11.2. **Stability of difference schemes – examples.** In this section we present some examples to illustrate the theory.

Example 1: explicit scheme for the heat equation. In this case, we saw that \( C_{-1} = C_1 = \sigma k/h^2 \) and \( C_0 = 1 - 2\sigma k/h^2 \). Hence the amplification matrix

\[
G(p, h, k) = \sum_{q=-1}^{1} e^{ipq} C_q = e^{-ip\sigma k/h^2} + (1 - 2\sigma k/h^2) + e^{ip\sigma k/h^2} \\
= 1 - 2\sigma k/h^2 + 2\sigma k/h^2 \cos(ph).
\]

For stability, we want \( |G| \leq 1 + Mk \). Now \( \cos(2\theta) = 1 - 2\sin^2 \theta \), so

\[
G = 1 - 2\sigma k/h^2 + 2(\sigma k/h^2)[1 - 2\sin^2(ph/2)] = 1 - 4(\sigma k/h^2)\sin^2(ph/2).
\]

Then \(-1 - Mk \leq G \leq 1 + Mk\) if

\[
-2 - Mk \leq -4(\sigma k/h^2)\sin^2(ph/2) \leq Mk.
\]

The right inequality is always true, since \( \sigma, k > 0 \). For the left inequality, we need

\[
(\sigma k/h^2)\sin^2(ph/2) \leq (1/2) + Mk/4, \quad \forall p.
\]

Since \( \sin^2(ph/2) \) will be arbitrarily close to 1, the stability condition becomes:

\[
\sigma k/h^2 \leq (1/2) + Mk/4.
\]

If we let \( h, k \to 0 \) in such a way that \( \sigma k/h^2 \) remains constant, then we obtain the stability restriction \( \sigma k/h^2 \leq 1/2 \).

Example 2: transport equation \( u_t + \alpha u_x = 0 \). If we consider the scheme:

\[
[U_{j+1}^{n+1} - U_j^n]/k + \alpha[U_{j+1}^n - U_j^n]/h = 0, \quad \text{i.e.,}
\]

\[
k^{-1}[U_{j+1}^{n+1} - (1 + \alpha k/h)U_j^n + \alpha k/hU_{j+1}^n] = 0,
\]

then \( C_0 = 1 + \alpha k/h \) and \( C_1 = -\alpha k/h \). Hence,

\[
G(p, h, k) = 1 + \alpha k/h - (\alpha k/h)e^{iph}.
\]

For \( p = \pi/h \),

\[
G(p, h, k) = 1 + 2\alpha k/h > 1 + Mk,
\]

no matter how \( k, h \to 0 \). Hence, the scheme is unstable. Recall that the CFL condition is also violated in this case.

Example 3: If, instead, we consider the scheme:

\[
[U_{j+1}^{n+1} - U_j^n]/k + \alpha[U_{j+1}^n - U_{j-1}^n]/h = 0, \quad \text{i.e.,}
\]

\[
k^{-1}[U_{j+1}^{n+1} - (1 - \alpha k/h)U_j^n - \alpha k/hU_{j+1}^n] = 0,
\]

then \( C_0 = 1 - \alpha k/h \) and \( C_{-1} = \alpha k/h \). Hence,

\[
G(p, h, k) = 1 - \alpha k/h + (\alpha k/h)e^{-iph}.
\]

For \( \lambda = \alpha k/h \) satisfying \( 0 \leq \lambda \leq 1 \),

\[
|G| \leq |1 - \alpha k/h + (\alpha k/h)e^{-iph}| \leq |1 - \lambda| + |\lambda e^{-iph}| \leq 1 - \lambda + \lambda \leq 1.
\]
Hence, the scheme is stable for $0 < \alpha k/h \leq 1$. For $\lambda > 1$ and $ph \approx \pi$, 
\[ |G| \approx |1 - 2\alpha k/h| = 2\lambda - 1 > 1, \]
so the method is unstable.

