Solutions to the Practice Problems

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Problem 1: Find the equation of the plane which passes through the points $P = (1,4,6), Q = (-2,5,-1), R = (1,-1,1)$. Also find the area of the triangle with those vertices.

Solution: $8x + 3y - 3z = 2$, $\frac{5}{2}\sqrt{82}$. To find the area, use the properties of the cross product: see section 12.4.

Problem 2: Find the length of one complete turn of the helix $r(t) = (2\sin(2t), 2\cos(2t), 3t)$.

Solution: 5π. See section 13.3, example 1, for a similar computation of arc length.

Problem 3: Find a parametric equation for the line passing through the point $(2,-7,5)$ which is parallel to the line $x = 2t+7$, $y = -3t+5$, $z = t+1$.

Solution: $\langle 2t+2, -3t-7, t+5\rangle$.

Problem 4: The three vectors $(1,1,0)$, $(0,1,2)$ and $(1,0,-1)$ are the edges of a parallelepiped. Find the volume of the parallelepiped.

Solution: 1. See section 12.4 for the properties of the cross product.

Problem 5: If $xyz = \sin(x^2+2y+z)$, find $\frac{\partial z}{\partial x}$.

Solution: $\dfrac{-2x\cos(x^2+2y+z)+yz}{\cos(x^2+2y+z)-xy}$. See section 14.6, Example 5 for a worked out example of implicit differentiation.

Problem 6: Find the tangential and normal components of acceleration for a particle moving with position function $r(t) = (t-\sin(t))\mathbf{i} - (1-\cos(t))\mathbf{j}$, for $0 \le t \le 2\pi$.

Solution: the tangential component $a_{\mathbf{T}} = \dfrac{\sin(t)}{\sqrt{2-2\cos(t)}}$, and the normal component $a_{\mathbf{N}} = \sqrt{1 - \dfrac{\sin^2(t)}{2-2\cos(t)}}$. See section 13.5. These formulas are not on your formula sheet!

Problem 7: Find an equation of the plane tangent to the surface $z = \sin(x^2+y)$ at the point $(1,-1,0)$.

Solution: $z = 0 + 2(x-1) + 1(y+1) = 2x + y - 1$. See section 14.4 for more about tangent planes.

Problem 8: Find an equation of the tangent plane to the surface $xy + yz + xz = 11$ at the point $(1, 2, 3)$.

Solution: $z = 11 - \frac{5}{3}(x-1) - \frac{4}{3}(y-2) = \frac{46}{3} - \frac{5}{3}x - \frac{4}{3}y$. This question uses implicit differentiation, as well as the equation for the tangent plane.

Problem 9: Use vectors to find the angle between the main diagonal of a cube and the diagonal of one of its faces (both diagonals emerging from a common vertex).

Solution: $\arccos(\frac{2}{\sqrt{6}})$. Note that I don't care about simplifying my answer beyond this point! See section 12.3 for the properties of the dot product.

Problem 10: Find the velocity and position of a particle that starts at the origin at time $t = 0$ with velocity $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and has acceleration $a(t) = t\mathbf{i} + \frac{t^2}{2}\mathbf{j} + t^2\mathbf{k}$.

Solution: the velocity vector $v(t) = (\frac{t^2}{2} + 1)\mathbf{i} + (\frac{t^3}{6} + 2)\mathbf{j} + (\frac{t^3}{3} + 1)\mathbf{k}$, and the position vector $r(t) = (\frac{t^3}{6} + t)\mathbf{i} + (\frac{t^4}{24} + 2t)\mathbf{j} + (\frac{t^4}{12} + t)\mathbf{k}$. See section 13.5 for the relationship of velocity, acceleration and position. Initial conditions for velocity and position are included in the statement of the problem - read it slowly and carefully and you should see them!

Problem 11: For the curve $r(t) = \frac{t^3}{3}\mathbf{i} + \frac{t^2}{2}\mathbf{j} + t\mathbf{k}$, find the unit tangent vector and the curvature.

