

4. CUBIC SPLINE APPROXIMATION

4.1. Cubic spline interpolation. We consider the problem of finding a C^2 piecewise cubic function $S(x)$ that satisfies $S(x_i) = f(x_i)$, $i = 0, \dots, n$ plus two additional conditions. These are usually taken to be either $S''(x_0) = f''(x_0)$ and $S''(x_n) = f''(x_n)$ or $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$. We will consider the first set of conditions.

We will obtain $S(x)$ by first obtaining $S''(x)$ and then integrating. Since $S''(x)$ is a continuous piecewise linear function, it is uniquely determined by the values $S''(x_i)$, $i = 0, \dots, n$. On the subinterval $[x_{i-1}, x_i]$, we can write it in the form

$$S''(x) = S_i''(x) = \frac{x_i - x}{h_i} S''(x_{i-1}) + \frac{x - x_{i-1}}{h_i} S''(x_i),$$

where $h_i = x_i - x_{i-1}$. Integrate twice on each subinterval to get

$$S_i(x) = \frac{h_i^2}{6} \left[\frac{x_i - x}{h_i} \right]^3 S''(x_{i-1}) + \frac{h_i^2}{6} \left[\frac{x - x_{i-1}}{h_i} \right]^3 S''(x_i) + A_i \frac{x_i - x}{h_i} + B_i \frac{x - x_{i-1}}{h_i}.$$

Note: Integrating twice introduces an arbitrary linear function that we represent as above.

$$\text{Now} \quad f(x_{i-1}) = S_i(x_{i-1}) = \frac{h_i^2}{6} S''(x_{i-1}) + A_i, \quad f(x_i) = S_i(x_i) = \frac{h_i^2}{6} S''(x_i) + B_i.$$

Hence,

$$\begin{aligned} S_i(x) &= \frac{h_i^2}{6} \left[\frac{x_i - x}{h_i} \right]^3 S''(x_{i-1}) + \frac{h_i^2}{6} \left[\frac{x - x_{i-1}}{h_i} \right]^3 S''(x_i) \\ &\quad + [f(x_{i-1}) - \frac{h_i^2}{6} S''(x_{i-1})] \frac{x_i - x}{h_i} + [f(x_i) - \frac{h_i^2}{6} S''(x_i)] \frac{x - x_{i-1}}{h_i}. \end{aligned}$$

We next determine the values of $S''(x_i)$ by the conditions that S' is continuous at each x_i , i.e., $S'_i(x_i) = S'_{i+1}(x_i)$, $i = 1, \dots, n-1$. Now

$$\begin{aligned} S'_i(x) &= -\frac{h_i}{2} \left[\frac{x_i - x}{h_i} \right]^2 S''(x_{i-1}) + \frac{h_i}{2} \left[\frac{x - x_{i-1}}{h_i} \right]^2 S''(x_i) \\ &\quad - [f(x_{i-1}) - \frac{h_i^2}{6} S''(x_{i-1})] \frac{1}{h_i} + [f(x_i) - \frac{h_i^2}{6} S''(x_i)] \frac{1}{h_i}. \end{aligned}$$

Then

$$\begin{aligned} S'_i(x_i) &= \frac{h_i}{3} S''(x_i) + \frac{h_i}{6} S''(x_{i-1}) + \frac{1}{h_i} [f(x_i) - f(x_{i-1})], \\ S'_{i+1}(x_i) &= -\frac{h_{i+1}}{3} S''(x_i) - \frac{h_{i+1}}{6} S''(x_{i+1}) + \frac{1}{h_{i+1}} [f(x_{i+1}) - f(x_i)]. \end{aligned}$$

Equating these quantities to insure continuity, we get:

$$\frac{h_i}{6} S''(x_{i-1}) + \frac{h_i + h_{i+1}}{3} S''(x_i) + \frac{h_{i+1}}{6} S''(x_{i+1}) = f[x_i, x_{i+1}] - f[x_{i-1}, x_i].$$

Thus, the $n-1$ quantities $S''(x_1), \dots, S''(x_{n-1})$ are determined by solving the linear system

$$\begin{pmatrix} (h_1 + h_2)/3 & h_2/6 & \cdots & \cdots \\ h_2/6 & (h_2 + h_3)/3 & h_3/6 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & h_{n-1}/6 & (h_{n-1} + h_n)/3 \end{pmatrix} \begin{pmatrix} S''(x_1) \\ S''(x_2) \\ \cdots \\ S''(x_{n-1}) \end{pmatrix} = \begin{pmatrix} f[x_1, x_2] - f[x_0, x_1] - h_1 f''(x_0)/6 \\ f[x_2, x_3] - f[x_1, x_2] \\ \cdots \\ f[x_{n-1}, x_n] - f[x_{n-2}, x_{n-1}] - h_n f''(x_n)/6 \end{pmatrix}$$

To check that the matrix is nonsingular, we can use the following result.