**Example 4:**
\[
\begin{align*}
[U_j^{n+1} - U_j^n]/k + \alpha [U_{j+1}^n - U_{j-1}^n]/(2h) &= 0, \quad \text{i.e.,} \\
k^{-1}[U_j^{n+1} - U_j^n + \alpha k/(2h)(U_{j+1}^n - U_{j-1}^n)] &= 0,
\end{align*}
\]
then $C_0 = 1$, $C_- = \alpha k/(2h)$, $C_1 = -\alpha k/(2h)$. Hence,
\[ G(p, h, k) = 1 + \alpha k/(2h)e^{-iph} - \alpha k/(2h)e^{iph} = 1 - i(ak/h) \sin(ph). \]
Then
\[ |G| = [1 + (\alpha^2 k^2/h^2) \sin^2(ph)]^{1/2} \approx [1 + \alpha^2 k^2/h^2]^{1/2} \]
for $ph \approx \pi/2$. If $k = O(h)$, then the method is unstable, while if $k = ch^2$, then
\[ |G| = [1 + \alpha^2 ck]^{1/2} \leq [1 + \alpha^2 ck + \alpha^4 c^2 k^2/4]^{1/2} = 1 + \alpha^2 ck/2. \]
Hence, the method is stable in this case. However, this is a bad scheme, since it requires a very small time step.

### 11.3. Three-level explicit schemes

A scheme that was mentioned earlier was the approximation of the wave equation $u_{tt} = c^2 u_{xx}$ by the method
\[
[U_j^{n+1} - 2U_j^n + U_j^{n-1})]/k^2 = c^2 [U_{j+1}^n - 2U_j^n + U_{j-1}^n]/h^2.
\]
If we set $\lambda = ck/h$ and introduce a new variable $V_j^n = U_j^{n-1}$. then we convert this scheme to a two level scheme for the vector $(U_j^n, V_j^n)$, i.e., we have
\[
U_j^{n+1} = (2 - 2\lambda^2)U_j^n + \lambda^2 (U_{j+1}^n + U_{j-1}^n) - V_j^n, \quad V_j^{n+1} = U_j^n.
\]
In matrix form, this becomes:
\[
\begin{pmatrix} U_{j+1}^{n+1} \\ V_{j+1}^{n+1} \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_j^{n+1} \\ V_j^{n+1} \end{pmatrix} + \begin{pmatrix} 2 - 2\lambda^2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_j^n \\ V_j^n \end{pmatrix} + \begin{pmatrix} \lambda^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{j+1}^n \\ V_{j+1}^n \end{pmatrix}.
\]
Hence, the amplification matrix for this method is
\[
G(p, h, k) = \begin{pmatrix} \lambda^2 [e^{-iph} + e^{iph}] + 2 - 2\lambda^2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2\lambda^2 \cos(ph) + 2 - 2\lambda^2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 - 4\lambda^2 \sin^2(ph/2) & -1 \\ 1 & 0 \end{pmatrix}.
\]
We next show that the von Neumann condition is satisfied if as $h, k \to 0$, $\lambda = ck/h \leq 1$. Let $\beta = 2 - 4\lambda^2 \sin^2(ph/2)$. Then the eigenvalues of the matrix $G$ are the roots of
\[
\det \begin{pmatrix} \beta - x & -1 \\ 0 & x \end{pmatrix} = x^2 - \beta x + 1 = 0, \quad \text{i.e.,} \quad x = (\beta \pm \sqrt{\beta^2 - 4})/2.
\]
Now for $0 \leq \lambda \leq 1$, $-2 \leq \beta \leq 2$ and so $\beta^2 - 4 \leq 0$. For $|\beta| < 2$, the roots are complex conjugates and so $|x|^2 = (\beta^2 + 4 - \beta^2)/4 = 1$. If $|\beta| = 2$, then $|x| = 1$. Hence, $\rho(G) \leq 1$ and so the von Neumann condition is satisfied. However,

