Sample problems for Exam 3

Mathematical Analysis - Math 411

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Problem 1. Let $f, g, h\colon I \to \mathbb{R}$, satisfy $f(x) \le g(x) \le h(x)$. Assume $f(a) = h(a)$ and $f'(a) = h'(a)$. Show g is differentiable at a.

Problem 2. Assume f is differentiable at a. Show

$$\lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a).$$

Also show that for any sequences $x_n < a < y_n$, with $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = a$,

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{x_n - y_n} = f'(a).$$

Problem 3. A function $f\colon I \to \mathbb{R}$ is said to be convex if $f(ta + (1-t)b) \ge tf(a) + (1-t)f(b)$, for all $a, b \in I$ and $0 \le t \le 1$. Show that if f is twice differentiable, then f is convex if and only if $f''(x) \ge 0$, for all $x \in I$.

[Hint: Show first that $f''(a) = \lim\limits_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}$].

Problem 4. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Assume $f(a) = f(b) = 0$. Show for every $k \in \mathbb{R}$, there exists a $c \in (a,b)$, such that

$$f'(c) = kf(c).$$

[Hint: Consider $g(x) = f(x)e^{-kx}$]

Problem 5. Assume $f\colon [0,\infty) \to \mathbb{R}$ is twice differentiable. Assume $|f''(x)| \le M$, for all $x \in [0,\infty)$ and $\lim\limits_{x\to\infty} f(x)$ exists, say, equal ℓ. Show $\lim\limits_{x\to\infty} f'(x) = 0$.

Problem 6. Let $f\colon \mathbb{R} \to (0,\infty)$ be differentiable. Assume

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Show there exists a $\gamma \in \mathbb{R}$ such that $f'(\gamma) = 0$.

Problem 7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function, satisfying $|f'(x)| \le M$, for all $x \in \mathbb{R}$. Show, there exists a constant $c \in \mathbb{R}$, such that the function $\varphi\colon \mathbb{R} \to \mathbb{R}$, given by

$$\varphi(x) := x + cf(x),$$

is a bijection from $\mathbb{R}$ onto $\mathbb{R}$, whose inverse, φ^{-1}, is also differentiable.

Problem 8. Assume I is closed bounded, $f\colon I \to I$ differentiable and that $|f'(x)| \le 1$. Show that f has a fixed point $\xi = f(\xi)$. Is such a fixed point necessarily unique?

[Hint: For each $\lambda < 1$, consider $f_\lambda(x) = \lambda f(x)$ and apply Banach Fixed Point Theorem].

Problem 9. Consider the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show p is a homomorphism (i.e., continuous bijection whose inverse is also continuous) from $\mathbb{R}$ onto itself, if and only if $a^2 \leq 3b$. Show p^{-1} is differentiable if and only if $a^2 < 3b$.

Problem 10. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a function. A continuous function y is said to be a viscosity solution to

$$y'(x) = F(y(x)), \tag{0.1}$$

if whenever $y(x) - \varphi(x)$ has a local maximum at a, for some differentiable function φ, there holds

$$\varphi'(a) = F(y(a)).$$

Show that if y is a classical solution to (0.1), i.e., a differentiable function that satisfies $y'(x) = F(y(x)) \forall x \in \mathbb{R}$, then y is also a viscosity solution. Also, show that if y is a viscosity solution that happens to be differentiable everywhere, then y is indeed a classical solution to (0.1).

Problem 11. Let $\mathcal{H}$ be a real Hilbert space and $f\colon [0,1] \to \mathcal{H}$ a differentiable path. Show $\mathrm{Im}(f)$ lies on a sphere $S_r(a) := \{X \in \mathcal{H} \bigm| |X - a| = r\}$ if and only if $f(0) \in S_r(a)$ and $f'(t)$ is perpendicular to $f(t) - a$ for all $t \in [0,1]$.

Problem 12. Let $\lambda\colon [a,b] \to \mathcal{H}$ be a closed path, that is $\lambda(a) = \lambda(b)$. Show $\lambda(t_0)$ is perpendicular to $\lambda'(t_0)$ for some $t_0 \in [a,b]$.

