

2. POLYNOMIAL INTERPOLATION

2.1. Interpolation error. We now turn to an analysis of the error $f(\bar{x}) - P_n(\bar{x})$, for $\bar{x} \neq x_0, \dots, x_n$. For the moment, consider \bar{x} fixed, and let P_{n+1} denote the polynomial of degree $\leq n+1$ interpolating $f(x)$ at x_0, x_1, \dots, x_n and \bar{x} . Using the Newton form of the interpolating polynomial, we know that

$$P_{n+1}(x) = P_n(x) + f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^n (x - x_j).$$

Since $P_{n+1}(\bar{x}) = f(\bar{x})$, we have by the above formula that

$$f(\bar{x}) = P_n(\bar{x}) + f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^n (\bar{x} - x_j),$$

and so a representation of the error is given by

$$(2.1) \quad f(\bar{x}) - P_n(\bar{x}) = f[x_0, \dots, x_n, \bar{x}] \prod_{j=0}^n (\bar{x} - x_j).$$

We next find an equivalent expression for $f[x_0, \dots, x_n, \bar{x}]$, valid when f is sufficiently smooth.

Definition: Suppose r is a non-negative integer. Then f is a function in $C^r[a, b]$ if f and its first r derivatives are continuous on $[a, b]$. So $C^0[a, b]$ denotes the space of continuous functions on $[a, b]$ and we shall use $C^{-1}[a, b]$ to denote functions which may be discontinuous on $[a, b]$.

Lemma 1. Let $f \in C^k[a, b]$ and x_0, \dots, x_k be distinct points in $[a, b]$. Then there exists a point $\xi \in (a, b)$ such that $f[x_0, x_1, \dots, x_k] = f^{(k)}(\xi)/k!$.

Proof. Let $P_k(x)$ denote the polynomial of degree $\leq k$ interpolating f at x_0, \dots, x_k and define $e_k(x) = f(x) - P_k(x)$. Observe first that $e_k(x)$ has at least $k+1$ distinct zeroes at the points x_0, \dots, x_k . Since f and therefore e_k is differentiable on (a, b) , we can use Rolle's theorem to conclude that between each two adjacent zeroes of $e_k(x)$, there exists at least one zero of $e'_k(x)$. Hence $e'_k(x)$ has at least k zeroes in (a, b) . Since f and therefore $e_k(x)$ is k times differentiable in (a, b) , we can continue this argument to conclude that $e''_k(x)$ has at least $k-1$ zeroes in (a, b) , and finally that $e_k^{(k)}(x)$ has at least one zero in (a, b) . If we denote that zero by the point ξ , then

$$0 = e_k^{(k)}(\xi) = f^{(k)}(\xi) - P_k^{(k)}(\xi).$$

Now by formula (5.2),

$$P_k(x) = \sum_{i=0}^k f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j) = f[x_0, \dots, x_k] x^k + \text{polynomial of degree } < k.$$

Hence $P_k^{(k)}(x) = f[x_0, \dots, x_k] k!$ for all x and so $f[x_0, \dots, x_k] = f^{(k)}(\xi)/k!$ for some $\xi \in (a, b)$. \square

Combining Lemma (2) with the representation of the interpolation error given by formula (6.1), we get the following result.

Theorem 1. *Suppose that $f \in C^{n+1}[a, b]$ and that $P_n(x)$ is a polynomial of degree $\leq n$ that interpolates f at the $n + 1$ distinct points $x_0, \dots, x_n \in (a, b)$. Then for all $x \in [a, b]$, there exists a point $\xi \in (a, b)$ (depending on x) such that*

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^n (x - x_j).$$

Proof. If x is equal to any of the interpolation points x_j , then the equation holds since both sides are zero. If x is not equal to any of the interpolation points, we have from the representation of the error given by (6.1) with $\bar{x} = x$, that

$$f(x) - P_n(x) = f[x_0, \dots, x_n, x] \prod_{j=0}^n (x - x_j).$$

Since the $n + 2$ points x_0, \dots, x_n, x are all distinct, we can apply Lemma (2) to conclude that $f[x_0, \dots, x_n, x] = f^{(n+1)}(\xi)/(n+1)!$ for some $\xi \in (a, b)$ (depending on x). Substituting this result gives the theorem. \square

Note that since ξ is not known explicitly, this formula can not be used to find the actual error. This is not surprising, since f can take on any value at non-interpolation points. However, the theorem can be used to find an upper bound on the interpolation error if we have more information about the way the derivatives of f behave. The following results follow directly from the theorem.

