Sample Problems for Exam 1

Calculus IV - Math 292
Honors Section

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1 First order differential equations

Problem 1. Solve the general first order linear differential equation

$$\frac{dy}{dt} + p(t)y = g(t),$$

using the integrating factor technic.

Problem 2. Solve

$$\frac{dy}{dt} + \frac{2y}{t} = \frac{\cos t}{t^2}, \quad y(\pi) = 0, \quad t > 0.$$

Problem 3. Solve the general separable first order differential equation

$$\Psi(y)\frac{dy}{dx} = \Phi(x).$$

Problem 4. Solve

$$\frac{dy}{dx} = \frac{x - e^x}{y + e^y}, \quad y(0) = 1.$$

Problem 5. Let $M, N \colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions defined on a simply connected domain Ω. Assume,

$$M_y = N_x.$$

Solve the differential equation

$$M(x, y) + N(x, y)y' = 0.$$

Fully justify all your steps.

Problem 6. Solve

$$y\sin(xy) + x\sin(xy)y' = 0, \quad y(\pi) = 1.$$

2 General Theory for Differential Equations

Problem 7. Consider an n^{th} order differential equation in $\mathbb{R}^m$ of the general form

$$\begin{cases} y^{(n)} = F(t, y, y', \cdots, y^{(n-1)}) \\ y(0) = y_0,\ y'(0) = y_1,\ \cdots,\ y^{(n-1)}(0) = y_{n-1}. \end{cases}$$

where $F\colon [0,1] \times \underbrace{\mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{n \text{ times}} \to \mathbb{R}^m$, and $y_0,\ y_1,\ \cdots,\ y_{n-1}$ are vectors in $\mathbb{R}^m$. Show, solving the above differential equation is equivalent to solving a first order differential equation in $\mathbb{R}^{n \cdot m}$. Carefully construct such a $1st$ order ODE.

Problem 8. Enunciate and prove Banach Fixed Point Theorem. Carefully define any "non-freshman" math elements involved on it.

Problem 9. Let B denote the closed unit ball in $\mathbb{R}^d$ and $\varphi\colon B \to B$ a map satisfying $\|\varphi(x) - \varphi(y)\| \le \|x - y\|$. Show φ has a fixed point in B.[1] Can you assure uniqueness?

Problem 10. Let $B_r(x_0)$ denote the closed ball centered at x_0 with radius r in $\mathbb{R}^N$ and $F\colon [a,b] \times B_r(x_0) \to \mathbb{R}^N$ be a continuous function. Assume

$$\|F(t,x) - F(t,y)\| \le L\|x - y\|, \quad \forall t \in [a,b] \text{ and } x, y \in B_r(x_0).$$

Let $\|F(t,x)\| \le M$, for all $(t,x) \in [a,b] \times B_r(x_0)$.

(a) Show that if $\delta - a \le \frac{r}{M}$[2], the operator

$$\Gamma(\gamma)(t) := x_0 + \int_a^t F(s, \gamma(s))ds$$

maps $C\big([a,\delta]; B_r(x_0)\big)$ into itself. Here $C\big([a,\delta]; B_r(x_0)\big)$ stands for the set of all continuous functions defined on $[a,\delta]$ whose image lies inside of $B_r(x_0)$.

(b) Show that if $\delta - a \le \frac{1}{2L}$, the operator Γ above defined satisfies

$$\|\Gamma(\gamma_1) - \Gamma(\gamma_2)\|_\infty \le \frac{1}{2}\|\gamma_1 - \gamma_2\|_\infty,$$

where $\|f\|_\infty := \max\limits_{t \in [a,\delta]} \|f(t)\|_{\mathbb{R}^N}$.

(c) Show Γ has a unique fixed point $u \in C\big([a,\delta]; B_r(x_0)\big)$ (you may assume $C\big([a,\delta]; B_r(x_0)\big)$ endowed with $\|\cdot\|_\infty$ is a complete metric space). Furthermore, such a fixed point is C^1 and is the only curve that solves

$$\begin{cases} u_t(t) = F(t, u(t)) \\ u(a) = x_0. \end{cases}$$

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[1] Hint: For any $\lambda < 1$, consider the map $\varphi_\lambda(\xi) = \lambda\varphi(\xi)$. It is worthwhile to point our that a much stronger result is true. Any continuous function from B into itself has a fixed point.

[2] Correction made by Avi, thanks!

Problem 11. Let $\Phi\colon \mathbb{R}^N \to \mathbb{R}^N$ be of class C^1. Show given any vector $x_0 \in \mathbb{R}^N$, there exists a unique curve defined on some interval $[0, \delta]$, that solves

$$\begin{cases} u_t(t) = \Phi\big(u(t)\big) \\ u(0) = x_0. \end{cases}$$

Problem 12.

