

Problem 1 (25pts). Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Assume

$$\Phi(X) \geq -1 - |X| + \epsilon X^2,$$

for some $\epsilon > 0$. Show there exists a point $\xi \in \mathbb{R}$ such that $\Phi'(\xi) = 0$.

Problem 2 (25pts). Let f be a bounded nonnegative integrable function in $[0, 1]$. Show the set $\mathcal{Z} := \{X \in [0, 1] \mid f(X) = 0\}$ is dense in $[0, 1]$ if and only if

$$\int_0^1 f(X) dX = 0.$$

Problem 3 (25pts). Let f_n be a sequence of differentiable functions on $[0, 1]$. Assume f'_n converges uniformly to zero and that $\{f_n(0)\}$ is Cauchy. Show that f_n converges uniformly to a constant.

Problem 4 (25pts). Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Assume that for any continuous non-decreasing function α ,

$$\int_0^1 f(X) d\alpha(X) = 0.$$

Prove that $f \equiv 0$.

Problem 5 (25pts). Let $f: (0, 1) \rightarrow \mathbb{R}$ be a differentiable function. Assume $f'(X) \neq e^X$ for all $X \in (0, 1)$. Show $\mathcal{P} := \{\xi \in (0, 1) \mid f(\xi) - e^\xi = 0\}$ has at most one point.

Problem 6 (25pts). Let $\varphi_k: [0, 1] \rightarrow \mathbb{R}$ be a sequence of bounded integrable functions. Assume φ_k converges uniformly to φ . Show φ is integrable and

$$\lim_{k \rightarrow +\infty} \int_0^1 \varphi_k(X) dX = \int_0^1 \varphi(X) dX.$$