following enumerative interpretation is immediate from the definitions:

Proposition 6.3.1. *Suppose that $\alpha \in H_{G_\mu}(X^\mu)$ is a class dual to a G_μ-stable submanifold Y in X^μ, that is, dual to the submanifold Y_{G_μ}, and* ev *meets Y_{G_μ} transversally. Then $Z_{G,X,\mu}(\alpha)$ is the number of points $[A,u] \in M(\Sigma,P,X,\mu)$ such that* $\operatorname{ev}([A,u]) \in Y_{G_\mu}$.

In the remainder of the section we suppose that the volume form vanishes, so that by Theorem 3.6.2 vortices with fixed holonomy correspond to parabolic holomorphic maps whose underlying parabolic bundle is semistable. Recall that a smooth projective variety X is *convex* if and only if for every holomorphic map $u : \mathbb{P}^1 \to X$, the pull-back u^*TX has vanishing higher cohomology; the convex case of Gromov-Witten theory is particularly simple. Unfortunately, the gauged Gromov-Witten theory still lacks transversality, because if the underlying bundle is non-trivial the associated bundle can be non-convex.

Definition 6.3.2. A set of markings $\mu \in \mathfrak{A}^n$ is *small* if the moduli space of parabolic bundles $M(\mathbb{P}^1,\mu)$ contains only parabolic bundles whose underlying bundle is trivial.

We remark that

(a) If μ is small, then $M(\mathbb{P}^1,\mu)$ is isomorphic to a geometric invariant theory quotient of a product of partial flag manifolds, corresponding to the markings μ.
(b) If all markings are generic, then $M(\mathbb{P}^1,\mu) \cong (G/B)^n /\!\!/ G$, where the polarization on the product of generalized flag varieties is determined by μ.
(c) Furthermore, for sufficiently small vortex parameter, the stability condition for vortices is that of the underlying bundle, so a holomorphic vortex is given by a stable parabolic structure on the trivial bundle over $\mathbb{P}^1$, and a holomorphic map $u : \mathbb{P}^1 \to X$.

Suppose that $Y = (Y_1,\ldots,Y_n)$ is a collection of G_μ-invariant complex submanifolds of X, and $\alpha = (\alpha_1,\ldots,\alpha_n)$ the dual class in $H_{G_\mu}(X^\mu)$. For $j = 1,\ldots,n$, let $\tilde{Y}_j = BY_j$ denote the flow-outs of these cycles by the action of the Borel. Assume that X is convex, the markings μ are small and generic, and $\Sigma = \mathbb{P}^1$. Then, under suitable transversality hypotheses, Theorem 3.6.2 and Proposition 6.3.1 give

Proposition 6.3.3. *With X,μ,ϵ as above, $Z_{\Sigma,X,\mu}(\alpha)$ is the number of equivalence classes of pairs (g,u), where $g \in (G/B)^n /\!\!/ G$, $u : \mathbb{P}^1 \to X$, for $p_1,\ldots,p_n \in \mathbb{P}^1$ generic distinct points,*

(a) $u(p_j) \in g_j \tilde{Y}_j, j = 1, \ldots, n$
(b) $[g] = ([g_1], \ldots, [g_n]) \in (G/P)^n$ *is in the semistable locus with respect to the markings* μ

and two pairs $(g, u), (g', u')$ *are equivalent if they are conjugate by the diagonal action of* G.

Let $\Sigma = \mathbb{P}^1$ and $E = \mathbb{P}^1 \times SL(n)$ denote the trivial $SL(n)$-bundle. Let $X = \mathbb{P}V^*$, the projectivization of the dual of the standard representation. Let $s_1, \ldots, s_b$ distinct points. For markings close to the identity and equal to some multiple of ω_1 the first fundamental weight, $M_{\mathbb{C}}(\Sigma, G) = (\mathbb{P}^{n-1})^b /\!\!/ SL(n)$. We assume that the holonomy parameters are chosen generically, so that the quotient is smooth. X^{g_j} is the union of a point and a projective line; we take for each class $Y_1, \ldots, Y_n$ the class of a point, and each $B_j Y_j$ is also a point. Thus

Corollary 6.3.4. *Under the above hypotheses,* $M(\Sigma, G, X, \mu)$ *is isomorphic to the set of of* $r_1, \ldots, r_b \in (\mathbb{P}^{n-1})^b$, *no more than* $b/2$ *equal, together with* $u : \mathbb{P}^1 \to \mathbb{P}^{n-1}$ *such that* $u(s_j)$ *is equal to* r_j, *quotiented by the action of* $SL(n)$.

For example, if $n = 2, b = 4$: The only possibility is the degree one invariant non-vanishing. Consider the case $[r_4] = u(s_4)$, for u the unique degree one map with $u(s_j) = [r_j], j = 1, 2, 3$. Thus the degree 1 invariant $Z_{\Sigma, X, \mu}(\alpha) = 1$, while the ordinary Gromov-Witten invariant for four fixed points on $\mathbb{P}^1$ and four point classes vanishes.

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