13. Finite element methods for parabolic problems

We consider the parabolic problem:

\[ u_t - \text{div}(p\nabla u) + qu = f, \quad (x, t) \in \Omega \times (0, T], \]
\[ u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T], \quad u(x, 0) = g(x), \quad x \in \Omega. \]

A variational formulation of this problem is to seek \( u(t) \in \tilde{H}^1(\Omega) \) such that

\[ (\partial u/\partial t, v) + a(u, v) = (f, v), \quad v \in \tilde{H}^1(\Omega), \]

where as in the elliptic case, \((\cdot, \cdot)\) denotes the \(L^2(\Omega)\) inner product and

\[ a(u, v) = \int_{\Omega} [p\nabla u \cdot \nabla v + quv] \, dx. \]

13.1. Continuous time Galerkin scheme. We first consider an approximation in which we discretize by finite elements in the spatial variable, but keep time continuous. Thus, we choose a finite dimensional subspace \( V_h \subset \tilde{H}^1(\Omega) \) and look for an approximation \( u_h(t) \in V_h \), \( t \in [0, T] \), satisfying:

\[ u_h(0) = g_h \text{ (an approximation to } g) \]
\[ (\partial u_h/\partial t, v) + a(u_h, v) = (f, v), \quad v \in V_h. \]

To see what is involved in solving this problem, we write \( u_h(t) = \sum_{j=1}^{m} \alpha_j(t) \phi_j(x) \). Inserting this into the variational equations, and choosing \( v \) to be each of the basis functions \( \phi_i \), we get

\[ \sum_{j=1}^{m} \alpha'_j(t) (\phi_j, \phi_i) + \sum_{j=1}^{m} \alpha_j(t) a(\phi_j, \phi_i) = (f, \phi_i), \quad i = 1, \ldots, m. \]

Let

\[ M_{ij} = (\phi_j, \phi_i), \quad A_{ij} = a(\phi_j, \phi_i), \quad F_i = (f, \phi_i), \quad \alpha = (\alpha_1, \ldots, \alpha_m)^T. \]

Our equations then have the form

\[ M\alpha'(t) + A\alpha = F, \]

a first order system of ordinary differential equations.

One can obtain a simple error estimate for this approximation scheme by comparing the approximate solution to the elliptic projection \( w_h(t) \in V_h \), satisfying

\[ a(u(t) - w_h(t), v_h) = 0, \quad v_h \in V_h. \]

We showed previously that if \( V_h \) consists of piecewise polynomials of degree \( \leq r \), and \( u \) is sufficiently smooth, then

\[ \|u(t) - w_h(t)\| + h\|u(t) - w_h(t)\|_1 \leq C h^{r+1}\|u(t)\|_{r+1}. \]

**Theorem 14.** If \( V_h \) consists of piecewise polynomials of degree \( \leq r \) and \( u \) is sufficiently smooth, then for \( t \geq 0 \),

\[ \|u(t) - u_h(t)\| \leq \|g - g_h\| + C h^{r+1}\left[\|g\|_{r+1} + \int_0^t \|u_t\|_{r+1} \, ds\right]. \]
Proof. We estimate the error by writing \( u - u_h = (u - w_h) + (w_h - u_h) \). From the above, we have
\[
\| u(t) - w_h(t) \| \leq C h^{r+1} \| u(t) \|_r + C h^{r+1} \| u(0) \| + \int_0^t \| u_t(s) \|_r \, ds \\
\leq C h^{r+1} \left[ \| g \|_r + \int_0^t \| u_t(s) \|_r \, ds \right].
\]

It thus remains to estimate \( \| u_h - w_h \| \). Using the continuous and discrete variational formulations and the definition of \( w_h(t) \), we get
\[
\left( \partial \left[ u_h - w_h \right] / \partial t, v \right) + a(u_h - w_h, v) = \left( \partial [u_h - u] / \partial t, v \right) + a(u_h - u, v) \\
+ \left( \partial [u - w_h] / \partial t, v \right) + a(u - w_h, v) = \left( \partial [u - w_h] / \partial t, v \right), \quad v \in V_h.
\]
Choosing \( v = u_h - w_h \), and observing that
\[
\| u_h - w_h \| \frac{d}{dt} \| u_h - w_h \| = \frac{1}{2} \frac{d}{dt} \| u_h - w_h \|^2 = (u_h - w_h, u_h - w_h),
\]
we get
\[
\| u_h - w_h \| \frac{d}{dt} \| u_h - w_h \| + \| u_h - w_h \|^2_E = (u_h - w_h, u_h - w_h) \leq \| (u_h - w_h, t) \| \| u_h - w_h \|.
\]
Hence,
\[
\frac{d}{dt} \| u_h - w_h \| \leq \| (u - w_h, t) \|.
\]
Integrating this equation between 0 and \( t \), we get
\[
\| u_h(t) - w_h(t) \| \leq \| u_h(0) - w_h(0) \| + \int_0^t \| (u_h - w_h)(s) \| \, ds \\
\leq \| u_h(0) - u(0) \| + \| u(0) - w_h(0) \| + \int_0^t \| (u - w_h)(s) \| \, ds \\
\leq \| g - g_h \| + Ch^{r+1} \left[ \| g \|_r + \int_0^t \| u_t(s) \|_r \, ds \right].
\]
Using the triangle inequality, and combining estimates, we then obtain
\[
\| u(t) - u_h(t) \| \leq \| u(t) - w_h(t) \| + \| u_h(t) - w_h(t) \| \\
\leq \| g - g_h \| + Ch^{r+1} \left[ \| g \|_r + \int_0^t \| u_t(s) \|_r \, ds \right].
\]
\( \square \)

