1. Find all eigenvalues and one eigenvector of the matrix $A = \begin{pmatrix} -4 & -9 \\ 2 & -5 \end{pmatrix}$. Note: the eigenvalues are small integers.

**Eigenvalues:**

$\begin{vmatrix} -4 & -9 \\ 2 & -5 \end{vmatrix} = (5-\lambda)(-4-\lambda) + 18$

$= -20 + 5\lambda + 4\lambda + \lambda^2 + 18$

$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$

$\lambda = -1$: $\begin{pmatrix} -4 & -9 \\ 2 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & -9 \\ 2 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & -9 \\ 2 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$, $x_1 + 3x_2 = 0$

We want solutions to $\begin{pmatrix} -3 & -9 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

**Eigenvector:** $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\lambda = 2$: $\begin{pmatrix} -4 & -9 \\ 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -6 & -9 \\ 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} -6 & -9 \\ 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$, $x_1 + \frac{3}{2}x_2 = 0$

**Eigenvector:** $\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$

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Note: the question only asked you to compute one eigenvector.

2. Suppose that $A$ is a $20 \times 20$ with rank($A$) = 19. Identify each of the statements about $A$ given below as true (T) or false (F) by circling the appropriate letter; if you are unsure, you may leave the statement unmarked. **Note:** Grading will be +1/2 point for each correct answer, -1/2 point for each incorrect answer, and 0 points for each unmarked statement (but not fewer than 0 points for this problem).

- T (F) The system $Ax = b$ has a solution for every $b$.
- T F For some vector $b$ the system $Ax = b$ will have infinitely many solutions.
- T F The columns of $A$ are linearly dependent.
- T F The homogeneous system $Ax = 0$ has a solution which is not the zero vector.
- T F $A$ has an inverse matrix $A^{-1}$ satisfying $AA^{-1} = A^{-1}A = I$.
- T F The reduced row-echelon form of $A$ is the identity matrix $I$.
- T F For some vector $b$ the system $Ax = b$ will have exactly two different solutions.
- T F The determinant of $A$ is zero.

See next page for explanations.
1. \( Ax = b \) will not have a solution if \( \text{Rank}(A|b) > \text{Rank}(A) \). In particular, if \( b \) has a non-zero 20th (last) component, the final row will correspond to the equation \( 0x_1 + 0x_2 + \ldots + 0x_{20} \neq 0 \), which can never be satisfied.

2. For some vector \( b \), \( Ax = b \) will have infinitely many solutions. Since \( \text{rank}(A) = 19 \), RREF \( (A) \) has 19 pivots. \( A \) has 20 columns, so one column must correspond to a free parameter. Every value of that parameter will yield a different solution, so we’ll have infinitely many solutions.

3. The columns of \( A \) are linearly dependent. Principle 5 in the online notes states that the columns are linearly dependent when \( Ax = 0 \) has a non-trivial solution. Since there is a solution to \( Ax = 0 \), and we will have a free parameter (see #2), there is a non-trivial solution.

4. The system \( Ax = 0 \) has a non-trivial solution. Note that this is equivalent to #3. Because there is a free parameter, there will be infinitely many solutions. There is only one all-zero solution, the rest must be non-trivial.

5. A does not have an inverse. The RREF of \( A \) only has 19 pivots, so \( \text{RREF}(A) \) is not \( I_{20} \). It follows that \( A \) has no inverse (we can see this from how we compute \( A^{-1} \)). We also know that because \( A \) does not have full rank, \( \det(A) = 0 \), so no inverse (see section 7.2).

6. \( \text{RREF}(A) \) is not \( I_{20} \). Note that this is equivalent to #5. \( \text{RREF}(A) \) has 19 non-zero rows, and one row of all zeros. But \( I_{20} \) has new all-zero rows.

7. For some vector \( b \), \( Ax = b \) does not have exactly 2 solutions. In fact, there are no vectors or matrices where this is true. There can be no solutions, one solution, or a solution with a free parameter (and \( \infty \) many solutions).

8. The determinant of \( A \) is zero. From the notes online, we see that we are in case 2. This only happens when \( \det(A) = 0 \). It also follows from #3: since the columns are linearly dependent, we know \( \det(A) \) must be 0. (see section 7.3).