Solution: The unit tangent vector $T(t) = \dfrac{\langle t^2, t, 1\rangle}{\sqrt{t^4 + t^2 + 1}}$. The curvature $k(t) = \dfrac{t^2 + 1}{(t^4 + t^2 + 1)^{3/2}}$. See section 13.4 for more information. These formulas are on your formula sheet.

Problem 12: Find the curvature of the curve $y = e^x$ (as a function of x). Also find the point on the curve where the curvature is maximum.

Solution: The curvature $k(x) = \dfrac{e^x}{(1 + e^{2x})^{\frac{3}{2}}}$. Maximum curvature at the point $\left(\dfrac{\ln(2)}{2}, \sqrt{2}\right)$. I used the formula for curvature of a function $y = f(x)$ that does not involve cross product (this formula is not on your sheet). But if we think of this curve as actually living in 3 dimensions we can use the generic formula. See section 13.4.

Problem 13: Find the angle between the planes $x + y + 2z = 1$ and $x - 2y + z = 1$. Also find an equation for the line of intersection of these two planes.

Solution: The angle between the planes is $\arccos(\frac{1}{6})$ (this is also the angle between their normal vectors, which is how I computed it). An equation for the line of intersection is $\langle 5t + 6, t + 1, -3t - 3\rangle$. To find a point on the line of intersection I picked an arbitrary value for y, and substituted it in to the two equations for the planes. This gave me two equations for x and z, which I was able to solve. I did this twice, for $y = 1$ and $y = 2$, to get the points $(6, 1, -3)$ and $(11, 2, -6)$. Then I wrote the parametric equation of the line that connects these two points. See section 12.5, exercise #59 for a similar problem.

Problem 14: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x + 2y)^2}{x^2 + y^2}$, or prove that the limit does not exist.

Solution: The limit does not exist. If we approach the point $(0,0)$ along the x axis, the limit becomes $\displaystyle\lim_{(x,0)\to(0,0)} \frac{(x+2(0))^2}{x^2+(0)^2} = \lim_{x\to 0}\frac{x^2}{x^2} = 1$. However, if we approach the point $(0,0)$ along the y axis, the limit becomes $\displaystyle\lim_{(0,y)\to(0,0)} \frac{(0+2y)^2}{(0)^2+y^2} = \lim_{y\to 0}\frac{4y^2}{y^2} = 4$. Since we get different values for the limit as we approach the origin from different directions, the limit cannot exist. See section 14.2 for a more detailed explanation of limits in 3 dimensions, and exercises 33-38 in 14.2 for more practice problems.

Problem 15: If $z = x^2\sin(y)$, $x = s^2+t^2$ and $y = 2st$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution: $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = 2x\cdot 2s + \cos(y)\cdot 2t = 4s(s^2+t^2) + 2t\cos(2st)$. Similarly, $\frac{\partial z}{\partial t} = 4t(s^2+t^2) + 2s\cos(2st)$. See section 14.6 for information on the chain rule.

Problem 16: If $z = f(x,y)$ and $x = u+v$ and $y = u-v$, show that

$$\left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2 = 4\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}.$$

Solution: From the definition of the chain rule, we have that

$$\left(\frac{\partial z}{\partial u}\right)^2 = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)^2 = \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + 2\frac{\partial z}{\partial z}\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial y}\right)^2,$$
$$\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 - 2\frac{\partial z}{\partial z}\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial y}\right)^2.$$

The difference between these two quantities is $4\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$. This problem is taken directly from the homework: section 14.6, #20.

Problem 17: Find dz if $z = xe^{\frac{y}{x}}$ - there is a typo here, since dz by itself doesn't mean anything in this context. I read this problem as "Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$."

Solution: $\frac{\partial z}{\partial x} = x\frac{-y}{x^2}e^{\frac{y}{x}} + 1e^{\frac{y}{x}} = e^{\frac{y}{x}}\left(\frac{-y}{x}+1\right)$ (note that we need the product rule). $\frac{\partial z}{\partial y} = x\frac{1}{x}e^{\frac{y}{x}} = e^{\frac{y}{x}}$ (note that we do not need the product rule here).

