We consider the approximation of the initial value problem:
\[
\beta \cdot \nabla u + \sigma u = f, \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_{in}(\Omega),
\]
where \( \beta \) is a unit vector, \( \sigma \geq c > 0 \) and
\[
\Gamma_{in}(\Omega) = \{ x \in \partial \Omega : \beta \cdot n < 0 \},
\]
where \( n \) is the unit outward normal to \( \Omega \). We shall assume that \( \Omega \) is a polygon and \( T_h \) is a triangulation of \( \Omega \). Then it is possible to order the triangles triangle by triangle.

Within a layer, \( u_h \in P_n(T) \) such that \( u_h^- = g_h \) on \( \Gamma_{in}(\Omega) \) and satisfies
\[
(\beta \cdot \nabla u_h + \sigma u_h, v)_T - \int_{\Gamma_{in}(T)} (u_h^+ - u_h^-) v \beta \cdot n \, ds = (f, v)_T, \quad v \in P_n(T),
\]
where for \( x \in \Gamma_{in}(T) \), \( u_h^+(x) = \lim_{\epsilon \to 0} u_h(x \pm \epsilon \beta) \). If we solve these equations using the ordering discussed above, then \( u_h^- \) is known at the time it is needed to compute the solution on triangle \( T \). On each triangle, we need to solve a simple square linear system of equations. Note that the solution produced is a piecewise polynomial, but one that is discontinuous across triangle edges. The key to the analysis of this method is the following identity.

**Lemma 9.** Assume that \( \beta \) is a constant vector. Then
\[
(\beta \cdot \nabla u, u)_T - \int_{\Gamma_{in}(T)} (u^+ - u^-) u^+ \beta \cdot n \, ds = \frac{1}{2} \int_{\Gamma_{out}(T)} (u^-)^2 |\beta \cdot n| \, ds + \frac{1}{2} \int_{\Gamma_{in}(T)} (u^+ - u^-)^2 |\beta \cdot n| \, ds - \frac{1}{2} \int_{\Gamma_{in}(T)} (u^-)^2 |\beta \cdot n| \, ds.
\]

One important implication of this identity is that it easily follows that the linear system on each triangle has a unique solution for \( \sigma > 0 \). To see this, we need only show that if \( f = 0 \) and \( u_h^- = 0 \), then \( u_h = 0 \). Choosing \( v = u_h^+ \) and using the above identity, we get
\[
\frac{1}{2} \int_{\Gamma_{out}(T)} (u^-)^2 |\beta \cdot n| \, ds + \frac{1}{2} \int_{\Gamma_{in}(T)} (u^+ - u^-)^2 |\beta \cdot n| \, ds + \sigma \| u_h \|^2_T = 0,
\]
and so \( u_h = 0 \).

In analyzing this problem, it is helpful to think of \( u_h \) as evolving in layers \( S_i \), defined by
\[
S_0 = \emptyset, \quad S_i = \{ T \in T_h : \Gamma_{in}(T) \subset \Gamma_{in}(\Omega - \cup_{j<i} S_j) \}, \quad j = 1, 2, \cdots.
\]
Within a layer, \( u_h \) can be developed in parallel. We can also define a sequence of fronts \( F_i \), to which \( u_h \) has advanced after it has been computed in \( \Omega_i = \cup_{j \leq i} S_j \).
In the case when \( f = 0 \), we also have a very simple stability analysis that can be expressed in the above terms.

**Theorem 16.**

$$\frac{1}{2}|u_h^-|_{F_i}^2 + \sigma \|u_h\|_{\Omega_i}^2 \leq \frac{1}{2}|u_h^-|_{\Gamma_{in}(\Omega)}^2.$$

**Proof.** Applying our identity with \( f = 0 \), we obtain

$$\frac{1}{2} \int_{\Gamma_{out}(T)} (u^-)^2 |\beta \cdot n| \, ds + \frac{1}{2} \int_{\Gamma_{in}(T)} (u^+ - u^-)^2 |\beta \cdot n| \, ds + \sigma \|u_h\|_{T}^2 = \frac{1}{2} \int_{\Gamma_{in}(T)} (u^-)^2 |\beta \cdot n| \, ds.$$

Summing over all the triangles in the layer \( S_i \) and omitting the positive jump terms, we get

$$\frac{1}{2}|u_h^-|_{F_i}^2 + \sigma \sum_{T \in S_i} \|u_h\|_{T}^2 \leq \frac{1}{2}|u_h^-|_{F_{i-1}}^2.$$

The theorem follows by iterating this inequality. \( \square \)

Also using this key identity, we are able to show that

$$\|u - u_h\|_{L^2(\Omega)} \leq C h^{n+1/2} \|u\|_{n+1}.$$

Note that this is not an optimal order error estimate, since the best approximation by polynomials of degree \( \leq n \), would be \( O(h^{n+1}) \).

15. **The finite volume method for elliptic problems**

We consider the approximation of the problem

$$- \text{div}(a \nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.$$

Finite volume methods are based on an approximation of the balance equation

$$- \int_{\partial b} a \nabla u \cdot n \, ds = \int_{b} f \, dx,$$

valid for any subdomain \( b \subset \Omega \), where \( n \) denotes the unit outward normal to the boundary of \( b \). Note this equation can be obtained by integrating the partial differential equation over the subdomain \( b \) and applying the divergence theorem. One approach to the finite volume method, and the one we will discuss, uses a finite element partition of \( \Omega \), where the approximate solution space consists of piecewise linear functions, a collection of vertex-centered control volumes, and a test space consisting of piecewise constant functions over the control volumes.

More precisely, we begin with a family of triangulations \( \{T_h\} \) of the domain \( \Omega \) and let

$$X_h = \{ v \in H_0^1(\Omega) : v|_T \in P_1, \quad \forall T \in T_h \}.$$

To construct the control volumes, we let \( z_T \) denote the barycenter of \( T \) and connect \( z_T \) to the midpoints of the edges of \( T \) with line segments. This partitions each triangle \( T \) into three quadrilaterals. We use the notation \( K_{z,T} \) to denote the quadrilateral in \( T \) which shares the vertex \( z \) of the triangle \( T \). To each vertex \( z \) of the triangulation, we associate a control
volume $b_z$ consisting of the union of the quadrilaterals $K_{z,T}$, where the union is taken over all triangles $T$ containing the vertex $z$.

The finite volume method is then to find $u_h \in X_h$ such that

$$- \int_{\partial b_z} a \nabla u_h \cdot n \, ds = \int_{b_z} f \, dx,$$

for all $z \in Z^0_h$, where $Z^0_h$ denotes the set of interior vertices of the mesh $T_h$.