Problem 1 (30pts)**.** Give the precise definition of[1]

(i) Metric Space:

(ii) Open set:

(iii) Closed set:

(iv) Compact set:

[1]For full credit, your answer must be carefully and formally stated.

(v) Convergent sequence:

(vi) Cauchy sequence and complete metric spaces:

(vii) Continuous and Differentiable functions:

(viii) Integrable functions:

Problem 2 (25pts)**.** Define $\mathcal{P}_f := \{X \subset \mathbb{N} \mid \text{Cardinality of } X < \infty\}$. Show $\mathcal{P}_f$ is enumerable.

Problem 3 (25pts)**.** Let (M, d) be a metric space and $A \subset M$. Show A is open if and only if $M \setminus A$ is closed.

Problem 4 (25pts)**.** Let M be a metric space and $K \subset V \subset M$, with K compact and V open. Show there is an $r > 0$ such that $\bigcup_{x \in K} B_r(x) \subset V$.

Problem 5 (30pts)**.** Let $p > 1$ and $\ell_p := \left\{ (a_n)_{n=1}^{\infty} \;\middle|\; \sum_{i=1}^{p} |a_n|^p < +\infty \right\}$. Define $\|(a_n)_{n=1}^{\infty}\|_p := \left(\sum_{i=1}^{p} |a_n|^p \right)^{1/p}$. Show $(\ell_p, \|\cdot\|_p)$ is a Banach space.

[2]

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[2] You don't have to show $\|\cdot\|_p$ defines a norm. Just that $(\ell_p, \|\cdot\|_p)$ is a complete vector space.

Problem 6 (30pts)**.** With the notation of previous problem, show $I\colon \ell_p \to \ell_q$, $I(a_n) := a_n$ is a well defined linear continuous map, provided $p \leq q$.

Problem 7 (30pts)**.** Let (M, d) be a complete metric space and $f\colon M \to M$ a function satisfying $d(f(x), f(y)) \leq \lambda d(x, y)$, for some $\lambda < 1$. Show f has a unique fixed point.
[Hint: Consider the sequence $a_0, f(a_0), f(f(a_0)), \cdots f^{(n)}(a_0), \cdots$]

Problem 8 (35pts). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function, satisfying $|f'(x)| \leq M$, for all $x \in \mathbb{R}$. Show, there exists a constant $c \in \mathbb{R}$, such that the function $\varphi\colon \mathbb{R} \to \mathbb{R}$, given by

$$\varphi(x) := x + cf(x),$$

is a differentiable bijection from $\mathbb{R}$ onto $\mathbb{R}$, whose inverse, φ^{-1}, is also differentiable.
[Hint: Consider previous problem]

Problem 9 (35pts)**.** Let $\mathrm{Lip}_K[0,1]$ denote the set of all Lipschitz functions on $[0,1]$ whose Lipschitz norm is at most K and $\varphi\colon \mathbb{R} \to \mathbb{R}_+$ be a continuous function. Consider the functional

$$\Phi(f) := \int_0^1 \varphi(f(\tau))d\tau.$$

Show that Φ attains its minimum over $\mathcal{E} := \{f \in \mathrm{Lip}_K[0,1] \mid f(0) = f(1) = 0\}$.
[Hint: Ascoli-Arzela Theorem]

Problem 10 (35pts)**.** Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Show

$$\lim_{n\to\infty} \left(\int_0^1 |f(\tau)|^n d\tau \right)^{1/n} \longrightarrow \max_{x\in[0,1]} |f(x)| := \|f\|_\infty.$$

[Hint: Show $\limsup \le \|f\|_\infty$ and $\liminf \ge \|f\|_\infty$. For the latter, given $\epsilon >$, $\exists I_\epsilon$ such that $|f|\big|_{I_\epsilon} > \|f\|_\infty - \epsilon$.]