Problem 18: Find two parallel planes, one containing the line $r_1(t) = (1,0,0) + t(1,1,1)$, and the other containing the line $r_2(t) = (1,1,0) + t(0,1,1)$.

Solution: The plane containing $r_1(t)$ is given by the equation $z - y = 0$, the plane containing $r_2(t)$ is given by the equation $z - y = -1$. Since the planes must be parallel they will have the same normal vector. They must also be parallel to the plane containing the vectors $\langle 1,1,1\rangle$ and $\langle 0,1,1\rangle$. Form this third plane I computed the normal vector to be $\langle 0,1,-1\rangle$, and then found the right constant term for each of the two planes.

Problem 19: a). Find the directional derivative of $f(x,y,z) = cos(xy) + e^{yz} - ln(xz)$ at the point $(1,0,2)$ i the direction from $(1,0,2)$ to $(2,3,4)$.

b). What direction gives the largest directional derivative of this function at $(1,0,2)$, and what is the value of this largest directional derivative?

Solution: a). $\frac{4}{\sqrt{14}}$. $\nabla f_{(1,0,2)} = \langle -1, 2, -0.5\rangle$, and we are lookng for the directional derivative in the direction $\langle 1,3,2\rangle$. See section 14.5 for the definition of the directional derivative.

b). The direction is $\langle \dfrac{-2}{\sqrt{21}}, \dfrac{4}{\sqrt{21}}, \dfrac{-1}{\sqrt{21}}\rangle$, and the value of the directional derivative is $\dfrac{10.5}{\sqrt{21}}$. This is an optimization problem with constraint. We want to optimize the function $F(x,y,z) = \nabla f_{(1,0,2)} \cdot \langle x,y,z\rangle = -x + 2y - 0.5z$ (from the definition of the directional derivative), subject to the constraint $x^2 + y^2 + z^2 = 1$. See section 14.8 on how to apply the method of Lagrange multipliers.

Problem 20: Find equations for the tangent plane and normal line to the surface $x^4y + y^4x + 2z^4x = 4$ at the point $(1,1,1)$.

Solution: The equation for the tangent plane is $7(x-1) + 5(y-1) + 8(z-1) = 0$, or equivalently, $7x + 5y + 8z = 20$. The normal line is given by $\langle 7t+1, 5t+1, 8t+1\rangle$. See Section 14.5 for an efficient method of getting the equation for the tangent plane when the function is defined in this way.

Problem 21: Find the absolute maximum and minimum values of the function $f(x,y,z) = x - 2y + 3z$, on the ellipsoid $x^2 + 2y^2 + 3z^2 = 24$.

Solution: The absolute max is 12, and it occurs at the point $(2,-2,2)$. The absolute min is -12, and it occurs at the point $(-2,2,-2)$. See section 14.8 on Lagrange multipliers.

Problem 22: Find all the critical points of $f(x,y) = x^3 - 6xy + y^3$, and classify them as local maxima, local minima, or saddle points.

Solution: The critical points are $(0,0)$ and $(2,2)$. $(0,0)$ is a saddle points, and $(2,2)$ is a local min (check the descriminant). See section 14.7.

Problem 23: Find ∇f if $f(x,y) = \frac{y-x^2}{4}$, and sketch the vector field ∇f.

Solution: $\nabla f = \langle \frac{-x}{2}, \frac{1}{4}\rangle$. You would sketch this by plotting vectors (i.e. substituting actual values in for x and y). Near the y axis your vectors should be nearly vertical (and pointing up), and at large distances from the y axis your vectors should be nearly horizontal and pointing in towards the y axis.

Problem 24: For the function $f(x,y) = x^2 - xy + y^2 - 3y$, find the absolute maximum and minimum values on the triangular region bounded by the lines $x = 0$, $y = 4$, and $y = x$.

