

1.4. Other approaches to approximation on domains with curved boundaries. A simple scheme for domains with curved boundaries is to define U_h as the solution of

$$-\Delta_h U_h(x, y) = f(x, y), \quad (x, y) \in \Omega_h^0, \quad U_h(x, y) = g(x', y'), \quad (x, y) \in \Omega_h^*,$$

where (x', y') is one of the neighbors of (x, y) on $\partial\Omega_h$. In this case, we only use the standard 5 point difference approximation to the Laplacian. The result is:

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \leq M_1 h + \frac{M_4 d^2}{96} h^2, \quad M_1 = \max_{\Omega} (\max |\partial u / \partial x|, |\partial u / \partial y|).$$

Note that the crude approximation of the boundary condition gives only an $O(h)$ error estimate.

An $O(h^2)$ approximation can be obtained by the following method, due to L. Collatz. Define $U_h(x, y)$ as the solution of

$$-\Delta_h U_h(x, y) = f(x, y), \quad (x, y) \in \Omega_h^0, \quad U_h(x, y) = g(x, y), \quad (x, y) \in \partial\Omega_h,$$

and for $(x, y) \in \Omega_h^*$, define $U_h(x, y)$ as the linear interpolate of the value of U_h at two neighbors of (x, y) , one in Ω_h^0 and one on $\partial\Omega_h$. For example, if $(x + h, y) \in \Omega_h^0$ and $(x - \alpha h, y) \in \partial\Omega_h$, define U_h at the point (x, y) by

$$U_h(x, y) = \frac{\alpha}{\alpha + 1} U_h(x + h, y) + \frac{1}{\alpha + 1} U_h(x - \alpha h, y).$$

The result is:

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \leq M_2 h^2 + \frac{M_4 d^2}{48} h^2, \quad M_2 = \max_{\Omega} (\max |\partial^2 u / \partial x^2|, |\partial u^2 / \partial y^2|).$$

1.5. Other boundary conditions. We next consider the boundary condition

$$\alpha(x, y)u(x, y) + \beta(x, y) \frac{\partial u}{\partial n}(x, y) = g(x, y).$$

Consider first the case of a point on a straight boundary, say $x = 1$, and $0 < y < 1$. At the boundary point $(1, y)$, an $O(h)$ approximation to $\partial u / \partial n = \partial u / \partial x$ is given by $[u(1, y) - u(1 - h, y)]/h$, so the boundary condition would be approximated by:

$$\alpha(1, y)u(1, y) + \beta(1, y)[u(1, y) - u(1 - h, y)]/h = g(1, y).$$

An $O(h^2)$ approximation to $\partial u / \partial x$ is given by the centered difference: $[u(1 + h, y) - u(1 - h, y)]/(2h)$. This introduces a new unknown at the point $1 + h, y$ outside the domain. Hence, we need an additional equation. Assuming that the solution is smooth and the partial differential equation holds on the boundary as well, we can use the 5 point difference approximation to the Laplacian applied at the boundary point, i.e., we have the equation

$$U_h(1 + h, y) + U_h(1 - h, y) + U_h(1, y + h) + U_h(1, y - h) - 4U_h(1, y) = h^2 f(1, y).$$

This equation can be used to eliminate the new unknown.

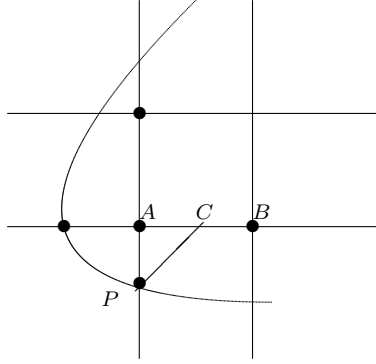
If the boundary is curved, draw the normal line through the point $P = (x, y)$ and assume it intersects a mesh line at a point C where C lies between the mesh points A and B . Then

we approximate $\partial u/\partial n(P)$ by $[u(P) - u(C)]/|P - C|$, where $|P - C|$ denotes the distance between P and C and $u(C)$ is defined by linear interpolation using $u(A)$ and $u(B)$, i.e.,

$$u(C) = \frac{|B - C|}{|B - A|}u(A) + \frac{|C - A|}{|B - A|}u(B).$$

Inserting this formula gives a linear relation equation involving $u(A)$, $u(B)$, and $u(P)$.

