Problem 1 (10pts)**.** Solve the general first order linear differential equation

$$\frac{dy}{dt} + p(t)y = g(t),$$

using the integrating factor technic.

Problem 2 (10pts)**.** Let $\alpha \in \mathbb{R}$ be a real number. Solve

$$\frac{dy}{dx} = y^{\alpha}, \quad y(0) = 1.$$

(Hint: Treat the case $\alpha = 1$ separately.)

Problem 3 (10pts)**.** Solve $\left(2xe^{x^2y} - \frac{1}{y}\right) + \left(e^{x^2y} + \frac{x}{y^2}\right)\frac{dy}{dx} = 0,\ y(0) = 2006.$

Problem 4 (12pts)**.** Consider the 3$^{\text{rd}}$ order linear differential equation

$$(\star) \quad \begin{cases} y^{(3)} - y'' - 2y' - 3y = 0. \\ y(0) = 0, \ y'(0) = 1, \ y''(0) = 2. \end{cases}$$

Find the 1$^{\text{st}}$ order differential equation in $\mathbb{R}^3$ equivalent to $(\star)$.

Problem 5 (12pts)**.** Solve the general homogeneous 2^{nd} order linear differential equation with constant coefficients

$$ay'' + by' + cy = 0.$$

Analyze all the three cases possible, i.e., $b^2 - 4ac > 0$, $b^2 - 4ac < 0$, and $b^2 - 4ac = 0$.

Problem 6 (10pts)**.** Solve the nonhomogeneous 2^{nd} order linear equation

$$y'' + y' - 2y = 2t, \qquad y(0) = 0, \quad y'(0) = 1.$$

Problem 7 (12pts)**.** Find the general solution of

$$y^{(6)} - 5y^{(5)} + 6y^{(4)} + 2y^{(3)} + 4y'' - 24y' + 16y = 0.$$

(Hint: The polynomial $p(t) = t^6 - 5t^5 + 6t^4 + 2t^3 + 4t^2 - 24t + 16$ can be factorized as $(t-1)(t-2)^3(t^2 + 2t + 2)$.)

Problem 8 (12pts)**.** Let $\{\phi_1(t), \phi_2(t)\}$ be a set of fundamental solutions of a 2^{nd} order linear differential equation $y'' + p(t)y' + q(t)y = 0$. Show the Graph(ϕ_1) cannot tangentially touch the Graph(ϕ_2).[1]

[1] Recall, the graph of a function f is defined as $\text{Graph}(f) := \{(t, f(t)) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$. Two curves γ_1, γ_2 in $\mathbb{R}^2$ is said to be tangential at $t = t_0$ if $\gamma_1(t_0) = \gamma_2(t_0)$ and $\gamma_1'(t_0) = \gamma_2'(t_0)$.

Problem 9 (12pts)**.** Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function and u_1 and u_2 global (i.e., defined on the whole real line) solutions of the differential equation

$$v_t(t) = F(v(t)).$$

Suppose there exist two points $t_1,\ t_2 \in \mathbb{R}$, such that $u_1(t_1) = x_0 = u_2(t_2)$. Show u_1 is a translation of u_2, i.e., show there exists a real number $a \in \mathbb{R}$ such that

$$u_1(t) = u_2(t + a), \quad \text{for all } t \in \mathbb{R}.$$

Problem 10 ($\star$Bonus$\star$ 10 extra pts)**.** Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz vector field in $\mathbb{R}^n$. Assume

$$F(x) \cdot x \leq 0,$$

for all $x \in \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ and $u(t)$ be the solution of

$$\begin{cases} u_t(t) = F\big(u(t)\big) \\ u(0) = x_0. \end{cases}$$

Prove $\|u(t)\| \leq \|x_0\|$, for all $t \geq 0$.