

## 251 FINAL REVIEW PROBLEMS

*The exam will cover the entire syllabus. The following problems are intended to improve your understanding of the course material and there is no guaranty that exam problems will resemble these ones.*

**Problem 1.** Find an equation of the plane which passes through the points  $P(1, 4, 6)$ ,  $Q(-2, 5, -1)$  and  $R(1, -1, 1)$  Also find the area of the triangle with theses vertices.

**Problem 2.** Find the length of one complete turn of the helix  $r(t) = (2 \sin 2t, 2 \cos 2t, 3t)$ .

**Problem 3.** Find parametric equations for the lune passing through the point  $(2, -7, 5)$  which is parallel to the line  $x = 2t + 7, y = -3t + 5$  and  $z = t + 1$ .

**Problem 4.** The three vectors  $(1, 1, 0)$ ,  $(0, 1, 2)$  and  $(1, 0 - 1)$  are the edges of a paral-lelepiped. Find the volume of the parallelepiped.

**Problem 5.** If  $xyz = \sin(x^2 + 2y + z)$ , find  $\frac{\partial z}{\partial x}$ .

**Problem 6.** Find the tangential and normal components of acceleration for a particle moving with position function,

$$r(t) = (t - \sin t)\mathbf{i} - (1 - \cos t)\mathbf{j}, \quad 0 \leq t \leq 2\pi >$$

**Problem 7.** Find an equation of the tangent plane to the surface  $z = \sin(x^2 + y)$  at the point  $(1, -1, 0)$ .

**Problem 8.** Find an equation of the tangent plane to the surface  $xy + yz + zx = 11$  at the point  $(1, 2, 3)$ .

**Problem 9.** Use vectors to find the angle between the main diagonal of a cube and the diagonal of one of its faces (both diagonals emerging from a common vertex).

**Problem 10.** Find the velocity and position of a particle that starts at the origin at time  $t = 0$  with velocity  $i + 2j + k$  and has acceleration  $a(t) = t\mathbf{i} + (t^2/2)\mathbf{j} + t^2\mathbf{k}$ .

**Problem 11.** For the curve  $r(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{j} + t\mathbf{k}$ , find the unit tangent vector and the curvature.

**Problem 12.** Find the curvature of the curve  $y = e^x$  (as a function of  $x$ ). Also find the point on the curve where the curvature is maximum.

**Problem 13.** Find the angle between the planes  $x + y + 2z = 1$  and  $x - 2y + z = 1$ , and also find equation(s) for the line of intersection of those planes.

**Problem 14.** Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{(x+2y)^2}{x^2+y^2}$  or prove that the limit does not exist.

**Problem 15.** If  $z = x^2 \sin y$ ,  $x = s^2 + t^2$  and  $y = 2st$ , find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$ .

**Problem 16.** If  $z = f(x, y)$  and  $x = u + v$  and  $yy = u - v$ , show that

$$\left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2 = 4 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}.$$

**Problem 17.** Find  $dz$  if  $z = xe^{y/x}$ .

**Problem 18.** Find two parallel planes, one containing the line  $r_1(t) = (1, 0, 0) + t(1, 1, 1)$  and the other containing the line  $r_2(t) = (1, 1, 0) + t(0, 1, 1)$ .

**Problem 19.** Find the directional derivative of  $f(x, y, z) = \cos(xy) + e^{yz} - \ln(xz)$  at  $(1, 0, 2)$  in the direction from  $(1, 0, 2)$  to  $(2, 3, 4)$ . What direction gives the largest directional derivative of this functional  $(1, 0, 2)$ , and what is the value of the largest directional derivative?

**Problem 20.** Find equations for the tangent plane and normal line to the surface  $x^4y + y^4x + 2z^4x = 4$  at the point  $(1, 1, 1)$ .

**Problem 21.** Find the absolute maximum and minimum values of the function  $f(x, y, z) = x - 2y + 3z$  on the ellipsoid  $x^2 + 2y^2 + 3z^2 = 24$ .

**Problem 22.** Find all the critical points of  $f(x, y) = x^3 - 6xy + y^3$ , and classify them as local maxima, local minima, or saddle points.

**Problem 23.** Find  $\nabla f$  if  $f(x, y) = \frac{y - x^2}{4}$ , and sketch the vector field  $\nabla f$ .

**Problem 24.** For the function  $f(x, y) = x^2 - xy + y^2 - 3y$ , find the absolute maximum and minimum values on the triangular region bounded by the lines  $x = 0$ ,  $y = 4$  and  $y = x$ .

**Problem 25.** Compute  $\iint_R 12x \, dA$ , where  $R$  is the triangle in the second quadrant ( $x \leq 0$ ,  $y \geq 0$ ) enclosed by the coordinate axes and the line  $2x - y + 2 = 0$ .