(a) (Gronwall's Inequality)[3] Let f, g be continuous functions on (a, b). Suppose

$$f(x) \leq K + \int_{x_0}^{x} g(s) f(s) ds,$$

for some positive constant K. Show

$$f(x) \leq K e^{\int_{x_0}^{x} g(s) ds}.$$

(b) Let F be a Lipschitz field in $\mathbb{R}^N$, say, $\|F(x) - F(y)\| \leq K\|x - y\|$. Let u_1 and u_2 be solutions of $u_i' = F(u_i)$ on some interval $[a, b]$. Show

$$\|u_1(t) - u_2(t)\| \leq \|u_1(a) - u_2(a)\| e^{K(t-a)}.$$

Notice that this result says that if two solutions started close, meaning $\|u_1(a) - u_2(a)\|$ is small, then $\|u_1(t) - u_2(t)\|$ will remain small for all $t \leq C$.

(c) Let F be a Lipschitz field in $\mathbb{R}^N$. For any $t \in \mathbb{R}$, define the flux operator $\Phi_t\colon \mathbb{R}^N \to \mathbb{R}^N$ as

$$\Phi_t(x) = u(t),$$

where u is the unique solution to $u_t = F(u)$, $u(0) = x$. Show Φ_t is a homeomorphism, i.e., Φ is a continuous bijection whose inverse is also continuous.

(d) Show, $\Phi_{t+s}(x) = \Phi_t \circ \Phi_s$, for all $t, s \in \mathbb{R}$.

Problem 13. Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function, i.e., $\|F(x) - F(y)\| \leq L\|x - y\|$. Show given a initial datum $x_0 \in \mathbb{R}^n$, the differential equation

$$u_t(t) = F(u(t)), \quad u(0) = x_0,$$

has a unique solution defined for all time $t \in \mathbb{R}^n$.

Problem 14. Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function and u a solution of the differential equation $u_t(t) = F(u(t))$. We define its trajectory by $\tau(u) = \{x \in \mathbb{R}^n \mid x = u(t) \text{ for some } t > 0\}$. Show that if u^1 and u^2 are solution of the differential equation $u_t^i(t) = F(u^i(t))$, $i = 1, 2$, then either $\tau(u^1) = \tau(u^2)$ or $\tau(u^1) \cap \tau(u^2) = \emptyset$.

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[3] Hint: Define $w = \int_{x_0}^{x} g(s) f(s) ds$. Show $w'(x) \leq K g(t) + g(x) w(x)$. Argue as in the integrating factor technic (problem 1) to obtain $f(x) \leq K + K e^{\int g} \int_{x_0}^{x} g(s) e^{-\int g} ds$. Use $g(s) e^{\int_s^x g(\lambda) d\lambda} = -\frac{d}{ds}\left(e^{\int_s^x g(\lambda) d\lambda}\right)$.

3 2nd and nth Order Differential Equations

Problem 15. Solve the general homogeneous $2nd$ order linear differential equation with constant coefficients

$$ay'' + by' + cy = 0.$$

Analyze all the three cases possible, i.e., $b^2 - 4ac > 0$, $b^2 - 4ac < 0$, or $b^2 - 4ac = 0$.

Problem 16. Let y_1 and y_2 solve the same homogeneous $2nd$ order linear differential equation, $y'' + p(t)y' + q(t)y = 0$ on (a, b). Show $W(y_1, y_2)(t) = c \exp\left[-\int p(t)dt\right]$. Use that to show that if there exits an interior point $t_0 \in (a, b)$, such that $y_1(t_0) = y_2(t_0) = 0$, then $y_1 = \lambda y_2$ for some $\lambda \in \mathbb{R}$.

Problem 17. Suppose y_1 and y_2 are solutions to the homogeneous problem $y'' + p(t)y' + q(t)y = 0$. Show

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} ds,$$

is a particular solution to the nonhomogeneous $y'' + p(t)y' + q(t)y = g(t)$.

Problem 18. Solve

$$y'' + 2y' + y = e^t,$$

by the method of undetermined coefficients and by the the method of variation of parameters.

Problem 19. Consider the $2nd$ order differential equation of the form

$$\begin{cases} x'' - \lambda x = 0 \\ x(0) = x(L) = 0. \end{cases}$$

where $L > 0$. Find all $\lambda \in \mathbb{R}$, for which the above equation has a non-identically zero solution.

Problem 20 (Viscosity solutions)**.** Consider a second order differential operator $L(f) := f'' + p(t)f' + q(t)f$. A function continuous function u is said to be a **viscosity solution** of $L(u) = g(t)$, provided the following two hypothesis happen:

1. If φ is a C^2 function touching u by above, i.e., $\varphi(t_0) = u(t_0)$ and $\varphi - u$ has a local minimum at t_0[4], then
$$L(\varphi)(t_0) \geq g(t_0).$$
2. If ψ is a C^2 function touching u by below, i.e., $\psi(t_0) = u(t_0)$ and $\psi - u$ has a local maximum at t_0, then
$$L(\phi)(t_0) \leq g(t_0).$$

(a) Show any classical solution, i.e., a C^2 function u for which $L(u)(t) = g(t)$ $\forall t$, is a viscosity solution.

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[4] corrected by Mathew, thanks!