Theorem 2. *Let $A = (a_{ij})$ be an $N \times N$ matrix. If $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$, $i = 1, 2, \dots, N$, then A is nonsingular.*

Proof. Suppose A is singular. Then there is a vector $x \neq 0$ such that $\sum_{j=1}^N a_{ij} x_j = 0$ for $i = 1, \dots, N$. Let k satisfy $|x_k| \geq |x_j|$, $j \neq k$. Note $|x_k| \neq 0$. Since $a_{kk} x_k = -\sum_{j \neq k} a_{kj} x_j$, $|a_{kk}| |x_k| \leq \sum_{j \neq k} |a_{kj}| |x_j|$. Hence

$$|a_{kk}| \leq \sum_{j \neq k} |a_{kj}| (|x_j|/|x_k|) \leq \sum_{j \neq k} |a_{kj}|.$$

Contradiction. □

4.2. Cubic spline basis functions. When the mesh points are equally spaced at a distance h apart, we can define basis functions for the space of cubic splines in the following simple way. We first define the function $B(x) \in C^2$ to satisfy: $B(x) = 0$ for $x < -2$ and $x > 2$,

$$\begin{aligned} B(-2) = 0, \quad B'(-2) = 0, \quad B''(-2) = 0, \\ B(2) = 0, \quad B'(2) = 0, \quad B''(2) = 0, \end{aligned}$$

and the condition $B(-1) + B(0) + B(1) = 1$. Note: a cubic spline on 4 subintervals has 7 degrees of freedom. Can show that $B(x)$ is given by:

$$\begin{aligned} B(x) = (x+2)^3/6, \quad -2 < x < -1, \quad B(x) = -x^3/2 - x^2 + 2/3, \quad -1 < x < 0, \\ B(x) = x^3/2 - x^2 + 2/3, \quad 0 < x < 1, \quad B(x) = -(x-2)^3/6, \quad 1 < x < 2, \end{aligned}$$

and $B(x) = 0$ for $x < -2$ and $x > 2$. To see this, note that $B(x)$ will have the form $A(2+x)^3$ for $-2 < x < -1$ and $C(2-x)^3$ for $1 < x < 2$. Hence

$$\begin{aligned} B(-1) = A, \quad B'(-1) = 3A, \quad B''(-1) = 6A, \\ B(1) = C, \quad B'(1) = -3C, \quad B''(1) = 6C, \end{aligned}$$

and so

$$\begin{aligned} B(x) = A + 3A(x+1) + 3A(x+1)^2 + D(x+1)^3, \quad -1 < x < 0, \\ B(x) = C - 3C(x-1) + 3C(x-1)^2 + E(x-1)^3, \quad 0 < x < 1. \end{aligned}$$

Then since $B \in C^2$, the following two expressions for each of the quantities $B(0)$, $B'(0)$ and $B''(0)$ must be equal.

$$\begin{aligned} B(0) &= 7A + D, & B'(0) &= 9A + 3D, & B''(0) &= 6A + 6D, \\ B(0) &= 7C - E, & B'(0) &= -9C + 3E, & B''(0) &= 6C - 6E. \end{aligned}$$

Solving these equations, we find: $C = A$, $D = -3A$, $E = 3A$. Finally, since $B(-1) + B(0) + B(1) = 1$, we find $A = 1/6$.

A graph of $B(x)$ is given below. To get a basis for the space of cubic splines defined on the partition $a = x_0 < \dots < x_n = b$, where $x_{i+1} - x_i = h$, we define for $i = -1, \dots, n+1$, the B-spline $B_i(x) = B([x - x_i]/h)$, where we restrict these functions to the interval $[a, b]$. This gives the required $n+3$ basis functions. Note that $B_i(x) \geq 0$. One can show that $\sum B_i(x) = 1$ (which partially explains the choice of the normalization condition $B(-1) + B(0) + B(1) = 1$).

Somewhat more complicated expressions (using ideas similar to those applied in the case of the piecewise cubic Hermite basis functions) are needed to define the B-splines when the mesh points are not equally spaced.

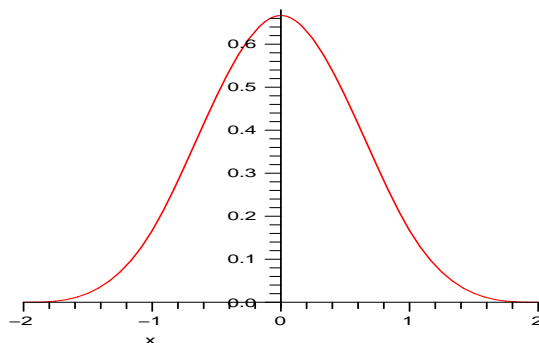


FIGURE 2. B-Spline

4.3. Error in cubic spline interpolation. One can derive the following error estimates for cubic spline interpolation. Again, we consider only the case when $S''(a) = f''(a)$ and $S''(b) = f''(b)$.

Theorem 3. *There exists constants C_0 , C_1 , and C_2 , independent of f , such that*

$$\begin{aligned} \max_{a \leq x \leq b} |f(x) - S(x)| &\leq C_0 h^4 M_4, & \max_{a \leq x \leq b} |f'(x) - S'(x)| &\leq C_1 h^3 M_4, \\ \max_{a \leq x \leq b} |f''(x) - S''(x)| &\leq C_2 h^2 M_4, \end{aligned}$$

where $M_4 = \max_{a \leq x \leq b} |f^{(4)}(x)|$.