**Problem 26.** Sketch the region of integration for the integral  $\int_0^1 \int_0^{3x} 3x^2 2xy dy dx$ . Write an equivalent integral with the order of integration reversed. Evaluate both integrals and check that they are equal.

**Problem 27.** Find the centroid of the region bounded by the  $y$ -axes and the parabola  $y = 1 - x^2$ .

**Problem 28.** Evaluate  $\int_{x=0}^2 \int_{y=0}^{\sqrt{4-x^2}} xy^2 dy dx$  by using polar coordinates.

**Problem 29.** Evaluate  $\int_{x=2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-x^2-y^2} dy dx$  by using polar coordinates.

**Problem 30.** Consider the integral  $\iiint_S (x^2 + y^2) dV$ , where  $S$  is that part of the ball  $x^2 + y^2 + z^2 \leq 9$  in the first octant. Set up the integral in two ways, first using cylindrical coordinates and then using the spherical coordinates. Use of these two to evaluate the integral.

**Problem 31.** Evaluate  $\iiint_S y dV$ , where  $S$  is the solid in the first octant cut off by the plane  $x + 2y + 2z = 4$ .

**Problem 32.** Compute  $\int_C xy dx + y^2 dy$ , where  $C$  is the quarter circle from  $(2, 0)$  to  $(0, 2)$  (centered at the origin.)

**Problem 33.** Compute  $\int_C (x + 2y) ds$ , where  $C$  is the straight line segment from  $(2, 0)$  to  $(0, 2)$ .

**Problem 34.** Compute  $\int_C xy dx + y^2 dy$ , where  $C$  is the straight line segment from  $(2, 0)$  to  $(0, 2)$ .

**Problem 35.** Find the Jacobian of the transformation  $x = 2u, y = 3v$ . Use this transformation as a first step to compute  $\iint_R x^2 dA$ , where  $R$  is the ellipse  $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$ .

**Problem 36.** Let  $C$  be the triangle in the plane with vertices  $(0, 1), (1, 0)$  and  $(1, 1)$ , oriented counter-clockwise.

(a) Evaluate  $\oint_C 2y^2 dx + 2x dy$  directly as a line integral.

(b) Evaluate the same line integral using Green's Theorem.

**Problem 37.** Use the curl operator to show that the field

$$F(x, y, z) = (3x^2y^2z + 1)\mathbf{i} + (2x^3yz + 2)\mathbf{j} + (x^3y^2 + 3)\mathbf{k}$$

is conservative. Then find a function  $f$  such that  $\nabla f = F$ . Then quickly calculate  $\int_C F \cdot dr$ , where  $C$  is the path  $r(t) = t^{10}\mathbf{i} + \arctan t\mathbf{j} + \ln(t + 1)\mathbf{k}$ ,  $0 \leq t \leq 1$ .

**Problem 38.** Let  $W$  be the solid region above the plane  $z = 3$  and below the paraboloid  $z = 4 - x^2 - y^2$ . Let  $S$  be the boundary surface of  $W$ . Let  $F(x, y, z) = (y + x)\mathbf{i} + (y - x)\mathbf{j}$ . Verify the Divergence Theorem

$$\iiint_W \operatorname{div}(F) dV = \iint_S F \cdot dS,$$

by computing both sides separately and showing that they are equal.

**Problem 39.** Use the Divergence Theorem to calculate  $\iint_S F \cdot ndS$ , where  $S$  is the (entire) surface of the cylinder enclosed by the surfaces  $x^2 + y^2 = 1$ ,  $z = 1$  and  $z = 3$ , and  $F = (x^3 + 2y, 2x + y^3, 3z)$ .

**Problem 40.** Find the surface area of that portion of the surface  $z = x^2 + y^2 + 9$  lying between the planes  $z = 10$  and  $z = 13$ .

**Problem 41.** Let  $F(x, y, z) = (x + y, y - x, z)$ .

(a) Compute  $\oint_C F \cdot ds$ , where  $C$  is the circle  $x^2 + y^2 = 4$  in the  $xy$ -plane, oriented clockwise.

(b) Compute  $\iint_S \operatorname{curl}(F) dS$ , where  $S$  is the upper half of the sphere  $x^2 + y^2 + z^2 = 4$ .

(c) Reconcile your answer to (a) and (b) with Stoke's Theorem.

**Problem 42.** Let  $S_1$  be the disk  $x^2 + y^2 \leq 1$  in the  $xy$ -plane, oriented with  $\mathbf{n}$  pointing up. Let  $S_2$  be the "northern hemisphere"  $x^2 + y^2 + z^2 = 1$ ,  $z \geq 0$ , also oriented with  $\mathbf{n}$  pointing up. Let  $F$  be any twice differentiable vector field. Give two explanations why

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(a) one using Stokes' Theorem.

(b) the other using the Divergence Theorem (first show that  $\operatorname{div}(\operatorname{curl}(F)) = 0$ ).