Solution: There are 7 critical points we need to check. From the interior of the triangle, we have the critical point $(1,2)$. Along the boundary $x = 0$ we are optimizing the function

$f(0,y) = y^2 - 3y$, and we get the critical point $(0, \frac{3}{2})$, along with the two endpoints, $(0,0)$ and $(0,4)$. On the boundary $y = 4$ we are optimizing $f(x,4) = x^2 - 4x + 4$, and we get the critical point $(2,4)$ along with the two endpoints $(0,4)$ and $(4,4)$. Along the boundary $y = x$ we are optimizing the function $f(x,x) = x^2 - 3x$, and we get the critical point $(\frac{3}{2}, \frac{3}{2})$, along with the endpoints $(0,0)$ and $(4,4)$.

Critical Point	Value of $f(x,y)$
$(1,2)$	$1 - 2 + 4 - 6 = -3$
$(0,0)$	0
$(0, \frac{3}{2})$	$\frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$
$(0,4)$	$16 - 12 = 4$
$(2,4)$	$4 - 8 + 16 - 12 = 0$
$(4,4)$	$16 - 16 + 16 - 12 = 4$
$(\frac{3}{2}, \frac{3}{2})$	$\frac{9}{4} - \frac{9}{4} + \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$

The absolute minimum is -3, occurring at $(1,2)$, and the absolute maximum is 4, occurring at both $(0,0)$ and $(4,4)$. See Example 5, in section 14.7 for a similar problem.

Problem 25: Compute $\int\int_R 12x\,dA$, where R is the triangle in the second quadrant ($x \leq 0, y \geq 0$) enclosed by the coordinate axes and the line $2x - y + 2 = 0$.

Solution: 20. I set up these bounds for the integral: $\int_{-1}^{0}\int_{0}^{2x+2} 12x\,dy\,dx$.

Problem 26: Sketch the region of integration for the integral $\int_0^1\int_0^{3x} 3x^2 \cdot 2xy\,dy\,dx$. Write an equivalent integral with the order of integration reversed. Evaluate both integrals and check that they are equal.

Solution: First, we evaluate the first integral: $\int_0^1\int_0^{3x} 6x^3y\,dy\,dx = \frac{9}{2}$. The region of integration is the triangle with vertices $(0,0), (1,0), (1,3)$. We can rewrite the integral as $\int_0^3\int_{\frac{y}{3}}^{1} 6x^3y\,dx\,dy$, which we evaluate to get $\frac{9}{2}$.

Problem 27: Find the centroid of the region bounded by the y-axis and the parabola $y = 1 - x^2$.

Solution: $(0, \frac{2}{5})$. The x-coordinate of the centroid is given by $\frac{\int\int_R x dA}{\int\int_R 1\,dA}$, and similarly, the y-coordinate is given by $\frac{\int\int_R y dA}{\int\int_R 1\,dA}$. We can describe this region algebraically with the inequalities $-1 \leq x \leq 1$, and $0 \leq y \leq 1 - x^2$.

$$\int_{-1}^{1}\int_{0}^{1-x^2} 1\,dy\,dx = \frac{4}{3}$$

$$\int_{-1}^{1}\int_{0}^{1-x^2} x\,dy\,dx = 0 \text{ - why might we have expected this?}$$

$$\int_{-1}^{1}\int_{0}^{1-x^2} y\,dy\,dx = \frac{8}{15}$$

So the x-coordinate is zero, and the y-coordinate is $\frac{8}{15} \cdot \frac{3}{4} = \frac{2}{5}$. This is the 2-dimesnional analog of the centroid as it is defined in section 15.3.

Problem 28: Evaluate $\displaystyle\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} xy^2\,dy\,dx$ by using polar coordinates.

Solution: $\frac{32}{15}$. When we change to polar coordinates, our new integral becomes

$$\int_0^{\frac{\pi}{2}}\int_0^2 r^4\cos(\theta)\sin^2(\theta)\,dr\,d\theta = \int_0^2 r^5\,dr \cdot \int_0^{\frac{\pi}{2}} \cos(\theta)\sin^2(\theta)\,d\theta$$

For more information on conversion to polar coordinates, see section 15.4.