Problem 13. Let $X\colon I \to \mathbb{R}^{n^2}$ be a differentiable path of matrices. Define $f(t) := X^k(t)$, for $k \geq 1$. Show that f is differentiable and compute $f'(t)$.

Problem 14. Let $f\colon (-\varepsilon, \varepsilon) \to O(n)$ be a differentiable path orthogonal matrices. Assume $f(0) = \mathrm{Id}$. Show $f'(0)$ is anti-symmetric. If $\det f \equiv 1$ show $\mathrm{Trace}(f'(0)) = 0$.

Problem 15. Let $f, g\ [0,1] \to \mathbb{R}$ continuous functions with $f \leq g$. Define $\phi(x) = f(x)$ is x is rational and $\phi(x) = g(x)$ is x is irrational. Show ϕ is integrable if and only if $f \equiv g$.

Problem 16. Let $P = \{t_0, t_1, \cdots t_n\}$ be a partition of $[a,b]$. We define $V(f,P) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$. A function is said to have bounded variation if $\sup\limits_{P} V(f,P) := \|f\|_{BV} < +\infty$ and we write $f \in BV[a,b]$.

1. Show that if f is monotone then $f \in BV[a,b]$. Also show that if f is Lipschitz, then $f \in BV[a,b]$.
2. If $f \in C^1$ then $\|f\|_{BV} = \int_a^b |f'(t)| dt$.
3. Show that $f(x) = x \sin(1/x)$ is continuous in $[0,1]$ but is not in $BV[0,1]$.
4. Show that if $f \in BV[a,b]$ then f is integrable. (Hint show that any BV function can be written as the difference of two monotone functions.)

Problem 17. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and $f(X+Y) = f(X)f(Y)$, for all $X, Y \in \mathbb{R}$. Show $f(X) = a^X$ for some $a > 0$ or $f \equiv 0$.

Problem 18. Let $f\colon [0,1] \to \mathbb{R}^d$ be an integrable path and $A \subset \mathbb{R}^d$ a convex subset.[1] Show $\int_0^1 f \in \overline{A}$.

Problem 19. Let $f, g\colon [0,1] \to \mathbb{R}^d$ be C^1 paths. Show

$$\int_a^b f(t) \cdot g'(t) dt = f \cdot g\Big|_a^b - \int_a^b f'(t) \cdot g(t) dt.$$

[1] A is convex if $a, b \in A$ implies $ta + (1-t)b \in A$ for all $t \in [0,1]$.

Problem 20. Let $f_k\colon [a,b] \to \mathbb{R}^d$ be a sequence of integrable paths. Assume f_k converges to f uniformly. Show that f is integrable and that $\lim_{k\to\infty} \int f_k = \int f$. Give examples to show that uniform convergence is necessary (that is pointwise convergence is not enough).

Problem 21 (Jensen's Theorem)**.** Let $J\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Show that

$$J\left(\int_0^1 f(t)dt\right) \leq \int_0^1 J(f(t))dt.$$

Problem 22. Let $f\colon [a,b] \to \mathbb{R}^d$ be an integrable path and $A\colon \mathbb{R}^d \to \mathbb{R}^d$ a linear map. Show that $A \circ f$ is integrable and

$$\int_a^b A(f(t))dt = A\left(\int_a^b f(t)dt\right).$$

Problem 23. Let $f, g\colon [a,b] \to (0,\infty)$ be continuous functions. Show

$$\left(\int_a^b f(t)g(t)dt\right)^2 = \int_a^b f^2(t)dt \times \int_a^b g^2(t)dt,$$

if and only if $f = cg$, for some constant $c \in (0,\infty)$.

Problem 24. Show $I\colon C[0,1] \to C[0,1]$ given by

$$I(f)(X) = \int_0^X f(t)dt,$$

is a compact[2] linear map. Compute $\|I\|$.

[2] A map is said to be compact if it maps bonded sets into relatively compact sets