Corollary 1. *Suppose the conditions of Theorem (4) are satisfied. If $\max_{a \leq \xi \leq b} |f^{(n+1)}(\xi)| \leq M_{n+1}$, then*

$$(2.2) \quad |f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |(x - x_0)(x - x_1) \cdots (x - x_n)|, \quad \text{for all } x \in [a, b]$$

and

$$(2.3) \quad \max_{a \leq x \leq b} |f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{a \leq x \leq b} |(x - x_0)(x - x_1) \cdots (x - x_n)|.$$

Let us now consider an application of these results to find a bound on the error in linear interpolation. Recall that the linear polynomial interpolating $f(x)$ at x_0 and x_1 is given by $P_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$. If $x \in [x_0, x_1]$ and $\max_{x_0 \leq \xi \leq x_1} |f''(\xi)| \leq M_2$, then we have by (6.2) with $a = x_0$, $b = x_1$ that

$$|f(x) - P_1(x)| \leq \frac{M_2}{2} |(x - x_0)(x - x_1)|, \quad \text{for all } x \in [a, b]$$

and by (6.3) that

$$\max_{x_0 \leq x \leq x_1} |f(x) - P_1(x)| \leq \frac{M_2}{2} \max_{x_0 \leq x \leq x_1} |(x - x_0)(x - x_1)| \leq \frac{M_2}{8} (x_1 - x_0)^2,$$

since the maximum occurs at the midpoint $(x_0 + x_1)/2$.

2.2. Divided differences for repeated points. Recall

$$f[x_0, x_1, \dots, x_k] = \sum_{i=0}^k \frac{f(x_i)}{\prod_{\substack{j=0 \\ j \neq i}}^n (x_i - x_j)}.$$

Note that $f[y_0, \dots, y_k] = f[x_0, \dots, x_k]$ if y_0, \dots, y_k is any reordering of x_0, \dots, x_k . So far, $f[x_0, \dots, x_k]$ has only been defined when the points x_0, \dots, x_k are distinct. We now wish to extend the definition to include the case of repeated points.

Example: $k = 1$. $f[x_0, x_1] = [f(x_1) - f(x_0)]/(x_1 - x_0)$, $x_1 \neq x_0$. If $f \in C^1$, then $\lim_{x_0, x_1 \rightarrow y} f[x_0, x_1] = f'(y)$. So we define $f[x_0, x_1] = f'(x_0)$ when $x_0 = x_1$. In general, define

$$f[x_0, \dots, x_k] = \frac{f^{(k)}(y)}{k!}, \quad \text{if } x_0 = x_1 = \dots = x_k = y.$$

With this interpretation, we can still use the Newton formula

$$P_n(x) = \sum_{i=0}^n f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

to describe the polynomial of degree $\leq n$ interpolating $f(x)$ at x_0, \dots, x_n , even when the points x_0, \dots, x_n are not necessarily distinct. The error is still given by the formula

$$f(x) - P_n(x) = f[x_0, \dots, x_n, x] \prod_{j=0}^n (x - x_j),$$

where by the interpolating polynomial we now mean that if the point z appears $k + 1$ times among x_0, \dots, x_n , then

$$P_n^{(j)}(z) = f^{(j)}(z), \quad j = 0, \dots, k.$$

Furthermore, if $f \in C^{n+1}(a, b)$ and $x_0, \dots, x_n, x \in [a, b]$, then one can show that

$$f[x_0, \dots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some ξ satisfying $\min(x_0, \dots, x_n, x) \leq \xi \leq \max(x_0, \dots, x_n, x)$.

Example: $f(x) = \ln x$. Calculate $f(1.5)$ by cubic interpolation using the data: $f(1) = 0$, $f'(1) = 1$, $f(2) = 0.693147$, $f'(2) = 0.5$. Take $x_0 = 1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 2$.

TABLE 2

Divided difference table

x_k	$f(x_k)$	$f[.]$	$f[.,.]$	$f[.,.,.]$
1	0			
		1		
1	0		-.306853	
		.693147		.113706
2	.693147		-0.193147	
		.5		
2	.693147			

Using the divided difference table, we get

$$\begin{aligned} P_3(x) &= f(1) + f[1, 1](x - 1) + f[1, 1, 2](x - 1)^2 + f[1, 1, 2, 2](x - 1)^2(x - 2) \\ &= 0 + 1(x - 1) + (-.306853)(x - 1)^2 + (.113706)(x - 1)^2(x - 2). \end{aligned}$$

So $P_3(1.5) = .409074$.

From the error formula, we have $\ln(x) - P_3(x) = f^{(4)}(\xi)(x - 1)^2(x - 2)^2/4!$. Hence,

$$|\ln(1.5) - P_3(1.5)| \leq \frac{1}{4!} \max_{1 \leq \xi \leq 2} \frac{6}{\xi^4} (.5)^4 = \frac{1}{64} = 0.015624$$

The actual error is .00361.