Problem 29: Evaluate $\displaystyle\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$ by using polar coordinates.

Solution: $\pi(1 - e^{-4})$. When we change to polar coordinates our new integral becomes

$$\int_0^{2\pi}\int_0^2 re^{-r^2}\,dr\,d\theta.$$

For more information on conversion to polar coordinates, see section 15.4.

Problem 30: Consider the integral $\int\int\int_S (x^2 + y^2)dV$, where S is the part of the ball $x^2 + y^2 + z^2 \leq 9$ that's in the first octant. Set up the integral in two ways, first using cylindrical coordinates and then using spherical coordinates. Use one of these two to evaluate the integral.

Solution: $\dfrac{81\pi}{5}$.

$$\text{Cylindrical Coordinates: } \int_0^{\frac{\pi}{2}}\int_0^3\int_0^{\sqrt{9-r^2}} r^3\,dz\,dr\,d\theta$$

$$\text{Spherical Coordinates: } \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 \rho^2\sin^2(\phi)\cdot\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

It's possible to evaluate the integral using either cylindrical or spherical coordinates - I did cylindrical. For more information on conversion to cylindrical and spherical coordinates, see section 15.4.

Problem 31: Evaluate $\int\int\int_S y\,dV$, where S is the solid in the first octant cut off by the plane $x+2y+2z=4$.

Solution: In the $x-y$ plane we have the triangle given by the x-axis, the y-axis, and the line $x+2y=4$. We can describe this algebraically with the inequalities $0\leq x\leq 4$, and $0\leq y\leq \frac{4-x}{2}$. Then we know z must be between 0 and $\frac{4-2y-x}{2}$. We set up the integral

$$\int_0^4\int_0^{\frac{4-x}{2}}\int_0^{\frac{4-2y-x}{2}} y\,dz\,dy\,dx = \frac{4}{3}.$$

This is very similar to exercise #17 in Section 15.3.

Problem 32: Compute $\int_C xy\,dx + y^2\,dy$, where C is the quarter circle from $(2,0)$ to $(0,2)$ (centered at the origin).

Solution: 0. We parameterize C as $\langle 2\cos(t), 2\sin(t)\rangle$, $0\leq t\leq \frac{\pi}{2}$ (note the orientation matches the statement of the problem). So $x(t)=2\cos(t)$, $x'(t)=-2\sin(t)$, $y(t)=2\sin(t)$, and $y'(t)=2\cos(t)$. The integrand will be

$$x(t)y(t)x'(t)+(y(t))^2y'(t) = -8\cos(t)\sin^2(t)+8\sin^2(t)\cos(t)=0,$$

and the integral of zero over anything is zero. This isn't quite the standard notation for line integrals - see section 16.2, example 4, for a worked out problem with this notation.

Problem 33: Compute $\int_C (x+2y)ds$, where C is the straight line segment from $(2,0)$ to $(0,2)$.

Solution: We parameterize C by $\langle 2-t, t\rangle$, for $0\leq t\leq 2$ (note that the orientation of the segment matches the statement of the problem). Our integral becomes

$$\int_0^2 (2-t+2t)\sqrt{(-1)^2+1^2}\,dt = \sqrt{2}\int_0^2 2+t\,dt = 6\sqrt{2}.$$

See section 16.2 for how to do line integrals of vector fields.

Problem 34: Compute $\int_C xy\,dx + y^2\,dy$, where C is the straight line segment from $(2,0)$ to $(0,2)$.

Solution: $\frac{4}{3}$. We parameterize C by $\langle 2-t, t\rangle$, for $0\leq t\leq 2$ (note that the orientation of the segment matches the statement of the problem). So $x(t)=2-t$, $x'(t)=-1$, $y(t)=t$, and $y'(t)=1$. The integrand becomes

$$x(t)y(t)x'(t)+(y(t))^2y'(t) = (2-t)t(-1)+t^2 = 2t^2-2t.$$

Then we have that $\int_0^2 2t^2 - 2t\,dt = \frac{4}{3}$. This isn't quite the standard notation for line integrals - see section 16.2, example 4, for a worked out problem with this notation.