\[
GG^* = \begin{pmatrix} \beta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \beta^2 + 1 & \beta \\ \beta & 1 \end{pmatrix},
\]

\[
G^*G = \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \beta & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta^2 + 1 & -\beta \\ -\beta & 1 \end{pmatrix},
\]

and so $GG^* \neq G^*G$, i.e., $G$ is not a normal matrix. Hence, the von Neumann condition does not imply stability. To investigate stability for this problem, one can verify directly that for $0 < \lambda < 1$,

\[
\|G^n(p, h, k)\| \leq K, \quad \forall p, h, k, \quad 0 \leq n \leq N, \quad \lambda = ak/h.
\]

For $\lambda = 1$, $\|G^n\| \to \infty$ and the method is not stable.

11.4. Stability of two-level implicit schemes. In the homogeneous case, ($f = 0$), a constant coefficient two-level implicit scheme may be written in the form

\[
\sum_{q=-Q}^{Q} B_q U^{n+1}(x + qh) = \sum_{q=-Q}^{Q} C_q U^n(x + qh).
\]

Again writing $U^n$ in terms of its Fourier series, i.e.,

\[
U^n(x) = \sum_{p=-\infty}^{\infty} \hat{U}^n(p) e^{ipx},
\]

we have

\[
\sum_{p} \sum_{q=-Q}^{Q} B_q e^{ipqh} \hat{U}^{n+1}(p) e^{ipx} = \sum_{p} \sum_{q=-Q}^{Q} C_q e^{ipqh} \hat{U}^n(p) e^{ipx}.
\]

Hence,

\[
H_1(p, h, k) \hat{U}^{n+1}(p) = H_0(p, h, k) \hat{U}^n(p), \quad \text{where}
\]

\[
H_1(p, h, k) = \sum_{q=-Q}^{Q} B_q e^{ipqh}, \quad H_0(p, h, k) = \sum_{q=-Q}^{Q} C_q e^{ipqh}.
\]

Setting $G(p, h, k) = H_1^{-1} H_0$, we get

\[
\hat{U}^{n+1}(p) = G(p, h, k) \hat{U}^n(p).
\]

The previous theory carries over directly to this case: the difference scheme is stable if and only if there exists a constant $K$ independent of $h$, $k$, and $p$ such that

\[
\max_{0 \leq n \leq N-1} \|G^n(p, h, k)\| \leq K, \quad \forall p, h, k.
\]

The von Neumann condition is again necessary for stability and is also sufficient if $G$ is a normal matrix.
Example: implicit scheme for the heat equation

\[ \frac{U_{j}^{n+1} - U_{j}^{n}}{k} = \sigma [U_{j+1}^{n+1} - 2U_{j}^{n+1} + U_{j-1}^{n+1}] h^2, \]

which we rewrite in the form

\[ k^{-1}[-(\sigma k/h^2)U_{j+1}^{n+1} + (1 + 2\sigma k/h^2)U_{j}^{n+1} - (\sigma k/h^2)U_{j-1}^{n+1}] = k^{-1}U_{j}^{n}. \]

Then

\[ B_{-1} = B_{1} = -\sigma k/h^2, \quad B_{0} = 1 + 2\sigma k/h^2, \quad C_{0} = 1. \]

Hence,

\[ H_{1} = -(\sigma k/h^2)(e^{iph} + e^{-iph}) + 1 + 2\sigma k/h^2 \]

\[ = 1 + 2\sigma k/h^2 - 2(\sigma k/h^2) \cos(ph/2) = 1 + (4\sigma k/h^2) \sin^2(ph/2). \]

Since \( H_{0} = 1 \), we get that

\[ 0 \leq G = 1/[1 + (4\sigma k/h^2) \sin^2(ph/2)] \leq 1, \]

and so the implicit method is unconditionally stable.