

### 13.6. Stability of linear multistep methods.

Definition: 1st and 2nd characteristic polynomial of a multistep method:

$$\rho(z) = z^{p+1} - \sum_{i=0}^p a_i z^{p-i}, \quad \sigma(z) = \sum_{i=-1}^p b_i z^{p-i}.$$

The linear multistep method is consistent if  $\rho(1) = 0$  and  $\rho'(1) = \sigma(1)$ . The first identity is obvious. The second implies

$$(p+1) - \sum_{i=0}^p (p-i)a_i = \sum_{i=-1}^p b_i.$$

Hence

$$p(1 - \sum_{i=0}^p a_i) + 1 - \sum_{i=0}^p (-i)a_i - \sum_{i=-1}^p b_i = 0.$$

Since the first term in the sum is zero, then so is the second.

Definition: The linear multistep method is said to be zero-stable (satisfy the root condition) if no root of the characteristic polynomial  $\rho(z)$  has modulus greater than one and if every root of modulus one is simple.

**Theorem 13.** *A necessary condition for convergence of a linear multistep method is that it be zero-stable.*

*Proof.* We only give the proof in the case that the roots of  $\rho(z)$  are real simple roots. If the method is convergent, then it is convergent for the IVP  $y' = 0$ ,  $y(0) = 0$ , whose solution is  $y(x) = 0$ . For this problem, the method becomes  $y_{n+1} = \sum_{i=0}^p a_i y_{n-i}$ . If the method is convergent, then by (i), for any  $x > 0$ ,

$$\lim y_n^h = 0, \quad n \rightarrow \infty, \quad h \rightarrow 0, \quad nh = x$$

for all solutions  $\{y_n\}$  of the difference equation satisfying (ii)  $\lim_{h \rightarrow 0} y_k(h) = 0$ ,  $k = 0, \dots, p$ . We first show that all roots have modulus  $\leq 1$ . Let  $z = r$  be a real root of  $\rho(z)$ . Then  $y_n = r^n$  is a solution of the difference equation and so is  $y_n = hr^n$ . Note that this second solution satisfies (ii). Hence, (i) must hold, i.e.,  $\lim_{n \rightarrow \infty} xr^n/n = 0$ . Now

$$\lim_{n \rightarrow \infty} xr^n/n = x \lim_{n \rightarrow \infty} r^n/n = 0$$

if  $0 \leq |r| \leq 1$ . If  $r > 1$ , using L'Hospital's rule,

$$x \lim_{n \rightarrow \infty} r^n/n = x \lim_{n \rightarrow \infty} r^n \ln r / 1 = \infty.$$

A similar result hold if  $r < -1$ . Hence for (i) to hold, we require  $|r| \leq 1$ . □

**Theorem 14.** *A necessary and sufficient condition for a linear multistep method to be convergent is that it be consistent and zero-stable.*

*Proof.* We have shown these conditions are necessary. The proof of sufficiency can be found in Henrici: Discrete Variable Methods in Ordinary Differential Equations. □

Question: How high an order can be achieved in a  $(p+1)$  step method if it is consistent and zero-stable? In seeking high order methods, we automatically get consistency; zero-stability poses a more difficult constraint. Recall that a  $(p+1)$  step method has  $(2p+3)$  coefficients,  $p+1$   $a_i$ s and  $p+2$   $b_i$ 's. If the method is explicit, this number is reduced by one. Hence, we can expect at most order  $2p+2$  for an implicit method and  $2p+1$  for an explicit method (recall that if a method has order  $r$ , it is exact for all polynomials of degree  $\leq r$ ). However, the following result is known.

**Theorem 15.** (*Dahlquist*) *No zero-stable  $p+1$  step linear multistep method can have order exceeding  $p+2$  when  $p$  is even or exceeding  $p+3$  when  $p$  is odd.*

A zero-stable  $p+1$  step method which has order  $p+3$  is called an optimal method. It can be shown that for an optimal method, all the roots of  $\rho(z)$  lie on the unit circle.

Example: Simpson's rule:  $y_{n+1} = y_{n-1} + (h/3)[f_{n+1} + 4f_n + f_{n-1}]$ . Since  $p = 1$ , this is a two-step method. The local truncation error is  $-(1/90)h^5 y^{(5)}(\xi)$ . It is a fourth order method, so Simpson's rule is an optimal method. However, we shall see that Simpson's rule has computational disadvantages that make it unsuitable as a general purpose method. These disadvantages are shared by all optimal order methods. Hence, we will not choose the coefficients in a multistep method solely to achieve maximum order.