Problem 35: Find the Jacobian of the transformation $x = 2u, y = 3v$. Use this transformation as a first step to compute $\int\int_R x^2 dA$, where R is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$.

Solution: The Jacobian is 6. When we apply the transformation, the ellipse becomes the unit circle and we can set up a new integral in the uv-plane.

$$\int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 4u^2 \cdot 6\,dv\,du.$$

Of course, we really want to convert to polar coordinates to evaluate this, which gives us

$$\int_0^{2\pi} \int_0^1 24r^3 \cos^2(\theta)\,dr\,d\theta = 6\pi.$$

See section 15.5 for more information and practice problems with change of variables.

Problem 36: Let C be the triangle in the plane with vertices $(0,1), (1,0)$, and $(1,1)$, oriented counter-clockwise.

a) Evaluate $\oint 2y^2\,dx + 2x\,dy$ directly as a line integral.

b) Evaluate the same line integral using Green's Theorem.

Solution:

a). To evaluate the line integral directly, we need to parameterize each of the three line segments that make up the sides of the triangle C. The vertical segment can be parameterized by $C_1(t) = \langle 1, t\rangle$, $0 \leq t \leq 1$, and the line integral over this piece contributes 2. The horizontal segment can be parameterized by $C_1(t) = \langle 1-t, 1\rangle$, $0 \leq t \leq 1$ (note the orientation!), and the line integral over this piece contributes -2. The diagonal segment can be parameterized by $C_1(t) = \langle t, 1-t\rangle$, $0 \leq t \leq 1$, and the line integral over this piece contributes $-\frac{1}{3}$, for a total value of $-\frac{1}{3}$.

b). To use Green's Theorem, we compute $\text{curl}_z(F) = 2 - 4y$. We integrate this over the entire triangle, and we get

$$\int_0^1 \int_{1-x}^1 (2-4y)\,dy\,dx = -\frac{1}{3}.$$

Part a) is a simpler version of section 16.2, example 5 (which happens to also be a lot like an exam question from the second midterm). Aren't you glad we have Green's Theorem to make this computation easier? Part b) is a lot like section 17.1, example 2.

Problem 37: Use the curl operator to show that the field

$$F(x,y,z) = (3x^2y^2z + 1)\mathbf{i} + (2x^3yz + 2)\mathbf{j} + (x^3y^2 + 3)\mathbf{k}$$

is conservative. Then find a function f such that $\nabla f = F$. Then quickly calculate $\int_C F \cdot dr$, where C is the path $r(t) = t^{10}\mathbf{i} + \arctan(t)\mathbf{j} + \ln(t+1)\mathbf{k}$, $0 \le t \le 1$.

Solution: $f(x,y,z) = x^3y^2z + x + 2y + 3z$, and $\int_C F \cdot dr = (\frac{\pi}{4})^2 \ln(2) + 1 + \frac{\pi}{2} + 3\ln(2)$. First we check that $F(x,y,z)$ is conservative. We know that it's defined everywhere, so all we have to do is check the partials. Note that when we compute the curl of a vector field $F = \langle F_1, F_2, F_3 \rangle$ we are really working with the cross partials:

$$\operatorname{curl}(F) = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$$

so when the curl is zero (in all components), we know that all the cross partials are equal. We can then find f by integrating F - see section 16.3, example 5 for a worked out example of how to do this part.

