KNOWN FAMILIES OF INTEGER SOLUTIONS OF $x^3 + y^3 + z^3 = n$

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Abstract. We survey the known polynomial families of solutions of the Diophantine equation $x^3 + y^3 + z^3 = n$. A search has been made for additional families by interpolating on subsets of the 5417 solutions found by Koyama, but no new polynomial solutions were discovered. Finally we mention Ramanujan's solution, which is one of generating functions.

1. Introduction

We will be interested in representations

$$n = x^3 + y^3 + z^3 \tag{1}$$

of various integers n as sums of three cubes. The cubic residues modulo 9 are $0, 1, 8$, so it follows by inspection of cases that for every integer solution to (1) we have $n \not\equiv \pm 4 \bmod 9$.

Any given solution can be written in one of the following forms for nonnegative a, b, c:

$$|n| = a^3 + b^3 + c^3,$$
$$|n| = a^3 + b^3 - c^3, \text{or}$$
$$|n| = c^3 - a^3 - b^3.$$

Therefore it suffices to consider nonnegative solutions to the equations $a^3 + b^3 = c^3 \pm n$ and $a^3 + b^3 + c^3 = n$. (For $n = 0$ it is a case of Fermat's last theorem that there are no integer solutions.)

In practice we need only search for *primitive* solutions, i.e. those for which $\gcd(a, b, c) \nmid n$, since the nonprimitive solutions for a fixed n are routinely obtained from the primitive solutions for its divisors.

Finding all nonnegative solutions to $a^3 + b^3 + c^3 = n$ for a given n is a finite computation, since necessarily $a, b, c \leq n$. However, finding all solutions to $a^3 + b^3 = c^3 \pm n$ is not.

2. Solutions for m^3, m^{12}, and $2m^9$

When $n = m^3$ is a cube there tends to be a comparatively large number of solutions to (1), and sometimes there are polynomial families among these solutions. For $n = 1$, the first few solutions are given by the following table, where $|x| \leq |y| \leq |z|$.

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n	x	y	z
1	-6	-8	9
1	9	10	-12
1	64	94	-103
1	-71	-138	144
1	73	144	-150
1	-135	-138	172
1	135	235	-249
1	334	438	-495
1	-372	-426	505
1	-426	-486	577

Some of these are given by the polynomial solution

$$(9t^3 + 1)^3 + (9t^4)^3 + (-9t^4 - 3t)^3 = 1, \tag{2}$$

the first few solutions of which are:

t	n	x	y	z
-5	1	-1124	5625	-5610
-4	1	-575	2304	-2292
-3	1	-242	729	-720
-2	1	-71	144	-138
-1	1	-8	9	-6
0	1	1	0	0
1	1	10	9	-12
2	1	73	144	-150
3	1	244	729	-738
4	1	577	2304	-2316
5	1	1126	5625	-5640

For other cubes, Davenport [2, p. 163–164] gives the Euler-Binet solutions to $x^3 + y^3 = z^3 + w^3$: Every rational solution is proportional to the solution

$$\begin{aligned} x &= 1 - (p - 3q)(p^2 + 3q^2), \\ y &= -1 + (p + 3q)(p^2 + 3q^2), \\ z &= (p + 3q) - (p^2 + 3q^2)^2, \\ w &= -(p - 3q) + (p^2 + 3q^2)^2, \end{aligned}$$

for some rational p, q. While this generates all integer solutions, it does not provide a finite way of computing all integer solutions for a given $n = w^3$. One can obtain specific polynomial families, however. For example, letting $p = 3q, t = -2q$ gives (2).

Lehmer [8] provides other polynomial families that can be obtained inductively from (2) as follows. Let

$$\begin{aligned} (x_0, y_0, z_0) &= (9t^4, -9t^4 + 3t, -9t^3 + 1), \\ (x_1, y_1, z_1) &= (9t^4, -9t^4 - 3t, 9t^3 + 1). \end{aligned}$$

Define

$$\begin{aligned}x_{k+1} &= 2(216t^6-1)x_k - x_{k-1} - 108t^4,\\ y_{k+1} &= 2(216t^6-1)y_k - y_{k-1} - 108t^4,\\ z_{k+1} &= 2(216t^6-1)z_k - z_{k-1} + 216t^4 + 4.\end{aligned}$$