(b) Show, if u is a C^2 viscosity solution of $L(u) = g(t)$, then, actually u is a classical solution, i.e., for all t, $L(u)(t) = g(t)$.

Problem 21. Consider the polynomial function $p(t) := t^n + a_{n-1}t^{n-1} + \cdots a_1 t + a_0$. Assume $p(t)$ can be uniquely factorized as

$$p(t) = \prod_{i=1}^{k}(t - \rho_i)^{\sigma_i} \times \prod_{j=1}^{m}\left(t^2 + 2\alpha_j t + (\alpha_j^2 + \beta_j^2)\right)^{\tau_j},$$

where $\rho_i, \alpha_i, \beta_i \in \mathbb{R}$ and $\sigma_i, \tau_i \in \mathbb{N}$. Solve the differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots a_1 y' + a_0 y = 0.$$

Problem 22. Let $p(t) = \prod_{i=1}^{5}(t-i)^i \times \prod_{j=1}^{3}\left(t^2+2jt+j^2+\frac{1}{j^2})\right)^j$. Find the general solution of $p(D)(u) = 0$, where $p(D)$ is the differential operator given by

$$p(D) := \prod_{i=1}^{5}(\frac{d}{dt} - i)^i \times \prod_{j=1}^{3}\big(\frac{d^2}{dt^2} + 2j\frac{d}{dt} + j^2 + \frac{1}{j^2})\big)^j.$$

Problem 23. Show the only solution to

$$\begin{cases} y^{(12)}(t) + y^{(8)}(t) + y^{(4)}(t) + y(t) = 0 \\ y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = y^{(5)}(0) = y^{(6)}(0) = 0 \\ y(1) = y'(1) = y''(1) = y'''(1) = y^{(4)}(1) = y^{(5)}(1) = y^{(6)}(1) = 0. \end{cases}$$

is the identically zero function.

4 General Theory for Linear Differential Equations

Problem 24. Let $M(n)$ denote the space of $n \times n$ matrices and $A\colon [0,\infty) \to M(n)$ be a continuous curve. Show, given a initial datum $x_0 \in \mathbb{R}^n$, the differential equation

$$u_t(t) = A(t)u(t) + b(t), \quad u(0) = x_0,$$

has a unique solution defined for all time $t \in \mathbb{R}$, where $b\colon \mathbb{R}^n \to \mathbb{R}^n$ is, say, continuous.

Problem 25. Consider a continuous curve $A\colon [0,\infty) \to M(n)$. Define $\mathcal{S}$ to be the space of all solutions of $u_t = A(t)u(t)$. Show $\mathcal{S}$ is an n-dimensional vector space.

Problem 26. Consider the linear differential equation

$$(P) \qquad\qquad u_t = A(t)u(t).$$

An $n \times n$ matrix $\phi(t)$ whose columns form a basis for $\mathcal{S}$ is called a fundamental matrix of (P).

(a) Show $\phi(t)$ is a fundamental matrix of (P), if and only if, $\phi'(t) = A(t)\phi(t)$ (matrix equation), and for some t_0, $\phi(t_0)$ is nonsingular.

(b) Let $\phi(t)$ and $\psi(t)$ be solutions of the matrix equation $X'(t) = A(t)X(t)$. Assume $\phi(t)$ is a fundamental matrix of (P). Show there exists a unique matrix C such that, for all t,

$$\psi(t) = \phi(t)C.$$

Furthermore, show C is nonsingular, if and only if ψ is a fundamental matrix of problem (P) as well.

(c) Let $\phi(t)$ be a fundamental solution of (P). Then the solution of

$$u_t(t) = A(t)u(t) + b(t), \quad u(t_0) = x_0,$$

is given by

$$\varphi(t) = \phi(t)\left[\phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \phi^{-1}(s)b(s)ds\right].$$

Problem 27. Assume $A(t)$ is anti-symmetric[5] for all t. Show for all fundamental matrix $\phi(t)$ of $x' = A(t)x$, one has

$$\phi(t)^T\phi(t) \equiv C,$$

where C is a constant matrix. In particular, if $\phi(t_0)$ is orthogonal, i.e., $\phi(t_0)^T\phi(t_0) = Id$, for some t_0, then $\Phi(t)$ is orthogonal for all $t \in I$.

Problem 28. Consider the homogeneous constant coefficients linear equation

$$(L) \qquad\qquad u' = Au.$$

Let $\phi(t)$ be a fundamental solution such that $\phi(0) = Id$, where Id stands for the identity matrix.

(a) Show $\phi(t+s) = \phi(t)\phi(s)$.

(b) Prove for any $t \in \mathbb{R}$, $\big[\phi(t)\big]^{-1} = \phi(-t)$

(c) Show the series

$$\sum_{n=1}^{\infty} \frac{t^k A^k}{k!},$$

converges to $\phi(t)$ in $\mathbb{R}$, uniformly over compacts.

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[5] A matrix A is said to be anti-symmetric, if $A^T = -A$, where A^T denotes the transposed of A.