Since the vector field is conservative, $\int_C F \cdot dr = f(C(1)) - f(C(0))$. We compute $C(0) = \langle 0,0,0 \rangle$, and $C(1) = \langle 1, \frac{\pi}{4}, \ln(2) \rangle$. $f(0,0,0) = 0$, which is easy, so all that we have to do is compute $f(1, \frac{\pi}{4}, \ln(2)) = (\frac{\pi}{4})^2 \ln(2) + 1 + \frac{\pi}{2} + 3\ln(2)$

Problem 38: Let W be the solid region above the plane $z = 3$ and below the paraboloid $z = 4 - x^2 - y^2$. Let S be the boundary surface of W. Let $F(x,y,z) = (y+x)\mathbf{i} + (y-x)\mathbf{j}$. Verify the Divergence Theorem

$$\int\int\int_W \operatorname{div}(F)\, dV = \int\int_S F \cdot dS$$

by computing both sides separately and showing that they are equal.

Solution:

LHS: We can set up this triple integral in cylindrical coordinates: $0 \le \theta \le 2\pi$, $0 \le r \le 1$, and $3 \le z \le 4 - r^2$. We need to compute $\operatorname{div}(F) = 1 + 1 + 0 = 2$, and then evaluate the integral

$$\int_0^{2\pi} \int_0^1 \int_3^{4-r^2} 2r\, dz\, dr\, d\theta = \pi$$

.

RHS: We need to do this surface integral in two pieces - one for the top part (the paraboloid) which we will call S_1, and one for the circular base which we will call S_2. We can parameterize S_1 in cylindrical coordinates with the function $\Phi(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 4 - r^2 \rangle$. This gives us the normal vector $n(r,\theta) = \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r \rangle$ (notice that this is the upward pointing normal, since r is always positive). Setting up this surface integral we have

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 \langle r\sin(\theta)+r\cos(\theta), r\sin(\theta)-r\cos(\theta), 0\rangle \cdot \langle 2r^2\cos(\theta), 2r^2\sin(\theta), r\rangle \, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 2r^3\sin(\theta)\cos(\theta)+2r^3\cos^2(\theta)+2r^3\sin^2(\theta)-2r^3\sin(\theta)\cos(\theta)\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 2r^3\, dr\, d\theta = \pi.
\end{aligned}$$

We need to set up the surface integral for S_2 as well. We use the parameterization $\Phi(x,y) = (x,y,3)$. When we compute the normal vector, we note that it only has a z component, and that our vector field on has x and y components, so that when we take their dot product we will get zero. In fact, this surface doesn't contribute anything to the flux. Given the geometry of the situation, why might we have expected that?

Problem 39: Use the Divergence Theorem to calculate $\int\int_S F\, dS$, where S is the (entire) surface of the cylinder enclosed by the surfaces $x^2+y^2=1$, $z=1$, and $z=3$, and $F = \langle x^3+2y, 2x+y^3, 3z\rangle$.

Solution: By the Divergence Theorem, we can compute instead the integral of $\mathrm{div}(F)$ over the volume of the cylinder. $\mathrm{div}(F) = 3x^2+3y^2+3$, and so we set up our integral in cylindrical coordinates

$$\int_0^{2\pi}\int_0^1\int_1^3 r(3r^2+3)\, dz\, dr\, d\theta = 9\pi$$

See section 17.3, example 2 for a similar problem.

Problem 40: Find the surface area of the portion of the surface $z = x^2+y^2+9$ lying between the planes $z=10$ and $z=13$.

Solution: $\frac{\pi}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$. To find the surface area we will integrate the constant function 1 over the surface. We parameterize the surface by $\Phi(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r^2+9\rangle$, on the interval $0 \le \theta \le 2\pi$, $1 \le r \le 2$. We compute the normal vector $n(r,\theta) = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r\rangle$, and setting up the integral we get:

$$\int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\, dr\, d\theta = \frac{\pi}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right).$$

Problem 41: Let $F(x,y,z) = (x+y, y-x, z)$.

a) Compute $\oint F \cdot ds$, where c is the circle $x^2+y^2=4$ in the xy-plane, oriented clockwise.

b) Compute $\int\int_S \mathrm{curl}(F)\, dS$, where S is the upper half of the sphere $x^2+y^2+z^2=4$ (i.e. $z \ge 0$), with outward pointing normal vector.

c). Reconcile your answers to parts a) and b) with Stoke's Theorem.