Lehmer proves that then $x_k^3+y_k^3+z_k^3=1$, and thus we obtain integer solutions to (1) with $n=1$ for $k\in\mathbb{N}, t\in\mathbb{Z}$. One computes

$$\begin{aligned}x_2 &= 3888t^{10} - 135t^4,\\ y_2 &= -3888t^{10} - 1296t^7 - 81t^4 + 3t,\\ z_2 &= 3888t^9 + 648t^6 - 9t^3 + 1,\end{aligned}$$

which produce the primitive solutions

t	n	x_2	y_2	z_2
−5	1	37968665625	−37867550640	−7583623874
−4	1	4076828928	−4055650572	−1016561087
−3	1	229571577	−226754730	−76054868
−2	1	3979152	−3816726	−1949111
−1	1	3753	−2676	−3230
0	1	0	0	1
1	1	3753	−5262	4528
2	1	3979152	−4148490	2032057
3	1	229571577	−232423416	76999654
4	1	4076828928	−4098117876	1021869505
5	1	37968665625	−38070050610	7603873876

Lehmer gives a closed-form expression for (x_k, y_k, z_k), and from this one sees for example that the degree of z_k is $6k-3$ for $k\geq 1$. However, not every solution $x^3+y^3+z^3=1$ can be obtained in this way: Of the 33 solutions with $|x|\leq|y|\leq|z|\leq 10000$, only 13 appear in the above tables, and larger values of k produce only larger solutions.

For $n=2$, the solution

$$(6t^3+1)^3-(6t^3-1)^3-(6t^2)^3=2 \tag{3}$$

is well known. (Precisely one solution for $n=2$ is known that is not given by this family: $2=1214928^3+3480205^3-3528875^3$. See [1], [4].) Multiply both sides of (3) by m^9, and apply the change of variable $t\mapsto t/m$ to obtain the more general solution

$$(6t^3+m^3)^3-(6t^3-m^3)^3-(6mt^2)^3=2m^9, \tag{4}$$

which is primitive for $\gcd(6t,m)=1$. If $\gcd(6t,m)>1$, then dividing (4) by $(\gcd(6t^3,m^3))^3$ gives a primitive solution. For example, for $l,k\geq 1$ the solutions

$$(3t^3+2^{3l-1}m^3)^3-(3t^3-2^{3l-1}m^3)^3-(2^l 3mt^2)^3=2^{9l-2}m^9, \tag{5}$$

$$(2t^3+3^{3k-1}m^3)^3-(2t^3-3^{3k-1}m^3)^3-(2\,3^k mt^2)^3=2\,3^{9k-3}m^9, \tag{6}$$

$$(t^3+2^{3l-1}3^{3k-1}m^3)^3-(t^3-2^{3l-1}3^{3k-1}m^3)^3-(2^l 3^k mt^2)^3=2^{9l-2}3^{9k-3}m^9 \tag{7}$$

are primitive for $\gcd(3t,2m)=1$, $\gcd(2t,3m)=1$, and $\gcd(t,6m)=1$ respectively. Equations (4)–(7) give polynomial families for $n=2$, 128, 1458, 65536, 93312, 3906250, 28697814, ...

An analogous procedure may be applied to (2) to obtain families of solutions for numbers of the form m^{12}. Multiplying both sides by m^{12} and applying the transformation $t \mapsto t/m$ gives

$$(9mt^3 + m^4)^3 - (9t^4 + 3mt)^3 + (9t^4)^3 = m^{12}, \tag{8}$$

which is primitive for $\gcd(3t, m) = 1$. In particular, for $3 \nmid m$ and $k \geq 1$,

$$(3^k mt^3 + 3^{4k-2}m^4)^3 - (t^4 + 3^{3k-1}m^3 t)^3 + (t^4)^3 = 3^{12k-6}m^{12} \tag{9}$$

is primitive for $\gcd(t, 3m) = 1$. Equations (8) and (9) give families of solutions for $n = 1$, 729, 4096, 2985984, 16777216, 244140625, 387420489, ...

3. Computations on Koyama's Table

Kenji Koyama [6] has generated a large table of integer solutions of

$$x^3 + y^3 + z^3 = n$$

for noncubes n in the range $1 \leq n \leq 1000$ and $|x| \leq |y| \leq |z| \leq 2^{21} - 1$. [6] consists of two tables: Table 1 (55 pages) contains the integer solutions, sorted by n, and Table 2 (2 pages) lists the number of primitive solutions found for each n in the search range.

