Show all work clearly and in order, and circle/box your final answers. Justify your answers algebraically whenever possible: sketch all relevant graphs and write down all relevant mathematics. You have 15 minutes to take this 10 point quiz.

1. (4 points) Consider the function \( f(x) = \begin{cases} x + 5; & x \neq 100 \\ 0; & x = 100 \end{cases} \)

   a. (3 pts) Evaluate \( \lim_{x \to 100} f(x) \), and explain why the limit does or does not exist.

   Is the function continuous at \( x = 100 \)?

   **Answer:** The limit equals 105, but since the limit does not equal the value of the function at \( x = 100 \), the function is not continuous at \( x = 100 \) (it fails #3 of the three criteria for continuity).

   **Explanation:** First, understand how the function behaves. \( f(x) \) is piecewise defined, and for any \( x \) other than \( x = 100 \) the value of \( x \) is given by the rule \( f(x) = x + 5 \). At \( x = 100 \) exactly, the value is defined to be zero. If it’s more comfortable to write this with inequalities, we can get rid of the “not equals” sign, and write \( f(x) \) this way:

   \[
   f(x) = \begin{cases} x + 5; & x < 100 \\ 0; & x = 100 \\ x + 5; & x > 100 \end{cases}
   \]

   As a rule, though, we expect students to be comfortable with the “not equals” sign showing up in the definition of a piecewise function.

   Once we understand the function, we can compute the limit. Since the function is piecewise-defined, we will compute both the LHL and the RHL, to make sure that they’re equal.

   \[
   \text{LHL: } \lim_{x \to 100^-} f(x) = \lim_{x \to 100^-} (x + 5) = 100 + 5 = 105. \\
   \text{RHL: } \lim_{x \to 100^+} f(x) = \lim_{x \to 100^+} (x + 5) = 100 + 5 = 105.
   \]

   Note that in both cases we can evaluate simply by plugging in (no need for extra algebra), and that the values we get are equal. Therefore, the two-sided limit exists: \( \lim_{x \to 100} f(x) = 105 \).

   For the second part of the question, we just need to check the three criteria for a function to be continuous at a point \( c = 100 \):

   1. \( f(100) \) exists
   2. \( \lim_{x \to 100} f(x) \) exists
   3. \( f(c) = \lim_{x \to 100} f(x) \)

   We get that \( f(100) = 0 \) from how the function is defined, and we get that \( \lim_{x \to 100} f(x) = 106 \) from the first part of the question. So the function satisfies the first two criteria. However, because \( 0 \neq 106 \), it fails the third criterion. As a result, we cannot be continuous at \( x = 100 \), in fact, we have a hole there.

   b. (3 pts) Evaluate \( \lim_{x \to 101} f(x) \), and explain why the limit does or does not exist.

   Is the function continuous at \( x = 101 \)?

   **Answer:** The limit equals 106, and since the value of the function at \( x = 100 \) is also 106, the function is continuous at \( x = 101 \) (it passes all of the three criteria for continuity).

   **Explanation:** Similarly to part a, we can compute the limit at \( x = 101 \) by comparing the LHL and the RHL:
We get that the limit exists, and equals 106. We compare this to the value of the function at $x = 101$, and see that in fact the function satisfies all three of our criteria.

In fact, though, we can make a more general argument that since 101 isn’t one of our “suspicious points”, that is, we don’t have division by zero at $x = 101$ and it’s not a place where the piecewise definition of $f(x)$ is changing, we will have to be continuous at $x = 101$.

2. (4 points) Find the value of the constants $C$ and $D$ such that $g(t)$ has no jump discontinuities. Are there any other discontinuities?

$$g(t) = \begin{cases} 
Ct^2 + 1; & t \leq 1 \\
2t + 1; & 1 < t < 2 \\
\frac{Dt}{t-9}; & t \geq 2 
\end{cases}$$

Answer: $C = 2$, $D = -\frac{35}{2}$, and there is a discontinuity, in fact a pole (asymptote), at $t = 9$

Explanation: Jump discontinuities occur when the LHL and the RHL at a point are not equal. In general, the places where we’re likely to have jump discontinuities are when the rule changes in a piecewise defined function. For $g(t)$, this happens at both $t = 1$ and $t = 2$.

First, consider a potential jump discontinuity at $t = 1$.

$$\text{LHL: } \lim _{t\to 1^{-}} g(t) = \lim _{t\to 1^{-}} Ct^2 + 1 = C + 1.$$  
$$\text{RHL: } \lim _{t\to 1^{+}} g(t) = \lim _{t\to 1^{+}} 2t + 1 = 2 + 1 = 3.$$  

The value of $C$ that prevents a jump discontinuity at $t = 1$ is the value of $C$ that makes the LHL and the RHL equal, or equivalently the value of $C$ that satisfies the equation $C + 1 = 3$. So we see that $C = 2$.

Now, consider a potential jump discontinuity at $t = 2$.

$$\text{LHL: } \lim _{t\to 2^{-}} g(t) = \lim _{t\to 2^{-}} 2t + 1 = 4 + 1 = 5.$$  
$$\text{RHL: } \lim _{t\to 2^{+}} g(t) = \lim _{t\to 2^{+}} \frac{Dt}{t-9} = \frac{2D}{-7}.$$  

The value of $D$ that prevents a jump discontinuity at $t = 2$ is the value of $D$ that makes the LHL and the RHL equal, or equivalently the value of $D$ that satisfies the equation $\frac{2D}{-7} = 5$. So we see that $2D = -35$, and $D = -\frac{35}{2}$.

To answer the last part of the question, we look for other “suspicious points”, i.e. places where there is a potential for discontinuity. These happen when we have division by zero, and when the rule of a piecewise defined function changes. We’ve already handled all of the places where the rule of the function changes, because we’ve chosen values for $C$ and $D$ that eliminate the jump discontinuities there.

We have the potential for division by zero at $t = 9$. We need to check that the rule that has division by zero is in fact in place when $t = 9$ (or equivalently, that $t = 9$ occurs in the interval $t \geq 2$), to see if the division by zero will actually happen. In this case 9 is in that interval, so the division by zero does occur. The function is undefined at $t = 9$, and so it can’t be continuous at $t = 9$ (it fails the first criterion).