Solutions:

a). 8π. We can parameterize the circle as $C(t) = \langle 2\cos(t), 2\sin(t), 0\rangle$, which lets us set up the integral

$$\int_0^{2\pi} 4\sin^2(\theta) + 4\cos^2(\theta) + 4\sin(\theta)\cos(\theta) - 4\cos(\theta)\sin(\theta)\,d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

b). -8π. The curl of F is $\langle 0, 0, -2\rangle$. We can parameterize the surface with spherical coordinates (in the standard way), which gives us a normal vector

$$n(\phi, \theta) = \langle 4\cos(\theta)\sin^2(\phi), 4\sin(\theta)\sin^2(\phi), 4\sin(\phi)\cos(\phi)\rangle.$$

Putting this together in the integral we get

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} -8\cos(\phi)\sin(\phi)\,d\phi\,d\theta = -8\pi.$$

c). This isn't surprising: Stoke's Theorem says that the integral of the flux of the curl through the surface should be the integral of the vector field around the boundary of the surface, but only when the boundary is oriented a particular way. What we actually computed in part a) was the line integral with the wrong orientation, so we would expect to be off by a factor of -1.

Problem 42: Let S_1 be the disk $x^2 + y^2 \leq 1$ in the xy-plane, oriented with $\mathbf{n}$ pointing up. Let S_2 be the "northern hemisphere" $x^2 + y^2 + z^2 = 1$, $z \geq 0$, also oriented with $\mathbf{n}$ pointing up. Let F be any twice differentiable vector field. Give two explanations why

$$\int\int_{S_1} \operatorname{curl}(F)\cdot dS - \int\int_{S_2} \operatorname{curl}(F)\cdot dS = 0.$$

a) Use Stoke's Theorem

b) Use the Divergence Theorem (first show that $\operatorname{div}(\operatorname{curl}(F)) = 0$).

Solution:

a). By Stoke's Theorem, the integral over the first surface should be the integral of F over the unit circle in the xy-plane: $x^2 + y^2 = 1$, oriented counterclockwise. Similarly, by Stoke's Theorem, the integral over the second surface should also be the integral of F over the unit circle in the xy-plane, oriented counterclockwise. Since these two integrals are equal, their difference must be zero.

b). If we think of S_1 and S_2 forming the boundary of a 3-dimensional shape W, we have by the Divergence Theorem

$$\int\int_{S_1} \operatorname{curl}(F)\cdot dS - \int\int_{S_2} \operatorname{curl}(F)\cdot dS = \int\int\int_W \operatorname{div}(\operatorname{curl}(F))\,dV.$$

Note that the negative sign comes from the fact that the normal vector for S_2 is pointing up and therefore inside the 3-d region, not outside. If we can show that $\operatorname{div}(\operatorname{curl}(F)) = 0$ then it follows that the integral on the left of the equal sign must be zero as well - zero integrated over anything must be zero - and we will have the result we want. Let $F = \langle F_1, F_2, F_3 \rangle$. We have that

$$\begin{aligned}
&\operatorname{div}(\operatorname{curl}(F)) = \operatorname{div}\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle \\
&= \frac{\partial^2 F_3}{\partial yx} - \frac{\partial^2 F_2}{\partial zx} - \frac{\partial^2 F_3}{\partial xy} + \frac{\partial^2 F_1}{\partial zy} + \frac{\partial^2 F_2}{\partial xz} - \frac{\partial^2 F_1}{\partial yz} \\
&= \left(\frac{\partial^2 F_3}{\partial yx} - \frac{\partial^2 F_3}{\partial xy} \right) + \left(\frac{\partial^2 F_2}{\partial xz} - \frac{\partial^2 F_2}{\partial zx} \right) + \left(\frac{\partial^2 F_1}{\partial zy} - \frac{\partial^2 F_1}{\partial yz} \right) \\
&= 0 + 0 + 0 = 0 \text{ (since the mixed partials are equal) .}
\end{aligned}$$