4. Approximation of Elliptic Variational Inequalities

We have previously considered approximation of variational equalities, i.e., problems of the form: Find $u \in V$ such that $a(u,v) = (f,v)$ for all $v \in V$. When $a(u,v) = a(v,u)$, this problem can also be formulated as a minimization problem, i.e., defining $J(v) = \frac{1}{2}a(v,v) - (f,v)$, we consider the problem: Find $u \in V$ such that $J(u) \leq J(v)$ for all $v \in V$.

To get a variational inequality, we let K be a closed convex subset of V (i.e., if $u, v \in K$, then for $0 \leq t \leq 1$, $(1-t)u + tv \in K$) and consider the problem: Find $u \in K$ such that $a(u, v-u) \geq (f,v)$ for all $v \in K$. When $a(u,v) = a(v,u)$, this problem is also equivalent to a minimization problem: Find $u \in K$ such that $J(u) \leq J(v)$ for all $v \in K$.

The canonical example of such a problem is the "obstacle" problem, in which $V = \mathring{H}^1(\Omega)$,

$$K = \{v \in V : v \geq \psi \text{ a.e. in } \Omega\}, \qquad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad (f,v) = \int_\Omega f\, v \, dx,$$

where $\psi(x)$ is a given function which we will assume $\in H^2(\Omega)$.

In the case of the variational equality, the variational equation says that $-\Delta u = f$ in Ω. In the case of the variational inequality describing the obstacle problem, this is no longer the case. Instead, we have that for any x, either $-\Delta u = f$ or $u = \psi$.

To get an approximation scheme, we let $V_h \subset V$ be finite dimensional and construct a closed convex subset K_h of V_h such that the following two conditions are satisfied: (i) Writing $v_h \in V_h$ as $v_h = \sum_{j=1}^{M} \beta_j \phi_j$, where $\{\phi_j\}$ are a basis for V_h, the set K_h should reduce to a finite number of constraints on the β_j; (ii) K_h should be a good approximation to K in a sense to be clearer in the error estimates.

In the case of the obstacle problem, a simple choice for the approximation scheme is to let V_h be the space of continuous piecewise linear functions with respect to a triangulation $\mathcal{T}_h$ of Ω (which we assume to be a convex polygon). We then define $K_h = \{v_h \in V_h : v_h \geq \psi_I \text{ for all } x \in \Omega\}$, where ψ_I is the interpolant of ψ in V_h. A key point here is that since v_h and ψ_I are piecewise linear, $v_h \in K_h$ is equivalent to requiring that $v_h(\boldsymbol{a}_i) \geq \psi_I(\boldsymbol{a}_i)$ for all vertices $\boldsymbol{a}_i$ of $\mathcal{T}_h$, so condition (i) above is satisfied.

The difficulty for the error analysis is that although $V_h \subset V$, K_h is not a subset of K. The reason for this is that $\psi_I(x)$ could be $< \psi(x)$, so requiring that $v_h \geq \psi_I$ does not guarantee that $v_h \geq \psi$.

In the case of variational equalities, we proved that

$$\|u - u_h\|_1 \leq \frac{M}{\alpha}\|u - v_h\|, \quad v \in V_h,$$

and so for C^0 piecewise linear functions, we get $\|u - u_h\|_1 \leq Ch\|u\|_2$. Letting A be the operator defined by the variational equation $(Au, v) = a(u,v)$ for all $v \in V$, we can show in the case of variational inequalities (see R. S. Falk, *Approximation of a Class of Variational*

Inequalities, Math. Comp., vol 28 (1974), 963-971) that

$$\|u-u_h\|_1 \le \left[\frac{M^2}{\alpha^2}\|u-v_h\|_1^2 + \frac{2}{\alpha}\|f-Au\|_0(\|u-v_h\|_0 + \|u_h-v\|_0)\right]^{1/2}, \quad v \in K, v_h \in K_h,$$

In the obstacle problem, $A = -\Delta$. If the constraint set were not present, $f - Au = 0$, so we recover exactly the previous estimate. However, despite the more complicated error estimate, we can still show that $\|u-u_h\|_1 \le Ch[\|u\|_2 + \|\psi\|_2]$. Choosing $v_h = u_I$, we know

$$\|u-u_I\|_1 \le Ch\|u\|_2, \qquad \|u-u_I\|_0 \le Ch^2\|u\|_2,$$

so the only tricky term to estimate is $\|u_h - v\|_0$, where $v \in K$. One can show that the choice $v = \max(u_h, \psi) \in K$. Now $u_h - v = 0$ when $u_h \ge \psi$ and $u_h - v = u_h - \psi$ when $u_h < \psi$. Since $u_h \ge \psi_I$, we can conclude that for any x, $|u_h - v| \le |\psi_I - \psi|$. Hence, $\|u_h - v\|_0 \le \|\psi - \psi_I\|_0 \le Ch^2\|\psi\|_2$. Combining these results, we have that $\|u-u_h\|_1 \le Ch[\|u\|_2 + \|\psi\|_2]$.

To solve the approximate problem, we can use the minimization formulation. Writing $v \in K_h = \sum_{j=1}^M \beta_j \phi_j(x)$, where the ϕ_j are the C^0 piecewise linear basis functions, i.e., the piecewise linear functions that are one at vertex $\boldsymbol{a}_j$ and zero at all the other vertices, we get

$$J(v) = \frac{1}{2}a(v,v) - (f,v) = \frac{1}{2}\sum_{i=1}^M\sum_{j=1}^M \beta_i\beta_j a(\phi_i,\phi_j) - \sum_{i=1}^M \beta_i(f,\phi_i) = \beta^T A\beta - \beta^T F,$$

where A is the matrix with entries $A_{ij} = a(\phi_i, \phi_j)$, F is the column vector with entries $F_i = (f, \phi_i)$ and β is the column vector with entries β_i. Hence J is a quadratic function of the β_i. The constraint set K_h consists of functions $v \in V_h$ such that $v \ge \psi_I$. The key fact is that since v and ψ_I are linear functions on each triangle, $v - \psi_I \ge 0$ on a triangle T if and only if $v - \psi_I \ge 0$ at the vertices of T. But at a vertex $\boldsymbol{a}_k$, $v(\boldsymbol{a}_k) = \sum_{j=1}^M \beta_j\phi_j(\boldsymbol{a}_k) = \beta_k$. Hence, the constraint set K_h is equivalent to the finite set of conditions $\beta_j \ge \psi_I(\boldsymbol{a}_j)$ for $j = 1, \ldots M$. This is a finite number of linear constraints on the β_j. Hence the finite dimensional problem becomes the minimization of a quadratic form subject to linear constraints, which can be solved by quadratic programming methods.