13.2. Fully discrete schemes. One way to get a fully discrete scheme is to combine the use of finite elements to discretize the spatial variable with a finite difference approximation in time. For example, if we approximate \( u_t \) by the backward Euler approximation, we get the scheme: Find \( U^n \in V_h \), satisfying \( U^0 = g_h \) and for \( n \geq 0 \)
\[
\left( [U^{n+1} - U^n] / k, v \right) + a(U^{n+1}, v) = (f^{n+1}, v) \quad \forall v \in V_h.
\]
Using the matrices defined previously, and defining $U^n(x) = \sum_{j=1}^{n} \alpha^n \phi_j(x)$, the discrete variational formulation above corresponds to the linear system

$$(M + kA)\alpha^{n+1} = M\alpha^n + kF^{n+1}, \quad n = 0, 1, \ldots.$$  

Another choice is the Crank-Nicholson-Galerkin method, which has the form: Find $U^n \in V_h$, satisfying $U^0 = g_h$ and for $n \geq 0$

$$([U^{n+1} - U^n]/k, v) + a([U^{n+1} + U^n]/2, v) = ([f^{n+1} + f^n]/2), \quad v \in V_h.$$

In this case, we get the linear system

$$(M + kA)\alpha^{n+1} = (M - kA)\alpha^n + k(F^{n+1} + F^n)/2, \quad n = 0, 1, \ldots.$$  

For the backward Euler method, we have the following error estimate ($t_n = nk$).

**Theorem 15.**

$$\|u(t_n) - U^n\| \leq \|g - g_h\| + Ch^{r+1} \left[ \|g\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1} \right] + k \int_0^{t_n} \|u_{tt}(s)\| \, ds, \quad n \geq 0.$$  

**Proof.** As before, we write $u(t_n) - U^n = (u(t_n) - W^n) + (W^n - U^n)$, where $W^n = w_h(t_n)$. From our previous result, we have

$$\|u(t_n) - W^n\| \leq Ch^{r+1} \left[ \|g\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1} \, ds \right].$$

To estimate $U^n - W^n$, we again use our continuous and discrete variational formulations, but this time obtaining

$$([U - W]^{n+1} - (U - W)^n)/k, v) + a((U - W)^{n+1}, v) = ([U - u]^{n+1} - (U - u)^n)/k, v) + a((U - W)^{n+1}, v) + a((u - W)^{n+1}, v) + a((u - W)^n, v) \equiv (\rho^n, v) \quad v \in V_h.$$

Choosing $v = (U - W)^{n+1}$, we get

$$\|U - W\|^{n+1} + k\|U - W\|^{n+1} \leq \|(U - W)^n, (U - W)^{n+1}\) + k(\rho^n, (U - W)^{n+1}) \leq \|\|(U - W)^n\| + k\|\rho^n\|\|\|(U - W)^{n+1}\|.$$  

Hence,

$$\|U - W\|^{n+1} \leq \|U - W\|^n + k\|\rho^n\|.$$  

Iterating this equation, we get

$$\|U - W\|^n \leq \|U - W\|^0 + k \sum_{j=0}^{n-1} \|\rho^j\|. $$
Now
\[
\rho^j = k^{-1} \int_{t_j}^{t_{j+1}} \left[ -(s - t_j)u_{tt}(s) + (u - w_h)_t(s) \right] ds
\]
\[
\leq k^{-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + \|(u - w_h)_t(s)\|] ds
\]
\[
\leq k^{-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + Ch^{r+1}\|u_t(s)\|_{r+1}] ds.
\]
Hence,
\[
k \sum_{j=0}^{n-1} \|\rho^j\| \leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + Ch^{r+1}\|u_t(s)\|_{r+1}] ds
\]
\[
\leq k \int_{t_0}^{t_n} \|u_{tt}(s)\| ds + Ch^{r+1} \int_{t_0}^{t_n} \|u_t(s)\|_{r+1} ds.
\]
The theorem follows by combining all these results. \qed