Approximation of $\partial u/\partial n$



1.6. Higher order approximations. To get higher order approximations to $\Delta u(x, y)$, we need to take more points at a larger distance from (x, y) . Using Taylor series expansions, we have

$$\begin{aligned} u(x \pm kh, y) = u(x, y) &\pm kh \frac{\partial u}{\partial x}(x, y) + \frac{k^2 h^2}{2} \frac{\partial^2 u}{\partial x^2}(x, y) \pm \frac{k^3 h^3}{6} \frac{\partial^3 u}{\partial x^3}(x, y) \\ &+ \frac{k^4 h^4}{24} \frac{\partial^4 u}{\partial x^4}(x, y) \pm \frac{k^5 h^5}{120} \frac{\partial^5 u}{\partial x^5}(x, y) + \frac{k^6 h^6}{6!} \frac{\partial^6 u}{\partial x^6}(\xi_{\pm}, y). \end{aligned}$$

Hence,

$$u(x + kh, y) + u(x - kh, y) - 2u(x, y) = 2 \frac{k^2 h^2}{2} \frac{\partial^2 u}{\partial x^2}(x, y) + 2 \frac{k^4 h^4}{24} \frac{\partial^4 u}{\partial x^4}(x, y) + O(h^6).$$

For $k = 1, 2$, this gives

$$\begin{aligned} u(x + h, y) + u(x - h, y) - 2u(x, y) &= h^2 \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{h^4}{12} \frac{\partial^4 u}{\partial x^4}(x, y) + O(h^6), \\ u(x + 2h, y) + u(x - 2h, y) - 2u(x, y) &= 4h^2 \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{16h^4}{12} \frac{\partial^4 u}{\partial x^4}(x, y) + O(h^6). \end{aligned}$$

Taking 16 times the first equation minus the second equation, we get

$$\begin{aligned} 16u(x + h, y) + 16u(x - h, y) - u(x + 2h, y) - u(x - 2h, y) - 30u(x, y) \\ = 12h^2 \frac{\partial^2 u}{\partial x^2}(x, y) + O(h^6). \end{aligned}$$

Taking a similar expansion in the y variable, we get

$$\begin{aligned} & \{16[u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h)] \\ & \quad - [u(x+2h, y) + u(x-2h, y) + u(x, y+2h) + u(x, y-2h)] - 60u(x, y)\} / (12h^2) \\ & \qquad \qquad \qquad = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) + O(h^4). \end{aligned}$$

Note that in the case of the unit square, this higher order approximation cannot be used at interior mesh points at a distance h from the boundary of Ω , since it would involve mesh points outside the domain. At these points, we can use the 5-point difference approximation without affecting the overall accuracy of the method.

On domains with curved boundaries, the situation is much more complicated. Use of the Shortly-Weller formula would decrease the rate of convergence. So instead, we consider finite element methods, which handle these difficulties in a more natural way.

1.7. More general elliptic operators. In this section, we develop difference approximations for the operator $\text{div}(a \text{grad } u)$ in two dimensions, where $a = a(x, y)$. Although we could expand this operator to $a\Delta u + \nabla a \cdot \nabla u$ and then apply difference approximations, this is not the best approach, since it does not lead to a symmetric matrix. To see a better approach, consider the one dimensional analogue, $(d/dx)(adu/dx)$. The basic idea is to use the approximation $dp/dx(x_j) \approx \frac{1}{h}[p(x_{j+1/2}) - p(x_{j-1/2})]$. Choosing $p = adu/dx$, we get

$$\begin{aligned} \frac{d}{dx} \left[a \frac{du}{dx} \right] (x_j) & \approx \frac{1}{h} \left\{ \left[a \frac{du}{dx} \right] (x_{j+1/2}) - \left[a \frac{du}{dx} \right] (x_{j-1/2}) \right\} \\ & \approx \frac{1}{h^2} \{ a(x_{j+1/2})[u(x_{j+1}) - u(x_j)] - a(x_{j-1/2})[u(x_j) - u(x_{j-1})] \}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{div}(a \text{grad } u) & \approx \frac{1}{h^2} \{ a(x_{j+1/2}, y_l)[u(x_{j+1}, y_l) - u(x_j, y_l)] - a(x_{j-1/2}, y_l)[u(x_j, y_l) - u(x_{j-1}, y_l)] \\ & \quad + a(x_j, y_{l+1/2})[u(x_j, y_{l+1}) - u(x_j, y_l)] - a(x_j, y_{l-1/2})[u(x_j, y_l) - u(x_j, y_{l-1})] \}. \end{aligned}$$