251 Multivariable Calculus Formulae Sheet

1. Vectors $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2)$, their angle and angle the cross product are

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\| \cdot \|v\|}\right), \qquad u \times v = \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right).$$

2. A curve $c(t) = (x(t), y(t), z(t)), a \leq t \leq b$, has length:

$$L = \int_a^b \|c'(t)\| dt,$$

the unit tangent $T(t)$, the unit normal vector $N(t)$ and the curvature $k(t)$:

$$\mathbf{T}(t) = \frac{r'(t)}{\|r'(t)\|}, \qquad \mathbf{N}(t) = \frac{T'(t)}{\|T'(t)\|}, \qquad k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}.$$

3. A plane at $\mathbf{P} = (x_0, y_0, z_0)$ with normal vector $\mathbf{N} = (a, b, c)$ has an equation:

$$(\mathbf{X} - \mathbf{P}) \cdot \mathbf{N} = 0, \qquad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

4. For $f(x, y, z)$, the linear approximation near (a, b, c)

$$f(a + h, b + k, c + l) \simeq f(a, b, c) + f_x(a, b, c)h + f_y(a, b, c)k + f_z(a, b, c)l.$$

5. Given a function, $f(x, y, z)$, and a vector-valued function, $(x(s, t), y(s, t), z(s, t))$,

$$\frac{\partial f}{\partial s}(x(s, t), y(s, t), z(s, t)) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}.$$

6. A point (a, b) is a critical point of a differentiable function f, if

$$\nabla f(a, b) = (f_x(a, b), f_y(a, b), f_z(a, b)) = 0.$$

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2 = \begin{cases} > 0 & \text{local max/min} \\ < 0 & \text{saddle point} \\ = 0 & \text{inconclusive} \end{cases}$$

7. Lagrange multiplier: Optimization with a constraint. If (x_0, y_0, z_0) is a critical point for a function $f(x, y, z)$ under a constraint equation $g(x) = 0$, then there is $\lambda \neq 0$ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \cdot \nabla g(x_0, y_0, z_0).$$

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Solving the above equation for (x_0, y_0, z_0).

8. The polar, cylindrical and spherical coordinates:

$$\text{(polar)} \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \qquad \text{(cylindrical)} \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \qquad \text{(spherical)} \begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases}.$$

9. Formula for change of variables: If $\Phi(u,v) = (x(u,v), y(u,v)) : D \to R$ is a map, then

$$\iint_R f(x,y)dxdy = \iint_D f(x(u,v), y(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$$

where

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}. \qquad \text{Ex: Jacobian deterninant} = \begin{cases} r & \text{polar} \\ r & \text{cylindrical} \\ \rho^2\sin\phi & \text{spherical} \end{cases}.$$

10. If $\mathbf{F}(x,y,z) = (F_1, F_2, F_3)$ is a gradient field on $\mathcal{D}$, then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

If $\mathcal{D}$ is also simply connected, then the cross partial are equal implies that F is a gradient field.

11. A function f and a vector F line integrals: $c(t) = (x(t), y(t), z(t)), a \le t \le b$,

$$\int_C f(c(t))|c'(t)|dt, \qquad \int_C \mathbf{F}\cdot d\mathbf{s} = \int_a^b \mathbf{F}(c(t))\cdot c'(t)dt.$$

12. If $\mathbf{F} = \nabla\phi$ on a domain $\mathcal{D}$, then for every oriented curve C in $\mathcal{D}$ with initial point P and terminal point Q,

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \phi(Q) - \phi(P).$$

If C is a closed curve, then $\oint_C \mathbf{F}\cdot d\mathbf{s} = 0$.

$$\text{Surface Area}(S) = \iint_D \|n(u,v)\|dudv, \qquad \Phi(u,v) : D \to S$$

$$\text{Surface Area}(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}dA \qquad \Phi(x,y) = (x, y, f(x,y)) : D \to S$$

$$\text{Green's Theorem: } \int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\text{Stok's Theorem: } \int_C \mathbf{F}ds = \iint_S \text{curl}(\mathbf{F})\cdot dS$$

$$\text{Divergence Theorem: } \iint_S \mathbf{F}\cdot dS = \iiint_W \text{div}(\mathbf{F})dV$$