Unable to find a digital version, I obtained a hard copy of the tables from the collections in the Center of American History at the University of Texas at Austin. Once scanned into a .pdf file, I performed major automated and manual formatting manipulation on the raw text to recover the tabular structure. Significant automated error correction was also necessary, as the data had not been digitized without errors. Specifically, I wrote a program to verify that each solution (n, x, y, z) in the table *was* in fact a solution; if a 4-tuple failed to satisfy (1), then the program attempted to guess the correct solution under the assumption that only one of the four numbers n, x, y, z had been read incorrectly. This generally worked very well, although in a few cases two distinct solutions were corrected into the same solution; I detected these when I compared Koyama's Table 2 with my own counts of solutions for each n (and in doing this I actually discovered more errors in Koyama's Table 2 than in my processed data). Because of the error correction techniques used, I do not know if any typos are present in Koyama's Table 1, since minor errors would have been automatically corrected. However, while verifying that all of the corrected solutions were indeed primitive, I did discover the erroneous inclusion of the nonprimitive solution

$$352 = -164930^3 - 193574^3 + 227276^3$$

in Koyama's Table 1. Excluding this solution, Koyama's table gives 5417 primitive solutions to (1). A (corrected) digital version of Koyama's data is available at `http://www.math.rutgers.edu/~erowland/data/koyama`.

Aside from the cases $n = 2$ and 128, no polynomial families of solutions to (1) are known for noncubes $1 \leq n \leq 1000$. I systematically searched for polynomial solutions for each noncube n in this range. For each n, my program looks at the set S of the first (at most) eight known solutions $s_1 = (x_1, y_1, z_1), \ldots, s_8 = (x_8, y_8, z_8)$, as given by Table 1. For each subset $S' \subset S$ and each 8-tuple $(\sigma_1, \ldots, \sigma_8)$ of permutations on $\{x, y, z\}$, it interpolates polynomials p_x, p_y, p_z on the three sequences $x(\sigma_1 s_{i_1}), \ldots, x(\sigma_k s_{i_k})$; $y(\sigma_1 s_{i_1}), \ldots, y(\sigma_k s_{i_k})$; and $z(\sigma_1 s_{i_1}), \ldots, z(\sigma_k s_{i_k})$ respectively, where $i_1 < \cdots < i_k$. If $n = (p_x(k+1))^3 + (p_y(k+1))^3 + (p_z(k+1))^3$, then a potential polynomial solution has been found.

Several positive results were returned in the search, but except for 2 and 128 these did not represent general solutions and indeed did not provide any numeric solutions not already appearing in Koyama's table. (For 2 and 128 the program found the known polynomial solutions but no additional polynomial solutions.) As a typical example of these false positives, consider the interpolating quadratic polynomials on the coordinates of the permutations $(-1, 3, 8)$, $(11, -4, -9)$, $(17, -10, -15)$ of the solutions $(-1, 3, 8)$, $(-4, -9, 11)$, $(-10, -15, 17)$ for $n = 538$. The interpolating polynomials obtained are

$$p_x(t) = -3t^2 + 21t - 19,$$
$$p_y(t) = t^2/2 - 17t/2 + 11,$$
$$p_z(t) = 11t^2/2 - 67t/2 + 36,$$

the values of which for the first few t are:

t	$p_x(t)$	$p_y(t)$	$p_z(t)$	$p_x(t)^3 + p_y(t)^3 + p_z(t)^3$
0	-19	11	36	41128
1	-1	3	8	538
2	11	-4	-9	538
3	17	-10	-15	538
4	17	-15	-10	538
5	11	-19	6	-5312
6	-1	-22	33	25288
7	-19	-24	71	337228

4. Ramanujan's Solution

It should finally be noted that Ramanujan discovered an infinite family of solutions to (1) for $n = \pm 1$ that is given not by polynomials but by generating functions. Let

$$\sum_{n=0}^{\infty} a_n x^n = \frac{9x^2 + 53x + 1}{x^3 - 82x^2 - 82x + 1},$$
$$\sum_{n=0}^{\infty} b_n x^n = \frac{-12x^2 - 26x + 2}{x^3 - 82x^2 - 82x + 1},$$
$$\sum_{n=0}^{\infty} c_n x^n = \frac{-10x^2 + 8x + 2}{x^3 - 82x^2 - 82x + 1}.$$

Then $a_n^3 + b_n^3 + c_n^3 = (-1)^n$. (See [5] for a proof.) The existence of this result, however, appears to hinge on special circumstances of the solution

$$(A^2 + 7AB - 9B^2)^3 + (2A^2 - 4AB + 12B^2)^3 = (2A^2 + 10B^2)^3 + (A^2 - 9AB - B^2)^3,$$

and it is not obvious how to generalize it or systematically search for similar results.

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