To understand this issue, consider the problem  $y' = -y$ ,  $y(0) = 1$ , whose exact solution is  $y(x) = e^{-x}$ . We apply the midpoint rule method  $y_{n+1} = y_{n-1} + 2hf_n$ , which in this case becomes  $y_{n+1} + 2hy_n - y_{n-1} = 0$ . Since this is a linear difference equation with constant coefficients, we solve it by first computing the roots of the characteristic polynomial  $\rho(z) = z^2 + 2hz - 1 = 0$ . Then  $z = -h \pm \sqrt{1+h^2}$ , so the general solution has the form

$$y_n = C_1(-h + \sqrt{1+h^2})^n + C_2(-h - \sqrt{1+h^2})^n.$$

Set  $y_0 = 1$  and leave  $y_1$  arbitrary for the moment and solve for  $C_1$  and  $C_2$ .

$$y_0 = 1 = C_1 + C_2, \quad y_1 = C_1(-h + \sqrt{1+h^2}) + C_2(-h - \sqrt{1+h^2}).$$

Then

$$C_1 = \frac{1}{2} + \frac{y_1 + h}{2\sqrt{1+h^2}}, \quad C_2 = \frac{1}{2} - \frac{y_1 + h}{2\sqrt{1+h^2}}.$$

Inserting this result, we get

$$y_n = \left(\frac{1}{2} + \frac{y_1 + h}{2\sqrt{1+h^2}}\right) (-h + \sqrt{1+h^2})^n + \left(\frac{1}{2} - \frac{y_1 + h}{2\sqrt{1+h^2}}\right) (-h - \sqrt{1+h^2})^n.$$

Note that the term  $|-h + \sqrt{1+h^2}| = |1 - h + O(h^2)| < 1$  and hence  $(-h + \sqrt{1+h^2})^n \rightarrow 0$  as  $n \rightarrow \infty$ . However,  $|-h - \sqrt{1+h^2}| > 1$ , so that unless we choose  $y_1$  so that  $\frac{1}{2} - \frac{y_1+h}{2\sqrt{1+h^2}} = 0$ ,  $\lim_{n \rightarrow \infty} y_n = \pm\infty$ .

The above is an example of numerical instability. The true solution  $e^{-x} \rightarrow 0$  as  $x = nh \rightarrow \infty$ , while for fixed  $h$  the approximate solution  $\rightarrow \infty$  as  $n \rightarrow \infty$ .

However, if we consider the convergence of the sequence  $\{y_n^h\}$  as  $h \rightarrow 0$ ,  $n \rightarrow \infty$  and  $x = nh$  and make the assumption that  $\lim_{h \rightarrow 0} y_1^h = y_0 = 1$ , then

$$\lim_{h \rightarrow 0} \left( \frac{1}{2} + \frac{y_1 + h}{2\sqrt{1+h^2}} \right) = 1, \quad \lim_{h \rightarrow 0} \left( \frac{1}{2} - \frac{y_1 + h}{2\sqrt{1+h^2}} \right) = 0.$$

Furthermore, for  $x = nh$ ,

$$\lim_{h \rightarrow 0, n \rightarrow \infty} (-h + \sqrt{1+h^2})^n = \lim_{h \rightarrow 0} [(-h + \sqrt{1+h^2})^{1/h}]^x.$$

Let  $y = \lim_{h \rightarrow 0} [(-h + \sqrt{1+h^2})^{1/h}]$ . Then

$$\ln y = \lim_{h \rightarrow 0} [\ln(-h + \sqrt{1+h^2})/h] = \lim_{h \rightarrow 0} \frac{\frac{2h}{2\sqrt{1+h^2}} - 1}{\sqrt{1+h^2} - h} = -1.$$

Hence,  $\ln y = -1$  so  $y = e^{-1}$ . Then

$$\lim_{h \rightarrow 0, n \rightarrow \infty} C_1(-h + \sqrt{1+h^2})^n = e^{-x}.$$

Thus, the first part of the solution of the difference equation gives an approximation to the true solution of the differential equation. One can easily show that  $|(-h - \sqrt{1+h^2})^n| \leq e^x$ . Hence, the second term is converging to zero, so the approximate solution is converging to the true solution.

To summarize, one root of the characteristic polynomial gives a solution that approximates the true solution. A second root gives a parasitic solution which for fixed  $h$  eventually blows up to give a bad overall approximation. Since the method converges, for any  $x$  and  $\epsilon$ , one can find a value of  $h$  such that  $|y_n^h - e^{-x}| < \epsilon$ . However, since the parasitic solution grows like  $e^x$ , this  $h$  would have to be impractically small for any reasonable size  $x$ .