**Problem 1** (25pts)**.** Let $\mathcal{C} := \{(t^1, t^2, ..., t^k) \in \mathbb{R}^k \mid 0 < t_i < 1 \text{ and } \sum_{i=1}^{k} t_i = 1\}$. Show $k$ sequences in a vector space $(\mathcal{E}, \|\cdot\|)$, say $\{\alpha_n^1\}_{n\geq 1}$, $\{\alpha_n^2\}_{n\geq 1}$, ..., $\{\alpha_n^k\}_{n\geq 1}$ converge to the same limit, i.e.,

$$\lim_{n\to\infty} \alpha_n^1 = \lim_{n\to\infty} \alpha_n^2 = \cdots = \lim_{n\to\infty} \alpha_n^2 = \cdots \lim_{n\to\infty} \alpha_n^k = \ell$$

*if and only if*, for any random choice of vectors $T_n = (t_n^1, t_n^2, \cdots, t_n^k) \in \mathcal{C}$, one has

$$\lim_{n\to\infty} \left[ \sum_{i=1}^{k} t_n^i \alpha_n^i \right] = \ell.$$

**Problem 2** (25pts)**.** Let $(M, d)$ be a metric space and $A$ a subset of $M$. Suppose that a sequence $\{p_n\}$ in $M$ converges to an interior point of $A$. Show that all, but possibly finitely many, points of the sequence are interior points of $A$.

**Problem 3** (25pts)**.** Let $F$ be a closed subset and $K$ be a compact subset of a locally compact metric space $(M, d)$. Show there exist a point $X_0 \in K$ and a point $Y_0 \in F$, such that

$$d(X_0, Y_0) = \inf \{d(X, Y) : X \in K \text{ and } Y \in F\}.$$

**Problem 4** (25pts)**.** Recall a metric space $(M, d)$ is said to be separable if there exists a dense sequence in $M$. Let $(M, d)$ be a compact metric space. Show that $M$ is complete and separable.

**Problem 5** (25pts). Show that a metric space $(M, d)$ is complete if, and only if, for any decreasing sequence of (nonempty) closed subsets $F_1 \supset F_2 \supset F_3 \supset \cdots$, with

$$\lim_{n\to\infty} [\text{diam } F_n] = 0,$$

we have $\bigcap_{n=1}^{\infty} F_n = \{p\}$.

**Problem 6** (25pts)**.** Let $(M, d)$ be a complete metric space and $\{\mathcal{A}_n\}$ a sequence of open and dense subsets of $M$. Show that $A := \bigcap_{n=1}^{\infty} \mathcal{A}_n$ is a dense subset of $M$.

[Hint: Let $B$ be a ball in $M$. Show $B \cap A$ is nonempty. For that problem 5 will help.]